Using Reachable Sets to Simulate Dynamic Games

Eugene Li and Bryan Yaya, May 6th, 2009

Outline

- Reachable Sets
- Collision Avoidance Example
  - Previous Work with Evader-Pursuer Dynamic Game
  - Hamilton-Jacobi PDE
- Enhancements
  - Variable Velocity
    - Different averages and deviations
    - Fixed Velocity for Pursuer and vice versa
    - Non-optimal Solutions
- References

Reachable Sets

- Complex systems require verification and validation
- Simplest method is simulation
  - Slow! Checks a single trajectory at a time
- Reachable sets allow us to capture behavior of entire groups of trajectories at once
- Types of reachable sets
  - Forwards – set of states that can be reached by the system given a set of initial states
  - Backwards – set of states that can give rise to trajectories given a set of target states

Reachable Sets (continued)

- Can be used in collision-avoidance example
- If pursuer is within this backwards reachable set from the evader, this states that there exists a way to catch the evader no matter what the evader does

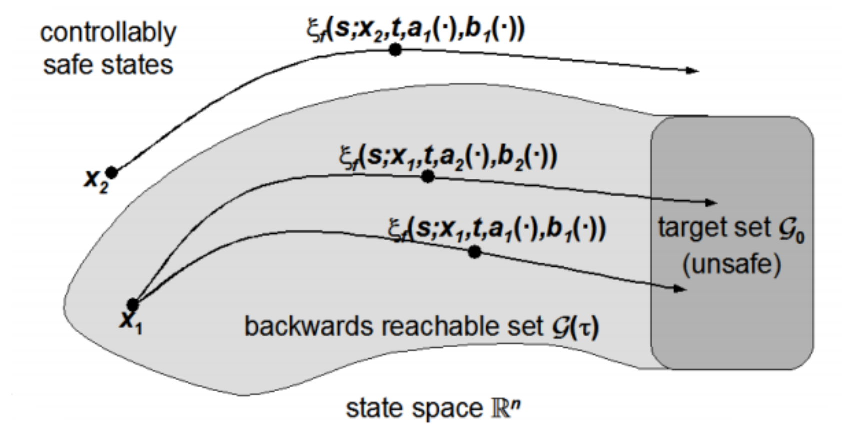


Diagram and consequent diagrams from Mitchell, et. al. 2005

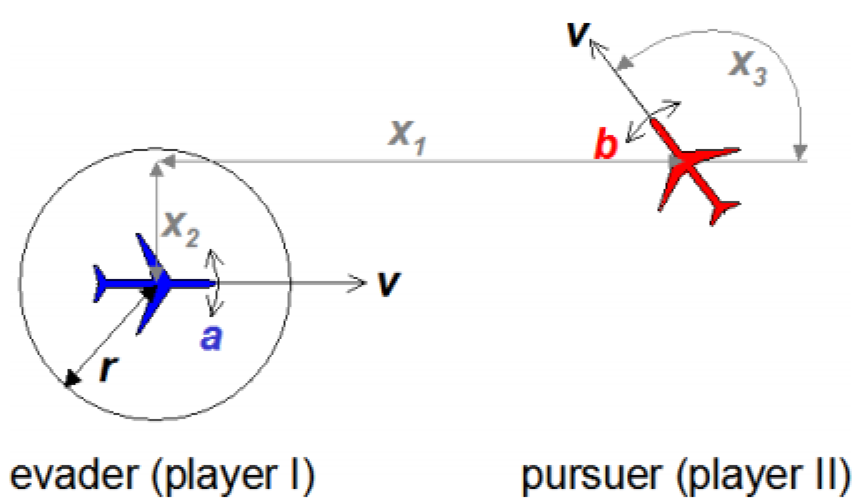
Collision Avoidance Example

- Ian Mitchell, Alexandre Bayen, Claire Tomlin posed a collision-avoidance problem with reachable sets as a way to solve them.
- Championed a new approach to use a Hamilton-Jacobi PDE to calculate these reachable sets.
- MATLAB toolbox created to more easily calculate HJI PDEs for different examples. Mitchell's site: http://www.cs.ubc.ca/~mitchell/ToolboxLS/index.html

Collision Avoidance Example (contd)

- System modeled by an ODE
- Pursuer wants to move within a radius of the evader
- Each vehicle modeled as point objects
- Fixed linear velocity
- Inputs as angular velocity

$$\dot{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -v_a + v_b \cos x_3 + u x_2 \\ v_b \sin x_3 - u x_1 \\ d - u \end{bmatrix} = f(x,u,d)$$



Collison Avoidance Example (contd)

- Assumption I
  - Inputs *a* and *b* are selected from compact sets A and B. Time runs from [-T, 0], where T is positive.
- Assumption II
  - System is uniformly continuous, bounded, and Lipschitz continuous in x for fixed u and d.
- Assumption III
  - Target set G_0 is closed and represented as the zero sublevel set where $G_0 = \{(x_1, x_2) \in \Re^2, x_3 \in [-\pi, \pi) \mid x_1^2 + x_2^2 \leq 5^2\}$

Collision Avoidance Example (contd)

- Evolution of backwards reachable set found by solving

$$D_t v(x,t) + \min[0, H(x, D_x v(x,t))] = 0 \quad \text{for } t \in [-T, 0], x \in \Re^n$$

where H(x,p) is

$$H(x,p) = \max_{u \in U} \min_{d \in D} p^T f(x,u,d)$$

- v(x,t) is an implicit surface representation of the backwards reachable set, where p is

$$p = \nabla v(x,t)$$

Collision Avoidance Example (contd)

- Evader chooses the input to maximize the optimal Hamiltonian, whereas pursuer tries to minimize

$$H^*(x,p) = \max_{u \in U} \min_{d \in D} \left[-p_1 v_1 + v_2 (p_1 \cos x_3 + p_2 \sin x_3) + (p_1 x_2 - p_2 x_1 - p_3) u + p_3 d \right]$$

- Therefore, define switching functions

$$s_1(t) = p_1(t) x_2(t) - p_2(t) x_1(t) - p_3(t)$$

$$s_2(t) = p_3(t)$$

- When these functions are not zero, the respective optimal inputs for the evader and pursuer are

$$u^* = \operatorname{sgn}(s_1)$$
$$d^* = -\operatorname{sgn}(s_2)$$

Enhancement: Velocity input

- Instantaneous change in velocity in the dynamic equations:

$$\dot{x}_1 = -v_1 + v_2 \cos x_3 + u x_2$$
$$\dot{x}_2 = v_2 \sin x_3 - u x_1$$
$$\dot{x}_3 = d - u$$

- Optimal Hamiltonian:

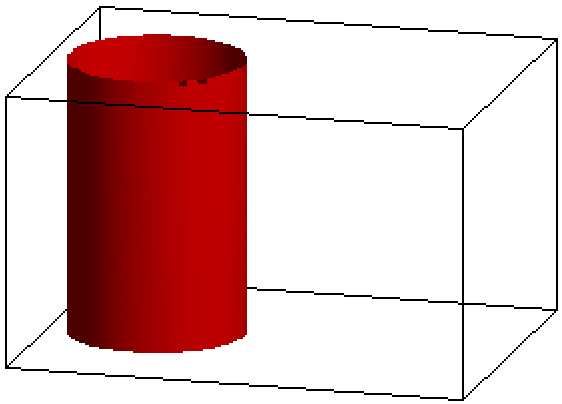
$$H^*(x,p) = \max_{v_1 \in V_1} \min_{v_2 \in V_2} \max_{u \in U} \min_{d \in D} \left[-p_1 v_1 + v_2 (p_1 \cos x_3 + p_2 \sin x_3) + (p_1 x_2 - p_2 x_1 - p_3) u + p_3 d \right]$$

- Switching functions and corresponding input selections:

$$s_1(t) = p_1(t) x_2(t) - p_2(t) x_1(t) - p_3(t)$$
$$s_2(t) = p_3(t)$$
$$s_3(t) = -p_1(t)$$
$$s_4(t) = p_1(t) \cos\left(x_3(t)\right) + p_2(t) \sin\left(x_3(t)\right)$$

$$u^* = \mathrm{sgn}(s_1)$$
$$d^* = -\mathrm{sgn}(s_2)$$
$$v_1^* = \begin{cases} v_{\max} & s_3 > 0 \\ v_{\min} & else \end{cases}$$
$$v_2^* = \begin{cases} v_{\min} & s_4 > 0 \\ v_{\max} & else \end{cases}$$

Solution



- Identical aircrafts:

$$u, d \in U = [-1, 1]$$

$$v_1, v_2 \in V = [1, 5]$$

Study the effects of different velocity input sets

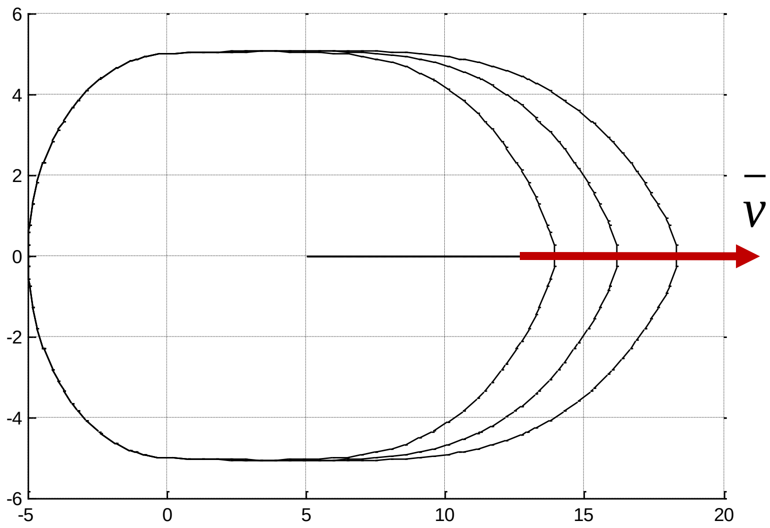
- Define the velocity sets of v_1 and v_2:

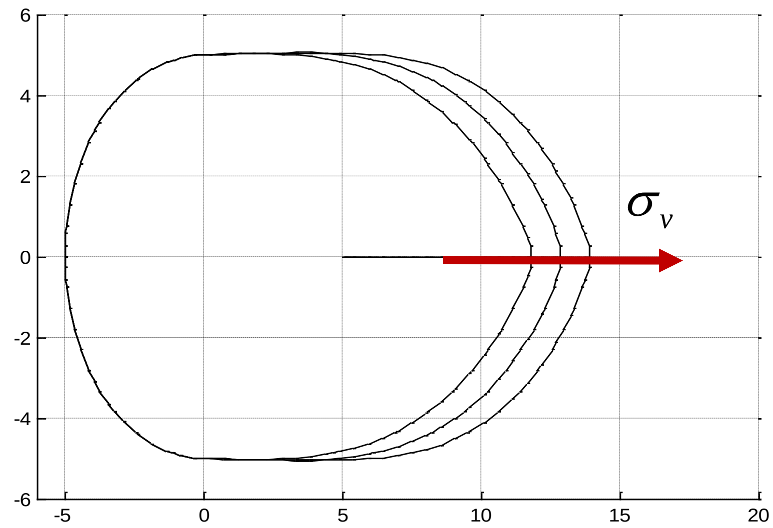
$$v_1, v_2 \in V = \bar{v} + \sigma_v$$

$\bar{v}$: mean

σ_v: deviation

- Declare identical aircrafts:
  - Vary the values of the mean and the deviation





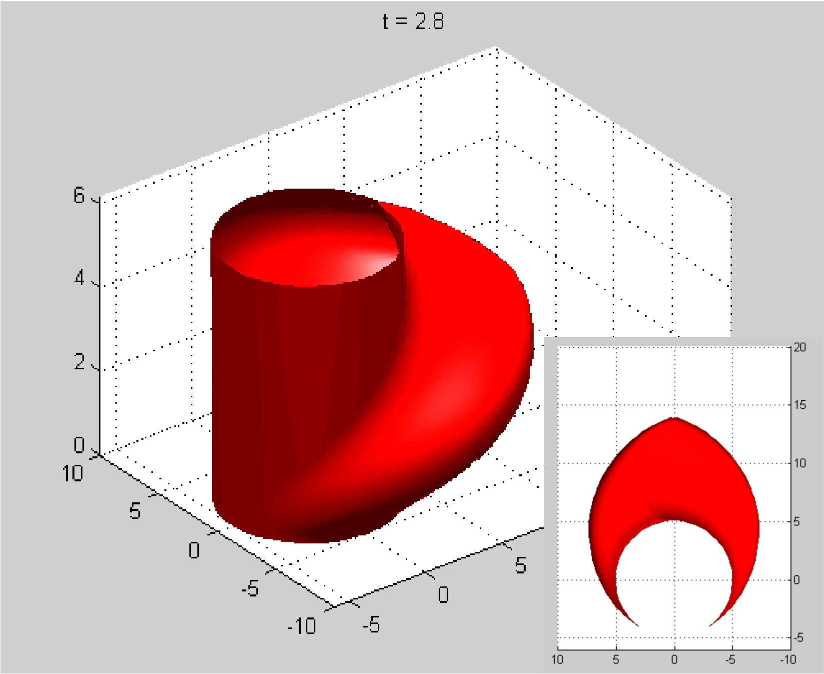
Set v_1 as a constant and v_2 as an input

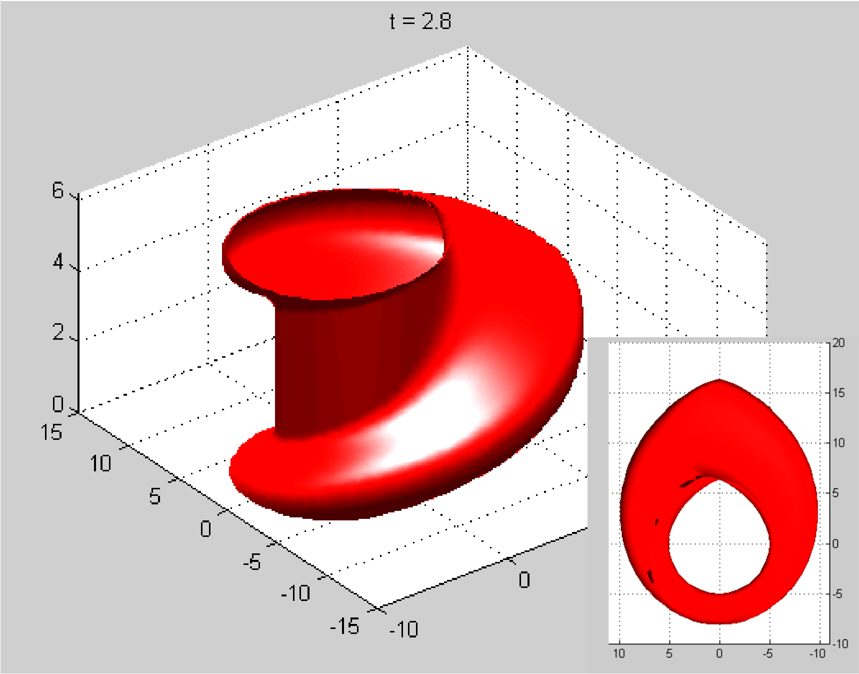
- Bounds on the velocity inputs; keep heading change inputs

$$v_1 \in V_1 = 3 \qquad u \in U = [-1, 1]$$

$$v_2 \in V_2 = 3 \pm 2 = [1, 5] \qquad d \in D = [-1, 1]$$

- Reachability set expands





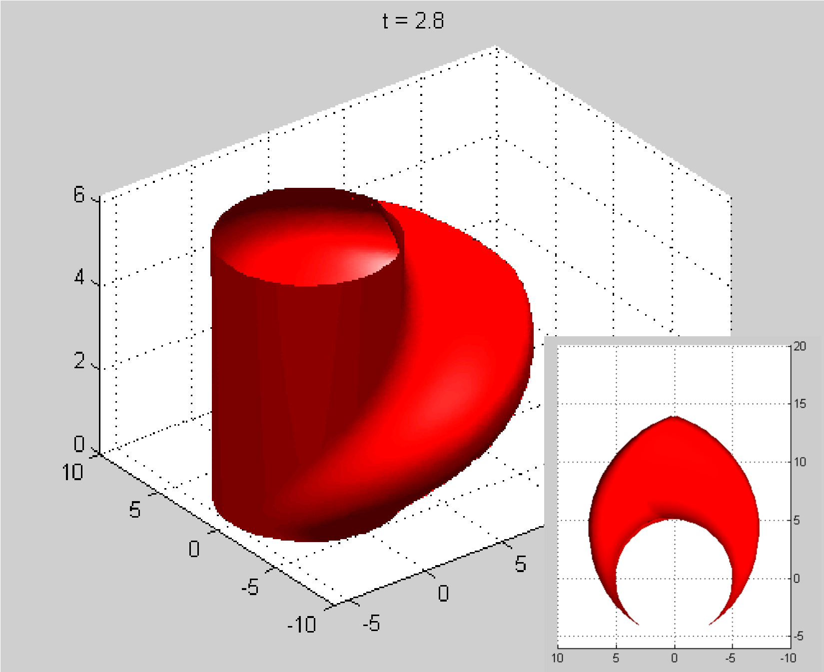
Set v_2 as a constant and v_1 as an input

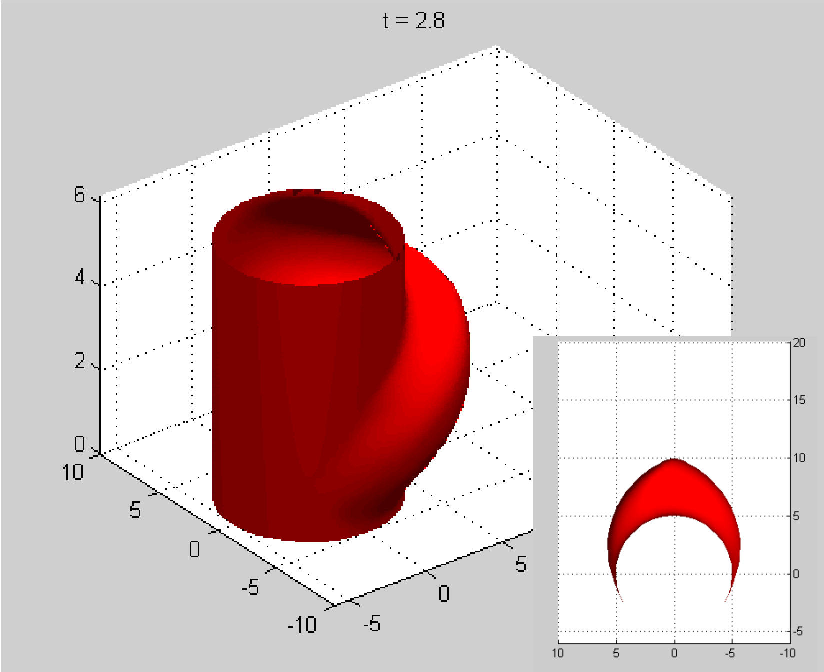
- Bounds on the velocity inputs; keep heading change inputs

$$v_1 \in V_1 = 3 \pm 2 = [1, 5] \qquad u \in U = [-1, 1]$$

$$v_2 \in V_2 = 3 \qquad d \in D = [-1, 1]$$

- Reachability set contracts

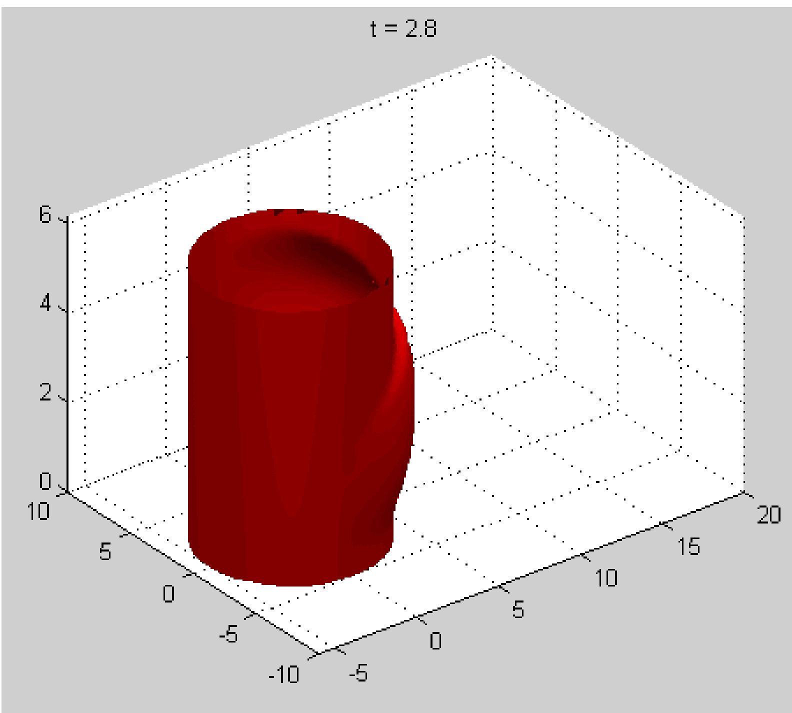


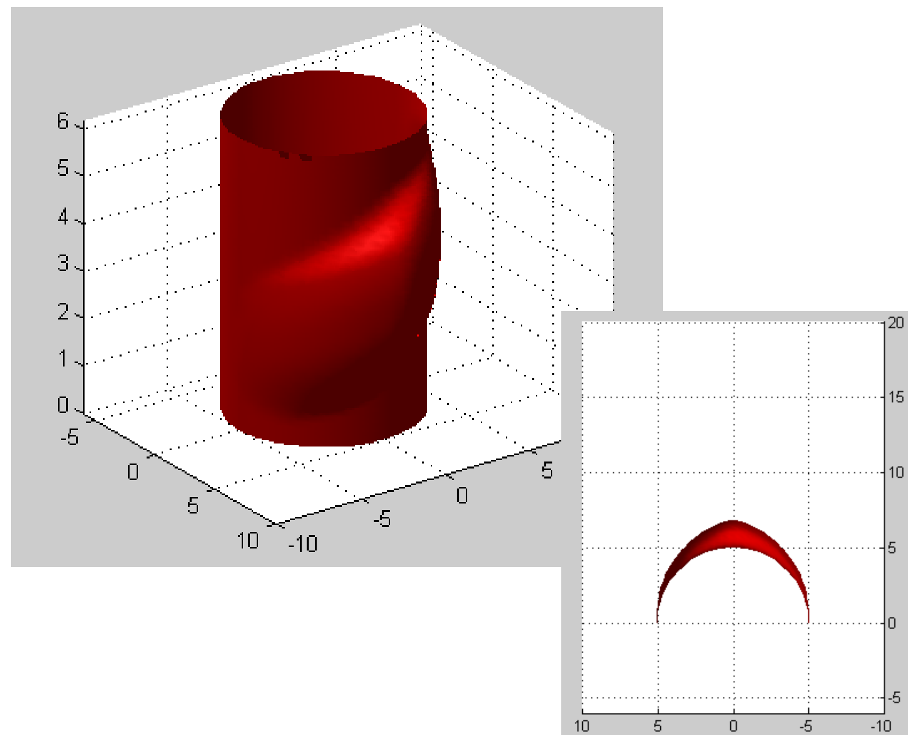


Non-optimal Solutions

- The pursuer's moves are not optimized in the Hamiltonian:

$$H^*(x,p) = \max_{v_1 \in V_1} \boxed{\max_{v_2 \in V_2}} \max_{u \in U} \min_{d \in D} \left[-p_1 v_1 + v_2 (p_1 \cos x_3 + p_2 \sin x_3) + (p_1 x_2 - p_2 x_1 - p_3) u + p_3 d \right]$$

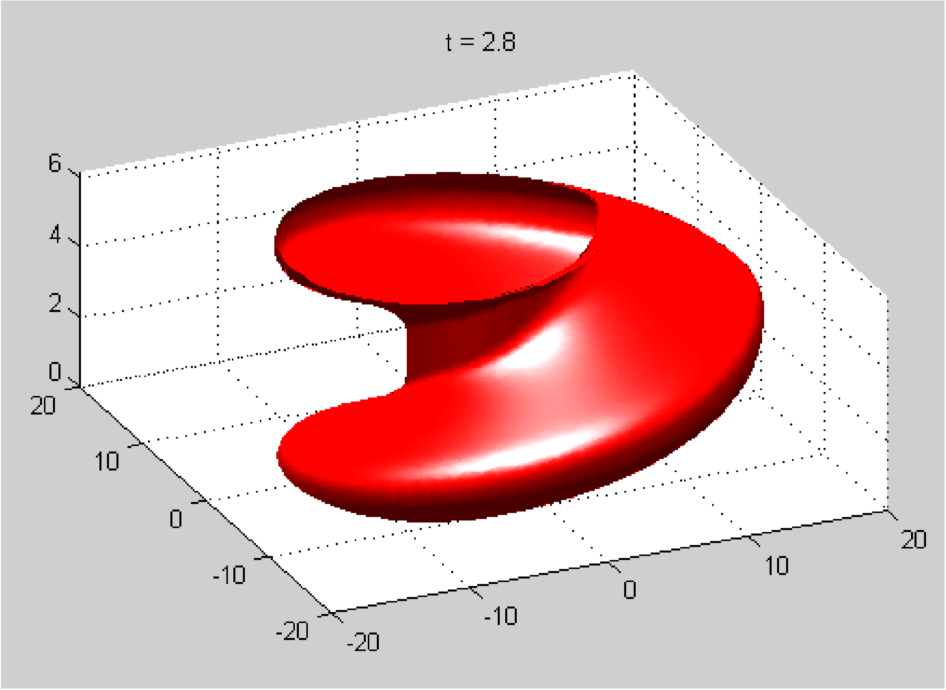


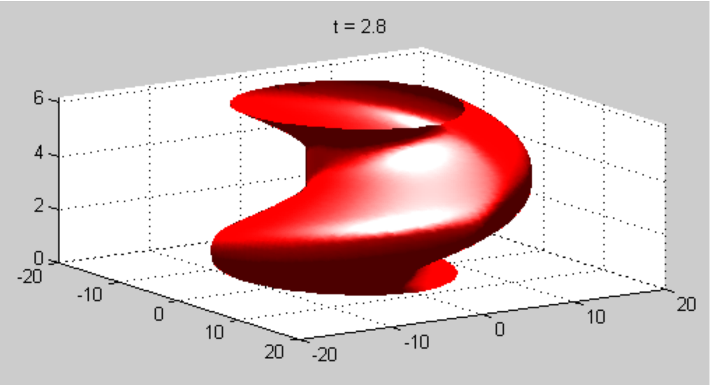


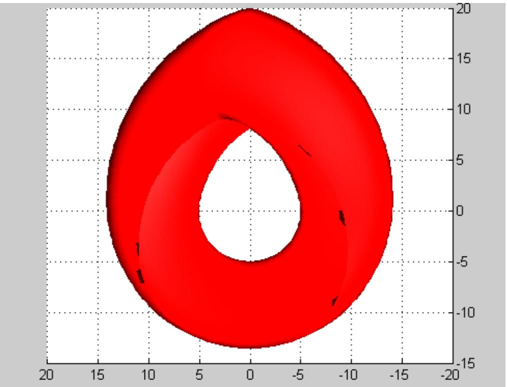
Non-optimal Solutions

- The evader's moves are not optimized in the Hamiltonian:

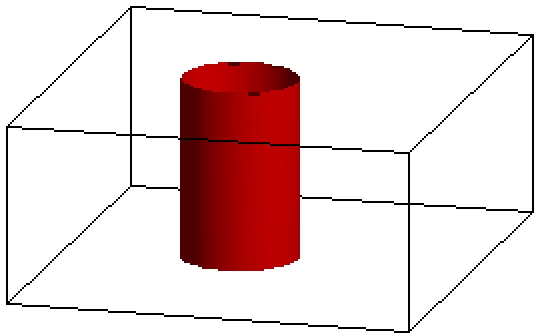
$$H^*(x,p) = \boxed{\min_{v_1 \in V_1}} \min_{v_2 \in V_2} \max_{u \in U} \min_{d \in D} \left[-p_1 v_1 + v_2 (p_1 \cos x_3 + p_2 \sin x_3) + (p_1 x_2 - p_2 x_1 - p_3) u + p_3 d \right]$$







Non-optimal Solution:



Future Work

- Influence of velocity inputs on the convergence of the reachability set.
- Utilize reachable sets to compute capture sets for guaranteed safety
  - Apply to more complicated system dynamics
  - Aggressive maneuvers for quad-rotor vehicles

References

- Ian M. Mitchell, Alexandre M. Bayen, Claire J. Tomlin , "A Time-Dependent Hamilton-Jacobi Formulation of Reachable Sets for Continuous Dynamic Games," *IEEE Transactions on Automatic Control*, vol. 50, no. 7, pp. 947-957, 2005
- Claire Tomlin, Ian Mitchell, Ronojoy Ghosh, "Safety Verification of Conflict Resolution Maneuvers," *IEEE Transactions on Intelligent Transportation Systems*, vol. 2, no. 2, pp 110-120, 2001
- "toolboxLS.pdf" tutorial file given in the ToolboxLS-1.1 program, pp 51-54.