CALIFORNIA CENTER FOR INNOVATIVE TRANSPORTATION
INSTITUTE OF TRANSPORTATION STUDIES
UNIVERSITY OF CALIFORNIA, BERKELEY

A hydrodynamic theory based statistical model of arterial traffic

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CCIT Research Report
UCB-ITS-CWP-2011-2

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Contents

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A. Hofleitner*, R. Herring† and A. Bayen‡

Abstract

In arterial networks, the dynamics of traffic flows are driven by the presence of traffic signals. A comprehensive model of the dynamics of arterial traffic flow is necessary to capture the specifics of arterial traffic and provide accurate traffic estimation. From hydrodynamic theory of traffic flow, we model the dynamics of arterial traffic under specific assumptions which are standard in transportation engineering. We use this flow model to develop a statistical model of arterial traffic. First, we derive an analytical expression for the spatial distribution of vehicles. This encompasses the fact that the average density of vehicles is higher close to the traffic signals because of the delays experienced by the vehicles. Second, we derive the probability distribution of *total* and *measured* delay (to be defined specifically in the document). The delays experienced by vehicles traveling on a link of the network depend on the time (from the beginning of the cycle) when they enter the link. We model the probability of delay for a path between two arbitrary points on the link. The probability distribution of *measured* delay takes into account the sampling scheme to derive the probability of the observed delay from probe vehicles sampled uniformly in time. Finally, we use the probability distribution of delays and a model of driving behavior to derive the probability distribution of travel times between any two arbitrary points on a link. The analytical derivations are parameterized by traffic variables (cycle time, red time, model of free flow speed, queue length and queue length at saturation). The models estimates queue length (and thus congestion levels), signal parameters and variability of driving behavior. We show that the probability distributions of travel times on an arterial links are quasi-concave. The probability distributions of travel times between any arbitrary location on the link are finite mixture distributions where each component represents a class of vehicles depending on the characteristics of its delay. We prove that each component of the mixture distribution is log-concave, which enables the use of specific optimization algorithm. The distributions derived in this report are used as fundamental building block for arterial traffic estimation using sparse travel time measurements from probe vehicles used in subsequent work.

*Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, aude@eecs.berkeley.edu, http://eecs.berkeley.edu/~aude

†Department of Industrial Engineering and Operations Research, University of California, Berkeley

‡Department of Civil and Environmental Engineering, Systems Engineering, University of California, Berkeley

1 Introduction

In numerous parts of the world, traffic congestion comes with important external costs due to added travel time, wasted fuel and increase in traffic accidents [23]. Numerous measures can be taken to address problems due to traffic congestion. An essential step for operations and planning is to create the ability to estimate and forecast traffic conditions with appropriate accuracy and reliability.

Historically, the design of highway traffic monitoring systems mostly relied on dedicated sensing infrastructure (loop detectors, radars, video cameras). When properly deployed, these data feeds provide sufficient information to reconstruct macroscopic traffic variables (flow, density, velocity) using traffic flow models developed in the literature [16, 21, 7]. However, for the secondary network or highways not covered by this infrastructure, traffic estimation models face challenges associated with probe vehicle data, which comes from various sources (fleet data, smartphones, RFID tags), each of them with specific issues (bias, noise, coverage).

Proof of concept studies have demonstrated the feasibility of designing highway traffic monitoring systems relying on probe data only [13, 26]. Arterials come with additional challenges: the underlying flow physics which governs them is more complex and highly variable (traffic lights with unknown cycles in general, turn movements, pedestrian traffic). Microscopic models have mainly focused on modeling single intersections (or a few intersections) relying on significant data availability assumptions (including signal timing, vehicle counts or high penetration rate of travel time measurements) [3]. While macroscopic flow models exist for the secondary network [11, 22], their parameters require site-specific calibration experiments. In addition, even if they were known, the complexity and statistical variability of the underlying flows make it challenging to perform estimation of the full macroscopic state of the system at low penetration rates of probe vehicles (which appears to be one of the few available data sources for arterial networks in the near future, at a global scale).

An important challenge in arterial traffic estimation is the characterization of travel time distributions. The study of speeds and travel time distributions is part of ongoing research that started in the 1950s with the emergence of flow-based traffic engineering [4]. This research area has been closely related to queuing theory and delay estimation for fixed cycle traffic lights. The arrival of vehicles is often modeled as a Poisson Process and the mean average delay and queue length at the end of the green time are derived using analytical expressions and numerical simulation [25, 17]. These results are generalized to arrival processes for which the number of arrivals per time interval is a discrete random variable [8, 6, 15]. The *probability generating function* (pgf) of overflow queue [8] was derived for general arrival distributions. The characterization of the stationary delay distribution was derived under simplification assumptions [2, 12] and with computational methods [18]. These articles model queues at traffic signals under stationary assumptions and numerically characterize the link delay distribution under specific assumptions. Analytical formulas of the mean delay are given in the Highway Capacity Manual [1] and related work [10]. They rely on static parameters of the road (number of lanes, average flow, cycle timing), rarely accessible on large scale networks. Dynamic estimation of the average delay and its variance has been derived for vertical queues [24], *i.e.* not modeling the location of where vehicles stop.

In light of the challenges of arterial traffic estimation, a statistical approach for characterizing the macroscopic state of traffic is well-suited toward designing a robust, scalable arterial traffic monitoring system. To our knowledge, an analytical characterization of horizontal queue dynamics and the corresponding travel time probability distributions between arbitrary locations on an arterial network is still an emerging field, for which few contributions exist.

We use the physics of traffic flows as a basis for designing probability distributions on the traffic variables. This work provides a hydrodynamic theory based statistical model of arterial traffic. We formulate specific assumptions on the physics of traffic flow which make the problem tractable, while keeping it realistic. From this theory, we derive the probability distribution of vehicle locations on arterial links, delimited by signalized intersections. Because of the presence of traffic signals, vehicles spend more time downstream of the links, where they experience delay. The probability distribution of vehicle locations enables the estimation of queue lengths, which is a measure of congestion. We use hydrodynamic theory to derive probability distributions of travel times between arbitrary points of the network. These distributions are characterized by a small set of parameters with direct physical interpretation (signal timing, queue length). When travel time measurements are available, one can estimate these parameters and thus estimate the probability distribution of travel times. Moreover, these parameters represent valuable information for traffic management entities. We prove the quasi concavity of link travel time distributions for future use in maximum likelihood estimator derivations. This feature is key to enable machine learning algorithms used in a companion article [14]

The rest of this work is organized as follows. In Section 2, we present traffic theory results derived from hydrodynamic models and queuing theory. We use these results in Sections 3 and 4 to derive probability distributions of vehicle measurement locations on an arterial link (Section 3) and delay distributions between two points on an arterial link (Section 4). Noticing that the travel time is the sum of the delay and the free flow travel time, we derive the probability distribution of travel times in Section 5.

2 Modeling

2.1 Traffic model

In traffic flow theory, it is common to model vehicular flow as a continuum and represent it with macroscopic variables of *flow* $q(x,t)$ (veh/s), *density* $\rho(x,t)$ (veh/m) and *velocity* $v(x,t)$ (m/s). The definition of flow gives the following relation between these three variables [16, 21]:

$$q(x,t) = \rho(x,t)\, v(x,t). \tag{1}$$

We will use this property throughout our derivations of traffic models.

For low values of density, experimental data shows that the velocity of traffic is relatively insensitive to the density; and all vehicles travel close to the so called free flow velocity of the corresponding road segment v_f. As density increases, there is a critical density ρ_c at which the flow of vehicles reaches the capacity $q_{\max}$ of the road. As the density of vehicles increases beyond ρ_c, the velocity decreases monotonically until it reaches zero at the maximal density $\rho_{\max}$. The maximal density can be thought of as the maximum number of vehicles that can physically fit

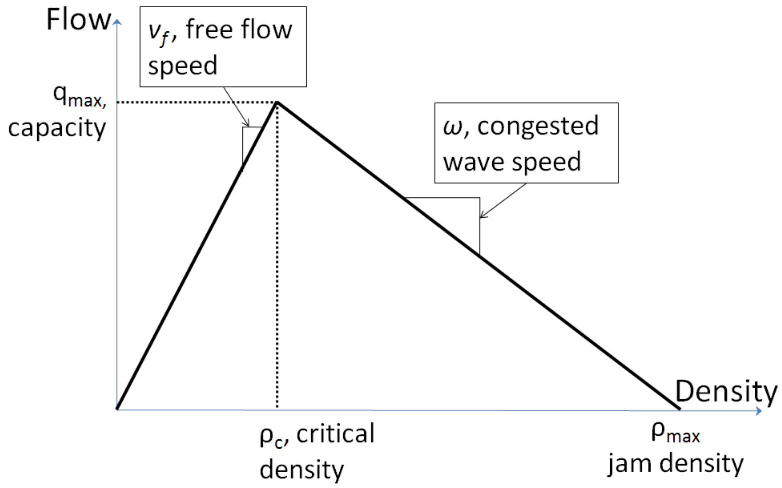

Figure 1: The fundamental diagram: empirically constructed relation between flow and density of vehicles.

per unit length, and at this density, vehicles are unable to move without additional space between vehicles. Experimental data indicates a decreasing linear relationship between flow and density, as the density increases beyond ρ_c. The slope of this line is referred to as the congested wave speed, noted w. This leads to the common assumption of a triangular *fundamental diagram* (FD) to model traffic flow dynamics [7].

The triangular FD (illustrated in Figure 1) is thus fully characterized by three parameters: v_f, the free flow speed (m/s); $\rho_{\max}$, the jam (or maximum) density (veh/m); and $q_{\max}$, the capacity (veh/m).

We note that ρ_c represents the boundary density value between (i) free flowing conditions for which cars have the same velocity and do not interact and (ii) saturated conditions for which the density of vehicles forces them to slow down and the flow to decrease. When a queue dissipates, vehicles are released from the queue with the maximum flow—capacity $q_{\max}$—which corresponds to the critical density $\rho_c = q_{\max}/v_f$.

For a given road segment of interest, the arrival rate at time t, i.e. the flow of vehicles entering the link at t, is denoted by $q_a(t)$. Conservation of flow relates it to the arrival density $\rho_a(t) = q_a(t)/v_f$.

2.2 Traffic flow modeling assumptions

We make the following assumptions on the dynamics of traffic flow and discuss their range of validity:

1. *Triangular fundamental diagram*: this is a standard assumption in transportation engineering.
2. *Stationarity of traffic*: during each estimation interval, the parameters of the light cycles (red and cycle time) and the arrival density ρ_a are constant. Moreover, we assume that there is not a consistent increase or decrease in the length of the queue, nor instability. With these assumptions, the traffic dynamics are periodic with period C (length of the light cycle). The

work is mainly focused on deriving travel time distributions for cases in which measurements are sparse. Constant quantities (for a limited period of time) do not limit the derivations of the model since we are here interested in trends rather than fluctuations.

3. *First In First Out* (FIFO) model: overtaking on the road network is neglected. When traffic is congested, it is generally difficult or impossible to pass other vehicles. In undersaturated conditions, vehicles can choose their own free flow speed, but we assume that the free flow speeds are similar enough that the "no overtaking" assumption is a good approximation.
4. *Model for differences in driving behavior*: the free flow pace (inverse of the free flow speed) is not the same for all vehicles: it is modeled as a random variable with vector of parameter θ_p—*e.g.* the free flow pace has a Gaussian or Gamma distribution with parameter vector $\theta_p = (\bar{p}_f, \sigma_p)^T$ where $\bar{p}_f$ and σ_p are respectively the mean and the standard deviation of the random variable.

2.3 Arterial traffic dynamics

In arterial networks, traffic is driven by the formation and the dissipation of queues at intersections. The dynamics of queues are characterized by shocks, which are formed at the interface of traffic flows with different densities.

We define two discrete traffic regimes: *undersaturated* and *congested*, which represent different dynamics of the arterial link depending on the presence (respectively the absence) of a remaining queue when the light switches from green to red. Figure 2 illustrates these two regimes under the assumptions made in Section 2.2. The speed of formation and dissolution of the queue are respectively called v_a and w. Their expression is derived from the Rankine-Hugoniot [9] jump conditions and given by

$$v_a = \frac{\rho_a v_f}{\rho_{\max} - \rho_a} \quad \text{and} \quad w = \frac{\rho_c v_f}{\rho_{\max} - \rho_c}. \tag{2}$$

Undersaturated regime. In this regime, the queue fully dissipates within the green time. This queue is called the *triangular queue* (from its triangular shape on the space-time diagram of trajectories). It is defined as the spatio-temporal region where vehicles are stopped on the link. Its length is called the maximum queue length, denoted $l_{\max}$, which can also be computed from traffic theory:

$$l_{\max} = R\frac{w v_a}{w - v_a} = R\frac{v_f}{\rho_{\max}}\frac{\rho_c \rho_a}{\rho_c - \rho_a}. \tag{3}$$

The duration between the time when the light turns green and the time when the queue fully dissipates is the *clearing time* denoted τ. We have

$$\tau = l_{\max}\left(\frac{1}{w} + \frac{1}{v_f}\right). \tag{4}$$

Replacing the $l_{\max}$ and w by their expressions derived in equations(3) and (2), we have

$$\tau = R\frac{\rho_a}{\rho_c - \rho_a}. \tag{5}$$

Congested regime. In this regime, there exists a part of the queue downstream of the triangular queue called *remaining queue* with length l_r corresponding to vehicles which have to stop multiple times before going through the intersection.

All notations introduced up to here are illustrated for both regimes in Figure 2.

Stationarity of the two regimes. Assumption 2 made earlier implies the periodicity of these queue evolutions (see Figure 2). As indicated by the slopes of the trajectories in the figure, when vehicles enter the link, they travel at the free flow speed v_f. The distance between two vehicles is the inverse of the arrival density $1/\rho_a$. The time during which vehicles are stopped in the queue is represented by the horizontal line in the queue. The length of this line represents the delay experienced at the corresponding location. The distance between vehicles stopped in the queue is the inverse of the maximum density $1/\rho_{\max}$. When the queue dissipates, vehicles are released with a speed v_f and a density ρ_c. The trajectory is represented by a line with slope v_f, the distance between two vehicles is $1/\rho_c$.

We next use these two discrete regimes to derive the pdf for the location of vehicles on a link and for the travel time along a link. The estimation of the distribution of vehicle location uses the individual measurement locations reported by the vehicles. The measurements being sent uniformly in time, vehicles are more likely to report their location where their speed is lower, where they experience delay. Because of the presence of traffic lights, vehicle are more likely to report their location on the downstream part of the link than on the upstream part. Section 3 develops a model for estimating vehicle distribution location. A probabilistic model based on the assumptions formulated in this section provides the pdf of delays (Section 4) and travel times (Section 5) between any two arbitrary locations on the network.

2.4 Notation

The list below summarizes the notation introduced earlier and to be used in the rest of the article. The parameters are specific for each network link j. The index j is omitted for notational simplicity.

- Model parameters:
  - ◇ Free flow pace, p_f (seconds/meter), inverse of the free flow velocity v_f. The free flow pace is a random variable. Its *probability distribution function* (p.d.f.) is denoted $\varphi^p(p)$; it models the different driving behavior by assuming a distribution of the free flow pace among the different drivers,
  - ◇ Cycle time, C (seconds),
  - ◇ Red time, R (seconds),
  - ◇ Length of the link, L (meters).
- Traffic state variables:
  - ◇ Clearing time τ,
  - ◇ Triangular queue length,
  - ◇ Remaining queue length, l_r.

This set of variables is sufficient to characterize the model and the time evolution of the state of traffic. The location x on a link corresponds to the distance from the location to the downstream intersection. From these variables, we can compute the other traffic variables, including velocity, flow, and density of vehicles at any x and time t and queue length. The remaining queue length l_r is specific to the congested regime ($l_r = 0$ in the undersaturated regime). Similarly, the existence of a

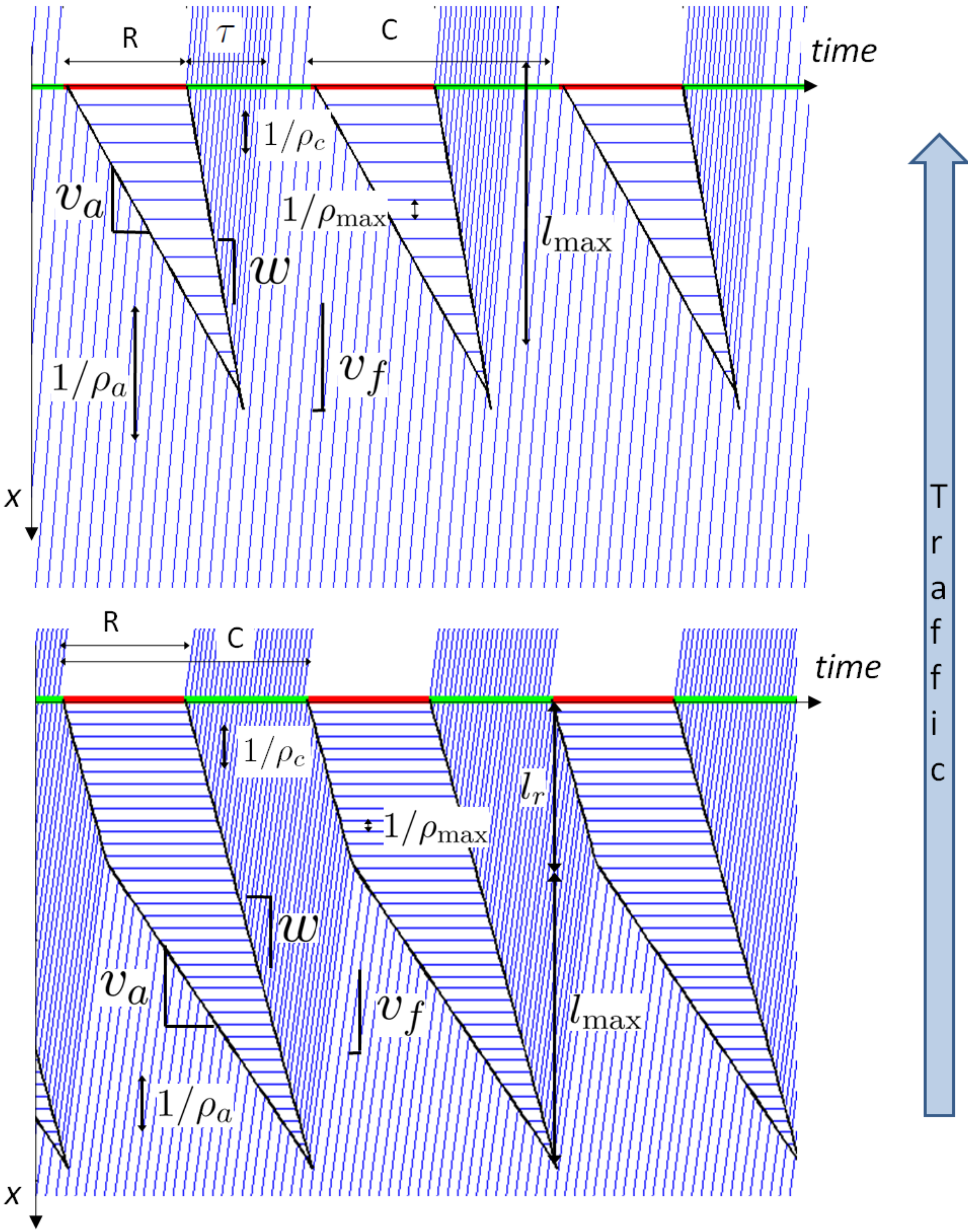

Figure 2: Space time diagram of vehicle trajectories with uniform arrivals under an undersaturated traffic regime (top) and a congested traffic regime (bottom).

clearing time is specific to the undersaturated regime (the clearing time is null and thus $\tau = C - R$ in the congested regime).

The undersaturated and congested regimes are labeled u and c respectively. In the following, we derive probability distribution of vehicle locations $f^s(x)$, $s \in \{u, c\}$ based on statistical analysis of queuing theory. We use the variable x to indicate the distance *to* the intersection, so the location of the intersection is at $x = 0$ and the start of the link is at $x = L$. The function f^s encodes the probability of a vehicle to be at location x, which depends on x because of the spatial heterogeneity of the density, due to the formation of queues at intersections, as can be seen in Figure 2. We also derive probability distributions for the delay δ_{x_1,x_2} and travel time y_{x_1,x_2} between two locations x_1 and x_2 on a link of the network, noted respectively $h(\delta_{x_1,x_2})$ and $g(y_{x_1,x_2})$. Using the stationarity assumption, we define temporal averages of the traffic variables. These averages are then taken over a light cycle C. For example, we define Z as the average number of vehicles present on a link, with index u (resp. c) for the undersaturated (resp. the congested) regime.

Finally, given that the term *density* has a very specific meaning in traffic theory, we use the term *probability distribution* to refer to a *probability density function.*

3 Modeling the spatial distribution of vehicles on an arterial link

In typical traffic monitoring systems relying on probe data, probe vehicles send periodic location measurements, which provide two sources of indirect information about the arterial traffic link parameters. (i) As the location measurements are taken uniformly over time, more densely populated areas *of the link* will have more location measurements. (ii) The time spent between two consecutive location measurements provides information on the speed at which the vehicle drove through the corresponding arterial link(s).

We use the traffic flow model presented in Section 2 to derive the probability distribution of vehicle locations (averaged over time), which corresponds to the probability distribution of measurement locations. The derivation relies on the computation of the average vehicle density over a cycle.

3.1 General case

Using the stationarity assumption, the density at location x is time periodic with period C. We define the average density $d(x)$ at location x as the temporal average of the density $\rho(x,t)$ at location x and time t.

$$d(x) = \frac{1}{C} \int_0^C \rho(x,t)\, dt$$

In practice, flow is never perfectly periodic of period C (even in stationary conditions), but we will assume that the above averaging over a duration C is a good proxy of a longer average.

According to the model assumptions, the density at location x and time t takes one of the three following values, numbered 1 to 3 for convenience: (1) $\rho_1 = \rho_{\max}$, when vehicles are stopped, (2) $\rho_2 = \rho_c$ when vehicles are dissipating from a queue, (3) $\rho_3 = \rho_a$ when vehicles arrive at the link and have not stopped in the queue.

The average density at location x is thus

$$d(x) = \sum_{i=1}^{3} \alpha_i(x)\rho_i$$

where $\alpha_i(x)$ represents the fraction of time that the density is equal to ρ_i at location x.

The probability distribution $f(x)$ of vehicle location at location x is proportional to the average density $d(x)$ at location x, with the proportionality constant given by $Z = \int_0^L d(x)\,dx$ so that

$$f(x) = \frac{d(x)}{\int_0^L d(x)\,dx}.$$

In the undersaturated and the congested regime, the computation of the $\alpha_i(\cdot)$, $i = 1 \dots 3$ enables the derivation of the probability distribution of vehicle locations.

3.2 Undersaturated regime

Upstream of the maximum queue length, the density remains constant at ρ_a throughout the whole light cycle. Downstream of the maximum queue length, the value of the density varies over time during the light cycle and takes one of the three density values ρ_1, ρ_2 and ρ_3.

Using the assumption that the FD is triangular and that the arrival density is constant, the average density increases linearly from ρ_a to the value it takes at the intersection, where $x = 0$. At the intersection, the density is $\rho_{\max}$ during the red time R. The density is ρ_c when the queue dissipates, *i.e.* during the clearing time $\tau = l_{\max}(\frac{1}{w} + \frac{1}{v_f})$. Replacing w and $l_{\max}$ by their expressions, the time during which the queue dissipates is $R\frac{\rho_a}{\rho_c - \rho_a}$. The rest of the cycle has a duration $C - R\frac{\rho_c}{\rho_c - \rho_a}$ and it has density ρ_a. The average density at the intersection is the sum of the arrival, maximum and critical densities, weighted by the fraction of the cycle during which each of the density is experienced. The average density at the intersection is:

$$\begin{aligned} d(0) &= \frac{1}{C}\left(\underbrace{R\rho_{\max}}_{\substack{\text{Red time } R \\ \text{at density } \rho_{\max}}} + \underbrace{R\frac{\rho_a}{\rho_c - \rho_a}\rho_c}_{\substack{\text{Clearing time } \tau \\ \text{at density } \rho_c}} + \underbrace{\left(C - \left(R + R\frac{\rho_a}{\rho_c - \rho_a}\right)\right)\rho_a}_{\substack{\text{Extra green-time } C-(R+\tau) \\ \text{at density } \rho_a}} \right) \\ &= \frac{R}{C}\rho_{\max} + \rho_a \end{aligned}$$

Given that the density grows linearly between the end of the queue and the intersection, the density at location x is given by

$$\begin{aligned} d(x) &= \rho_a & &\text{if } x \geq l_{\max} \\ d(x) &= \rho_a + \tfrac{R}{C}\rho_{\max}\tfrac{l_{\max} - x}{l_{\max}} & &\text{if } x \leq l_{\max}, \end{aligned}$$

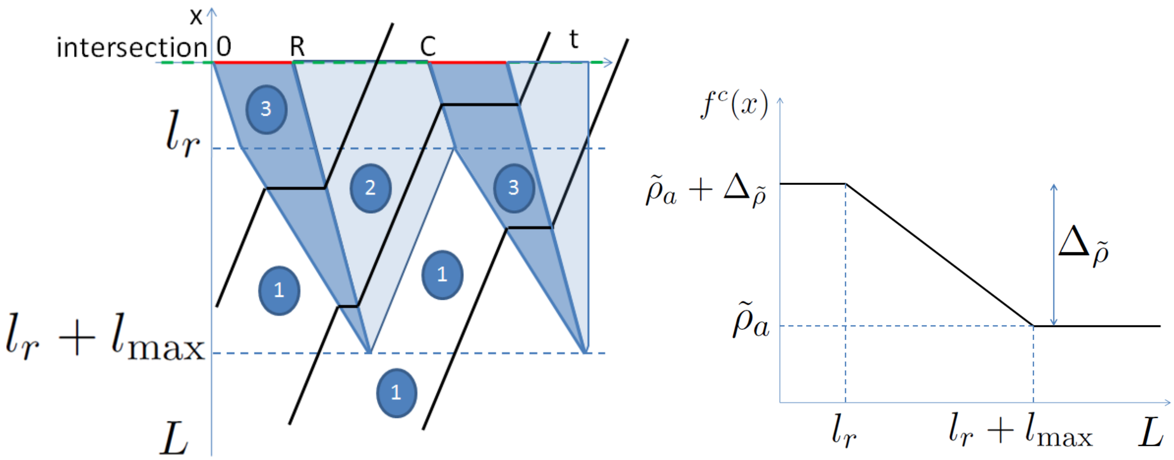

Figure 3: **Left**: The estimation of vehicle spatial distribution on a link is derived from the queue dynamics of the traffic flow model. The space-time plane represents the space-time domain in which density of vehicles is constant. Domain 1 represents the arrival density ρ_a, domain 2 represents the critical density ρ_c and domain 3 represents the maximum density $\rho_{\max}$. **Right**: Using the stationarity assumption, we compute the average density at location x and normalize to derive the probability distribution of vehicle locations on the link.

which can be summarized as

$$d(x) = \rho_a + \frac{R}{C}\rho_{\max}\frac{\max(l_{\max} - x, 0)}{l_{\max}}.$$

We introduce the normalization constant Z_u, which is defined by $Z_u = \int_0^L d(x)\,dx$ and represents the temporal average of the number of vehicles on the link. Its explicit value is given by $Z_u = L\rho_a + \frac{l_{\max}}{2}\frac{R}{C}\rho_{\max}$. The normalized density of vehicles as a function of the position on the link, defined by $f^u(x) = d(x)/Z_u$ is thus equal to

$$\begin{array}{lcll} f^u(x) & = & \frac{\rho_a}{Z_u} & \text{if } x \geq l_{\max} \\ f^u(x) & = & \frac{\rho_a}{Z_u} + \frac{R}{C}\rho_{\max}\frac{l_{\max}-x}{l_{\max}Z_u} & \text{if } x \leq l_{\max} \end{array}.$$

When vehicles report their location arbitrarily in time, this function represents the probability of receiving a measurement at location x.

3.3 Congested regime

In the congested regime, the average density is constant upstream of the maximum queue length—equal to ρ_a—and increases linearly until the remaining queue. In the remaining queue, it is constant and equal to $\frac{R}{C}\rho_{\max} + (1 - \frac{R}{C})\rho_c$. The different spatio-temporal domains of the density regions are illustrated Figure 3 (left). The probability distribution of vehicle locations is:

$$
\begin{array}{lcll}
f^c(x) & = & \frac{\rho_a}{Z_c} & \text{if } x \geq l_{\max} + l_r \\
f^c(x) & = & \frac{\rho_a}{Z_c} + \left(\frac{R}{C}\rho_{\max} + \left(1 - \frac{R}{C}\right)\rho_c - \rho_a\right)\frac{-x+l_{\max}+l_r}{l_{\max}Z_c} & \text{if } x \in [l_r,\, l_{\max} + l_r] \\
f^c(x) & = & \frac{R}{C}\frac{\rho_{\max}}{Z_c} + \left(1 - \frac{R}{C}\right)\frac{\rho_c}{Z_c} & \text{if } x \leq l_r
\end{array} . \tag{6}
$$

where Z_c is the normalizing constant that ensures that the integral of the function on $[0, L]$ equals 1. We have

$$
Z_c = L\rho_a + \left(\frac{l_{\max}}{2} + l_r\right)\left(\frac{R}{C}\rho_{\max} + \left(1 - \frac{R}{C}\right)\rho_c - \rho_a\right).
$$

Notice that the undersaturated regime is a special case of the congested regime, in which the remaining queue length l_r is equal to zero. In the remainder of this report, we consider the congested regime as the general case for the spatial distribution of vehicle location. This distribution is fully determined by three independent parameters. We choose the following parameterization: the remaining queue length l_r, the triangular queue length $l_{\max}$ and the normalized arrival density $\tilde{\rho}_a = \rho_a/Z_c$ to specify the distribution f^c. Using this parameterization, the probability distribution of vehicle location is illustrated in Figure 3 (right) and reads:

$$
\begin{array}{lcll}
f^c(x) & = & \tilde{\rho}_a & \text{if } x \geq l_{\max} + l_r \\
f^c(x) & = & \tilde{\rho}_a + \frac{(l_r + l_{\max}) - x}{l_{\max}}\Delta_{\tilde{\rho}} & \text{if } x \in [l_r,\, l_{\max} + l_r] \\
f^c(x) & = & \tilde{\rho}_a + \Delta_{\tilde{\rho}} & \text{if } x \leq l_r
\end{array} ,
$$

$$
\text{with } \Delta_{\tilde{\rho}} = \frac{1 - \tilde{\rho}_a L}{l_{\max}/2 + l_r}. \tag{7}
$$

The expression of $\Delta_{\tilde{\rho}}$ above, can be obtained easily by noticing that $\int_0^L f^c(x)\,dx = 1$ or by direct computation from Equation (6), by replacing Z_c by its expression (Equation (7)) and ρ_a by $Z_c\tilde{\rho}_a$.

Remark: The undersaturated regime is a special case of the congested regime in which $l_r = 0$.

4 Modeling the probability distribution of delay among the vehicles entering the link in a cycle

The travel time experienced by vehicles traveling on arterial networks is conditioned on two factors. First, the traffic conditions, given by the parameters of the network, dictate the state of traffic experienced by all the vehicles entering the link. Second, the time (after the beginning of a cycle) at which each vehicle arrives at the link determines how much delay will be experienced in the queue due to the presence of a traffic signal and the presence of other vehicles. Under similar traffic conditions, drivers experience different travel times depending on their arrival time. Using the assumption that the arrival density (and thus the arrival rate) is constant, arrival times are uniformly distributed on the duration of the light cycle. This allows for the derivation of the pdf of delay, which depends on the characteristics of the traffic light and the traffic conditions as defined in Section 2.4.

In this work, we assume that we receive travel time measurements from vehicles traveling on the network. The vehicles are sampled uniformly in time and they send tuples of the form (x_1, t_1, x_2, t_2) where x_1 is the location of the vehicle at t_1 and x_2 is the position of the vehicle at t_2. This is representative of taxi fleets or truck delivery fleets which typically send data every minute in urban networks. We consider all the tuples sent by the vehicles independently. For example, we assume that the sampling strategy is such that we cannot reconstruct the trajectories of vehicles from the tuples (*e.g.* at each sampling time, the vehicles send tuples with a defined probability).

4.1 *Total delay* and *measured delay* between locations x_1 and x_2

We consider a vehicle traveling from location x_1 to location x_2 and sending its location x_1 at time t_1 and its location x_2 at time t_2. We call *measured delay from* x_1 to x_2, *experienced in the time interval* $[t_1, t_2]$, in short "measured delay from x_1 to x_2", the difference between the travel time of the vehicle $(t_2 - t_1)$ and the travel time that the vehicle would experience between x_1 and x_2 without the presence of other vehicles nor signals. For a vehicle with free flow pace p_f, we call free flow travel time between x_1 and x_2, the quantity $y_{f;x_1,x_2} = p_f(x_1 - x_2)$, representing the travel time between x_1 and x_2 if the vehicle is not slowed down or stopped on its trajectory. The delay experienced between x_1 and x_2 is the difference between the travel time y_{x_1,x_2} of the vehicle between x_1 and x_2—not necessarily at free flow speed—and the free flow travel time $y_{f;\,x_1,x_2}$. In this model, vehicles are either stopped or driving at the free flow speed. The measured delay from x_1 to x_2, experienced in the time interval $[t_1, t_2]$ is the cumulative stopping time between t_1 and t_2.

We call *total delay* from x_1 to x_2 the cumulative stopping time of the vehicle on its trajectory from x_1 (from the first time it joined the queue, if the vehicle was in the queue at x_1) to x_2 (until the time it left the queue, if it was in the queue at x_2). In particular, if the vehicle stops at x_1 or at x_2 the total delay from x_1 to x_2 covers the full delay experienced during the stop, without taking into account the sampling scheme. Note that for vehicles sampled at x_1 and x_2 that do not stop at x_1 nor at x_2 the total delay is equal to the measured delay. For vehicles stopping in x_1 or in x_2, the measured delay is less than or equal to the total delay experienced by the vehicle (Figure 4.2 (right)).

To gain more insight in the difference between *measured* and *total* delay, we can study a simple case. Let a vehicle be sampled every 30 seconds. Assume that the vehicle stops at the traffic signal $(x = 0)$ and that the duration of the red time is 40 seconds. The vehicle sends its locations x_1 at t_1 and x_2 at $t_2 = t_1 + 30$. We do not receive additional information on the trajectory prior to t_1 or past t_2. The measured delay is at most 30 seconds (sampling rate); the total delay is 40 seconds. As a general remark, a vehicle reporting its delay during a stop reports a delay that is less than or equal to the total delay experienced on the trajectory, it represents the delay experienced between the two sampling times.

Using the modeling assumptions defined in Section 2.1, we derive the pdf of the *measured* and the *total* delay between any two locations x_1 and x_2. Given two sampling locations x_1 and x_2, the probability distribution of the total (resp. measured) delay δ_{x_1,x_2} is denoted $h^t(\delta_{x_1,x_2})$ (resp. $h^m(\delta_{x_1,x_2})$). We use the stationarity and constant arrival assumptions to derive the speed of

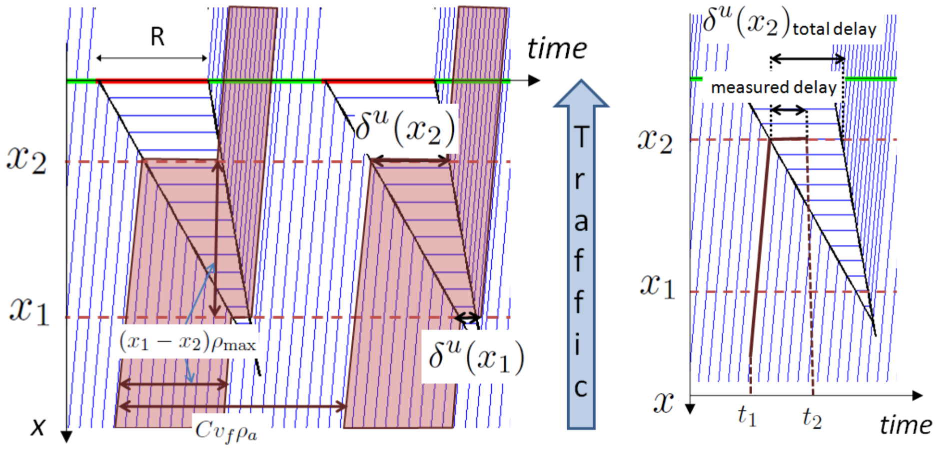

Figure 4: (**Left**) The proportion of delayed vehicles $\eta^u_{x_1,x_2}$ is the ratio between the number of vehicles joining the queue between x_1 and x_2 over the total number of vehicles entering the link in one cycle. The trajectories highlighted in purple represent the trajectories of vehicles delayed between x_1 and x_2. (**Right**) The vehicles reporting their location during a stop at x_2 experience a delay $\delta \in [0, \delta^u(x_2)]$ in the time interval $[t_1, t_2]$. This delay is less than or equal to the total delay ($\delta^u(x_2)$) experienced on the trajectory.

formation and dissolution of the queue, respectively denoted v_a and w (2). Under the stationarity assumption, the traffic variables are periodic with period C. For each arrival time, we compute the delay corresponding to the trajectory of the vehicle (Figure 2). The arrivals being uniform, we can compute the probability distribution of delays.

4.2 Probability distribution of the total and measured delay between x_1 and x_2 in the undersaturated regime

Pdf of the *total* delay between x_1 and x_2: In the undersaturated regime, we call $\eta^u_{x_1,x_2}$, the fraction of the vehicles entering the link during a cycle that experience a delay between x_1 and x_2. The remainder of the vehicles entering the link in a cycle travels from x_1 to x_2 without experiencing any delay. The proportion $\eta^u_{x_1,x_2}$ of vehicles delayed between x_1 and x_2 in a cycle, is computed as the ratio of vehicles joining the queue between x_1 and x_2 over the total number of vehicles entering the link in one cycle (Figure 4.2, left). The number of vehicles joining the queue between x_1 and x_2 is the number of vehicles stopped between x_1 and x_2: $\left(\min(l_{\max}, x_1) - \min(l_{\max}, x_2)\right)\rho_{\max}$. The number of vehicles entering the link is $v_f C \rho_a$. The proportion of vehicles delayed between x_1 and x_2 is thus:

$$\eta^u_{x_1,x_2} = (\min(x_1, l_{\max}) - \min(x_2, l_{\max}))\frac{\rho_{\max}}{v_f C \rho_a}.$$

The total stopping time experienced when stopping at x is denoted by $\delta^u(x)$ for the undersaturated regime. Because the arrival of vehicles is homogenous, the delay $\delta^u(x)$ increases linearly with x. At the intersection ($x = 0$), the delay is maximal and equals the duration of the red light R. At the end of the queue ($x = l_{\max}$) and upstream of the queue ($x \geq l_{\max}$), the delay is null. Thus the expression of $\delta^u(x)$:

$$\delta^u(x) = R\left(1 - \frac{\min(x, l_{\max})}{l_{\max}}\right).$$

Given that the arrival of vehicles is uniform in time, the distribution of the location where the vehicles reach the queue between x_1 and x_2 is uniform in space. For vehicles reaching the queue between x_1 and x_2, the probability to experience a delay between locations x_1 and x_2 is uniform. The uniform distribution has support $[\delta^u(x_1), \delta^u(x_2)]$, corresponding to the minimum and maximum delay between x_1 and x_2.

The *total* delay experienced between x_1 and x_2 is a random variable with a mixture distribution with two components. The first component represents the vehicles that do not experience any stopping time between x_1 and x_2 (mass distribution in 0), the second component represents the vehicles reaching the queue between x_1 and x_2 (uniform distribution on $[\delta^u(x_1), \delta^u(x_2)]$). We note $\mathbf{1}_A$ the indicator function of set A,

$$\mathbf{1}_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$

We note $\mathrm{Dir}_{\{a\}}(\cdot)$ the Dirac distribution centered in a, used to represent the mass probability. The pdf of total delay between x_1 and x_2 (Figure 6, left) reads:

$$h^t(\delta_{x_1,x_2}) = (1 - \eta^u_{x_1,x_2})\mathrm{Dir}_{\{0\}}(\delta_{x_1,x_2}) + \frac{\eta^u_{x_1,x_2}}{\delta^u(x_2) - \delta^u(x_1)}\mathbf{1}_{[\delta^u(x_1),\delta^u(x_2)]}(\delta_{x_1,x_2})$$

The cumulative distribution function of total delay $H^t(\cdot)$ reads:

$$H^t(\delta_{x_1,x_2}) = \begin{cases} 0 & \text{if} \quad \delta_{x_1,x_2} < 0 \\ (1 - \eta^u_{x_1,x_2}) & \text{if} \quad \delta_{x_1,x_2} \in [0, \delta^u(x_1)] \\ (1 - \eta^u_{x_1,x_2}) + \eta^u_{x_1,x_2}\frac{\delta_{x_1,x_2} - \delta^u(x_1)}{\delta^u(x_2) - \delta^u(x_1)} & \text{if} \quad \delta_{x_1,x_2} \in [\delta^u(x_1), \delta^u(x_2)] \\ 1 & \text{if} \quad \delta_{x_1,x_2} > \delta^u(x_2) \end{cases}$$

Pdf of the *measured* delay between x_1 and x_2: because of the sampling scheme, the measured delay differs from the total delay experienced by the vehicles.

In the following, i refers to the upstream or the downstream measurement locations ($i \in \{1, 2\}$). When sending their location x_i, some vehicles are stopped at this location. These vehicles may not report the full delay associated with location x_i (Figure 4.2, right). In particular, a vehicle stopped at x_1 when sending its location at time t_1 will only report the delay experienced after t_1. Similarly, a vehicle stopped at x_2 when sending its location at time t_2 will only report the delay experienced before t_2.

- For a measurement received at x_i, the probability that it comes from a vehicle stopped in the queue is $\frac{\delta^u(x_i)}{C}$, which is the ratio of the time spent by the stopped vehicle at x_i over the duration of the cycle.

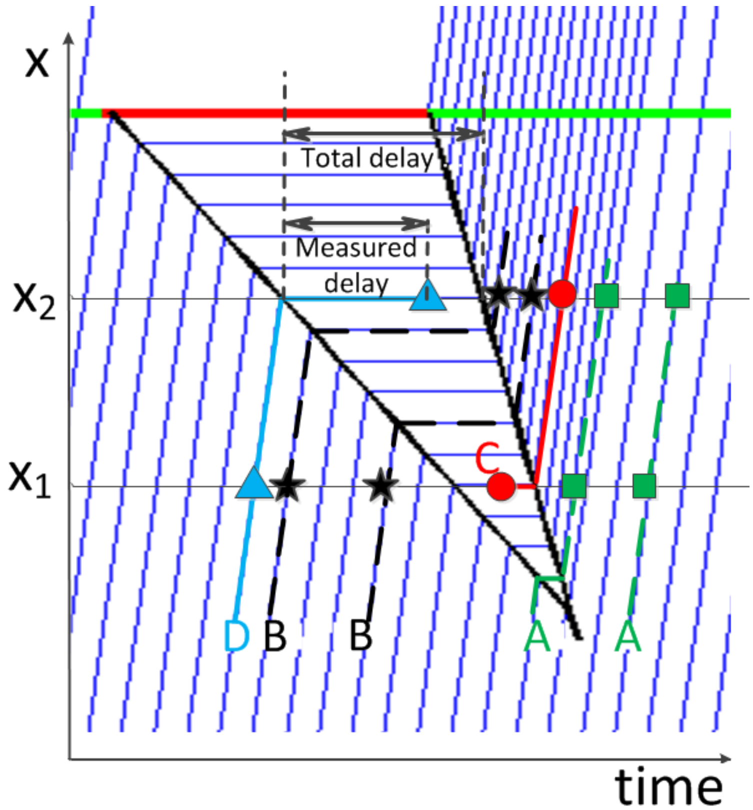

Figure 5: Classification of the trajectories depending on the stopping location. We consider vehicles traveling between the two measurement points x_1 and x_2, sampled uniformly in time.

- For a vehicle stopped at x_2, observed at t_2 coming from a previous observation point x_1 (at t_1), the probability of a stopping time experienced by the vehicle until t_2 is uniform between 0 and $\delta^u(x_2)$, since the vehicle is sampled arbitrarily during its stopping phase.

We classify the vehicles traveling from x_1 to x_2 (where x_1 and x_2 are measurement points) depending on the locations of their stop with respect to x_1 and x_2. This classification of the trajectories is also illustrated in Figure 5:

A) Vehicles do not experience any delay between the measurement points x_1 and x_2.

B) Vehicles reach the queue at x with $x \in (x_2, x_1)$. These vehicles are not stopped when they send their location at x_1 and x_2.

C) Vehicles reach the queue at x_1, where they report their location at time t_1. At t_1, the vehicle was already stopped (or was just stopping) and the measurement only accounts for the delay occurring after t_1, which is less than or equal to $\delta^u(x_1)$. Because of the uniform sampling in time, the reported delay has a uniform distribution on $[0, \delta^u(x_1)]$

D) Vehicles reach the queue at x_2, where they report their location at time t_2. At t_2, the vehicle is still stopped and the measurement only represents the delay occurring up to t_2, which is less than or equal to $\delta^u(x_2)$. Because of the uniform sampling in time, the reported delay has a uniform distribution on $[0, \delta^u(x_2)]$

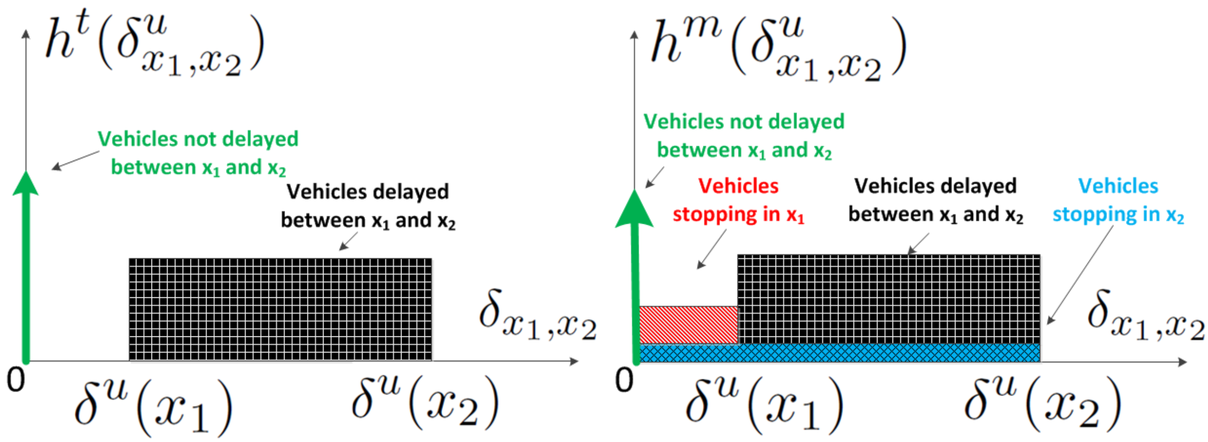

Figure 6: (**Left**) Probability distribution of the total delay between x_1 and x_2 in the undersaturated regime. (**Right**) Probability distribution of the measured delay between x_1 and x_2 in the undersaturated regime. Vehicles are assumed to be sampled uniformly in time.

We denote by s_{x_i} the event "*vehicle stops at location* x_i". Denoting by $\mathcal{P}(A)$ the probability of event A, we have $\mathcal{P}(s_{x_i}) = \frac{\delta^u(x_i)}{C}$. The notation $\bar{s}_{x_i}$ represents the event "*vehicle does not stop at location* x_i". The notation $(\bar{s}_{x_1}, \bar{s}_{x_2})$ represents the event "*vehicles do not stop at location* x_1 *nor* x_2". We assume that the events $\bar{s}_{x_1}$ and $\bar{s}_{x_2}$ are independent. The probability of event $(\bar{s}_{x_1}, \bar{s}_{x_2})$ reads:

$$\begin{aligned} \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2}) &= \mathcal{P}(\bar{s}_{x_1})\mathcal{P}(\bar{s}_{x_2}) && \text{Independence assumption} \\ &= (1 - \mathcal{P}(s_{x_1}))(1 - \mathcal{P}(s_{x_2})) && \text{Complementary events} \end{aligned}$$

The event $(\bar{s}_{x_1}, \bar{s}_{x_2})$ corresponds to trajectories of type A (vehicles do not stop between x_1 and x_2) and trajectories of type B (vehicles stop strictly between x_1 and x_2 but neither in x_1 nor in x_2). Among the vehicles stopping at none of the measurement points, a fraction $\eta^u_{x_1,x_2}$ is delayed between x_1 and x_2 (trajectories of type B) and a fraction $1 - \eta^u_{x_1,x_2}$ does not experience delay between x_1 and x_2 (trajectories of type A). Given that we receive a delay measurement between locations x_1 and x_2, the probability that it was sent by a vehicle with a trajectory of type A is $\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})(1 - \eta^u_{x_1,x_2})$. Similarly, the probability that it was sent by a vehicle with a trajectory of type B is $\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\eta^u_{x_1,x_2}$.

Given that a measurement is received at location x_i, the probability that this measurement is sent by a vehicle that joined the queue at x_i is proportional to the delay experienced at location x_i. Given successive measurements at locations x_1 and x_2, the probability that a vehicle reports its location x_i ($i \in \{1, 2\}$) while being stopped at this location is denoted ζ_{x_i}. From this definition and given that we receive a delay measurement between locations x_1 and x_2, the probability that it was sent by a vehicle with a trajectory of type C is ζ_{x_1}. The probability that it was sent by a vehicle with a trajectory of type D is ζ_{x_2}. Note that vehicles cannot stop both at x_1 and x_2 (they stop only once in the queue); thus $\mathcal{P}(s_{x_1}, s_{x_2}) = 0$. Given that a vehicle was sampled at x_1 and x_2, we have:

$$\underbrace{\zeta_{x_1}}_{\substack{\text{Prob. that the veh}\\ \text{stopped at } x_1 \text{ only}}} + \underbrace{\zeta_{x_2}}_{\substack{\text{Prob. that the veh}\\ \text{stopped at } x_2 \text{ only}}} + \underbrace{\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})}_{\substack{\text{Prob. that the veh}\\ \text{stopped neither at } x_1 \text{ nor at } x_2}} + \underbrace{\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})}_{\substack{\text{Prob. that the veh}\\ \text{stopped both at } x_1 \text{ and at } x_2\\ (=0)}} = 1$$

The probability of stopping either at x_1 or at x_2 is $1 - \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$ (complementary of stopping neither at x_1 nor at x_2). Among these vehicles, the proportion that stops in x_1 is proportional to the delay experienced in x_1. We have:

$$\begin{cases} \zeta_{x_i} & \propto \quad \delta^u(x_i),\ i \in \{1,2\} \\ \zeta_{x_1} + \zeta_{x_2} & = \quad 1 - \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2}) \end{cases} \Rightarrow \zeta_{x_i} = \left(1 - \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\right) \frac{\delta^u(x_i)}{\delta^u(x_1) + \delta^u(x_2)} \quad i \in \{1,2\}$$

The probability distribution of measured delay is a finite mixture distribution, in which each component is a mass probability or a uniform distribution. The theoretical probability distribution function is illustrated Figure 6, right. It is the sum of the following terms that also refer to Figure 5:

(A) a mass probability in 0 with weight $(1 - \eta^u_{x_1,x_2})\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$, representing the vehicles that do not reach the queue between x_1 and x_2,

(B) a uniform distribution on $(\delta^u(x_1), \delta^u(x_2))$ with weight $\eta^u_{x_1,x_2}\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$, representing the vehicles that reach the queue strictly between x_1 and x_2,

(C) a uniform distribution on $[0, \delta^u(x_1)]$ with weight ζ_{x_1}, representing the vehicles that stop in x_1,

(D) a uniform distribution on $[0, \delta^u(x_2)]$ with weight ζ_{x_2}, representing the vehicles that stop in x_2.

The pdf of the measured delay is related to the pdf of the total delay as:

$$h^m(\delta_{x_1,x_2}) = \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})h^t(\delta_{x_1,x_2}) + \frac{\zeta_{x_1}}{\delta^u(x_1)}\mathbf{1}_{[0,\delta^u(x_1)]}(\delta_{x_1,x_2}) + \frac{\zeta_{x_2}}{\delta^u(x_2)}\mathbf{1}_{[0,\delta^u(x_2)]}(\delta_{x_1,x_2})$$

4.3 Probability distribution of the measured delay between x_1 and x_2 in the congested regime

In the congested regime, the delay distribution can be computed using a similar methodology as for the undersaturated regime, by deriving the delay experienced between x_1 and x_2 for each arrival time. We call n_s the maximum number of stops experienced by the vehicles in the remaining queue between the locations x_1 and x_2. The delay experienced at location x when reaching the triangular queue at x is readily derived from the expression of the delay in the undersaturated regime. The delay experienced when reaching the remaining queue is the duration of the red time R. The expression of the delay at location x is then

$$\delta^c(x) = \begin{cases} R & \text{if } x \leq l_r \\ R\,\frac{l_r + l_{\max} - x}{l_{\max}} & \text{if } x \in [l_r, l_r + l_{\max}] \\ 0 & \text{if } x \geq l_r + l_{\max} \end{cases}$$

The details of the derivation are given in Appendix A and illustrated in Figures 8–11. Note that to satisfy the stationarity assumption, the distance traveled by vehicles in the queue in the duration of a light cycle is $l_{\max}$.

We summarize the derivations, classified depending on the location of the positions x_1 and x_2 with respect to the remaining and triangular queue lengths:

1. x_1 *Upstream* – x_2 *Remaining* (Figure 8): The location x_1 is upstream of the queue and the location x_2 is in the triangular queue. We define the critical location x_c by $x_c = x_2 + n_s l_{\max}$. Vehicles reaching the triangular queue upstream of x_c stop n_s times in the remaining queue. On the road segment $[x_1, x_2]$, vehicles reaching the triangular queue downstream of x_c stop $n_s - 1$ times in the remaining queue. The vehicles experience a delay uniformly distributed on $[\delta_{\min}, \delta_{\max}]$ with $\delta_{\min} = (n_s - 1)R + \delta^c(x_c)$ and $\delta_{\max} = n_s R + \delta^c(x_c) = \delta_{\min} + R$. The probability distribution of total delay reads:

$$h^t(\delta_{x_1,x_2}) = \frac{1}{\delta_{\max} - \delta_{\min}} \mathbf{1}_{[\delta_{\min},\delta_{\max}]}(\delta_{x_1,x_2}), \qquad \begin{aligned} \delta_{\min} &= \delta^c(x_c) + (n_s - 1)R \\ \delta_{\max} &= \delta^c(x_c) + n_s R \end{aligned}$$

2. x_1 *Triangular* – x_2 *Triangular* (Figure 9): Both locations x_1 and x_2 are upstream of the remaining queue (in the triangular queue or upstream of the queue). Given that the path is upstream of the remaining queue, this case is similar to the undersaturated regime, where derivations are updated to account for the fact that the triangular queue starts at $x = l_r$. We adapt the notation from Section 4.2 and denote by $\eta^c_{x_1,x_2}$ the fraction of the vehicles entering the link in a cycle that experience delay between locations x_1 and x_2.

$$\eta^c_{x_1,x_2} = \frac{\min(x_1 - l_r, l_{\max}) - \min(x_2 - l_r, l_{\max})}{l_{\max}}$$

This delay is uniformly distributed on $[\delta^c(x_1), \delta^c(x_2)]$. The reminder do not stop between x_1 and x_2. The probability distribution of total delay reads:

$$h^t(\delta_{x_1,x_2}) = (1 - \eta^c_{x_1,x_2})\mathrm{Dir}_{\{0\}}(\delta_{x_1,x_2}) + \frac{\eta^c_{x_1,x_2}}{\delta^c(x_2) - \delta^c(x_1)} \mathbf{1}_{[\delta^c(x_1),\delta^c(x_2)]}(\delta_{x_1,x_2})$$

3. x_1 *Remaining* – x_2 *Remaining* (Figure 10): Both locations x_1 and x_2 are in the remaining queue. We define the critical location x_c by $x_c = x_2 + (n_s - 1)l_{\max}$. The vehicles reaching the queue between x_1 and x_c stop n_s times in the remaining queue between x_1 and x_2, their stopping time is $n_s R$. The reminder of the vehicles stop $n_s - 1$ times in the remaining queue and their stopping time is $(n_s - 1)R$. The probability distribution of total delay reads:

$$h^t(\delta_{x_1,x_2}) = \frac{x_1 - x_c}{l_{\max}} \mathrm{Dir}_{\{n_s R\}}(\delta_{x_1,x_2}) + \left(1 - \frac{x_1 - x_c}{l_{\max}}\right) \mathrm{Dir}_{\{(n_s-1)R\}}(\delta_{x_1,x_2})$$

4. x_1 *Triangular* – x_2 *Remaining* (Figure 11): The upstream location x_1 is in the triangular queue and the downstream location x_2 is in the remaining queue. We define the critical location x_c by $x_c = x_2 + n_s l_{\max}$.

◇ If $x_1 \geq x_c$, a fraction $(x_1 - x_c)/l_{\max}$ of the vehicles entering the link in a cycle join the triangular queue between x_1 and x_c. They stop once in the triangular queue and n_s times in the remaining queue. Among these vehicles, the stopping time is uniformly distributed on $[\delta^c(x_1) + n_s R,\, \delta^c(x_c) + n_s R]$. A fraction $(x_c - l_r)/l_{\max}$ of the vehicles entering the link in a cycle join the triangular queue between x_c and $l_{\max}$. Among these vehicles, the stopping time is uniformly distributed on $[\delta^c(x_c) + (n_s - 1)R,\, n_s R]$. The remainder of the vehicles reach the remaining queue between l_r and $x_1 - l_{\max}$ and their stopping time is $n_s R$. The probability distribution of total delay reads:

$$\begin{array}{rll} h^t(\delta_{x_1,x_2}) = & \frac{x_1-x_c}{l_{\max}} \frac{\mathbf{1}_{[\delta^c(x_1)+n_sR,\, \delta^c(x_c)+n_sR]}(\delta_{x_1,x_2})}{\delta^c(x_c)-\delta^c(x_1)} & \text{Vehicles stopping between } x_1 \text{ and } x_c \\ + & \frac{x_c-l_r}{l_{\max}} \frac{\mathbf{1}_{[\delta^c(x_c)+(n_s-1)R,\, n_sR]}(\delta_{x_1,x_2})}{R-\delta^c(x_c)} & \text{Vehicles stopping between } x_c \text{ and } l_r \\ + & \left(1 - \frac{x_1-l_r}{l_{\max}}\right) \mathrm{Dir}_{\{n_sR\}}(\delta_{x_1,x_2}) & \text{Vehicles stopping between } l_r \text{ and } x_1-l_{\max} \end{array}$$

◇ If $x_1 \leq x_c$, a fraction $(x_1 - l_r)/l_{\max}$ of the vehicles entering the link in a cycle join the triangular queue between x_1 and l_r. They stop once in the triangular queue and $n_s - 1$ times in the remaining queue. Among these vehicles, the stopping time is uniformly distributed on $[\delta^c(x_1) + (n_s - 1)R,\, n_s R]$. A fraction $1 - (x_c - l_r)/l_{\max}$ of the vehicles entering the link in a cycle join the remaining queue between l_r and $x_c - l_{\max}$. The stopping time of these vehicles is $n_s R$. The remainder of the vehicles experiences a stopping time of $(n_s - 1)R$. The probability distribution of total delay reads:

$$\begin{array}{rll} h^t(\delta_{x_1,x_2}) = & \frac{x_1-l_r}{l_{\max}} \frac{\mathbf{1}_{[\delta^c(x_1)+(n_s-1)R,\, n_sR]}(\delta_{x_1,x_2})}{R-\delta^c(x_1)} & \text{Vehicles stopping between } x_1 \text{ and } l_r \\ + & \left(1 - \frac{x_c-l_r}{l_{\max}}\right) \mathrm{Dir}_{\{n_sR\}}(\delta_{x_1,x_2}) & \text{Vehicles stopping between } l_r \text{ and } x_c-l_{\max} \\ + & \frac{x_c-x_1}{l_{\max}} \mathrm{Dir}_{\{(n_s-1)R\}}(\delta_{x_1,x_2}) & \text{Vehicles stopping between } x_c-l_{\max} \text{ and } x_1-l_{\max} \end{array}$$

These cases represent the pdf of total delay. From the results derived in Section 4.2, we derive the pdf of measured delay. From the previous derivations, we have:

$$\begin{array}{rcl} \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_1}) & = & (1 - \mathcal{P}(\bar{s}_{x_1}))(1 - \mathcal{P}(\bar{s}_{x_2})) \\ \zeta_{x_i} & = & \left(1 - \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\right) \frac{\delta^u(x_i)}{\delta^u(x_1)+\delta^u(x_2)} \quad i \in \{1, 2\} \end{array}$$

It is the sum of the following terms:

(i) the delay distribution given that the vehicles stop neither in x_1 nor in x_2, with weight $\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$,

(ii) the delay probability distribution given a stop in x_1, with weight ζ_{x_1},

(iii) the delay probability distribution given a stop in x_2, with weight ζ_{x_2}.

We summarize the different components of the delay distribution, described as a mixture distribution for all the different cases in Table 1.

Case	Trajectories	Weight	Dist.	Support
<u>Case 1</u> $x_1 \geq l_r + l_{\max}$, $x_2 \leq l_r$, $x_c = x_2 + n_s l_{\max}$	Does not stop at x_2	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$	Unif.	$[(n_s-1)R + \delta^c(x_c),$ $n_s R + \delta^c(x_c)]$
	Stop at x_2	$\zeta_{x_2} = 1 - \mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})$	Unif.	$[(n_s-1)R + \delta^c(x_c),$ $n_s R + \delta^c(x_c)]$
<u>Case 2</u> $x_1 \geq l_r$, $x_2 \geq l_r$	No stop between x_1 and x_2	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $(1-\eta^c_{x_1,x_2})$	Mass	$\{0\}$
	Reach the (triangular) queue between x_1 and x_2	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\eta^c_{x_1,x_2}$	Unif.	$[\delta^c(x_2), \delta^c(x_1)]$
	Stop at x_1	ζ_{x_1}	Unif.	$[0,\, \delta^c(x_1)]$
	Stop at x_2	ζ_{x_2}	Unif.	$[0,\, \delta^c(x_2)]$
<u>Case 3</u> $x_1 \leq l_r$, $x_2 \leq l_r$, $x_c = x_2 + (n_s-1)l_{\max}$	Reach the (remaining) queue between x_1 and x_c	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_1 - x_c}{l_{\max}}$	Mass	$\{n_s R\}$
	Reach the (remaining) queue between x_c and $x_1 - l_{\max}$	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_c - x_1 + l_{\max}}{l_{\max}}$	Mass	$\{(n_s-1)R\}$
	Stop at x_1	ζ_{x_1}	Unif.	$[(n_s-1)R,\, n_s R]$
	Stop at x_2	ζ_{x_2}	Unif.	$[(n_s-1)R,\, n_s R]$
<u>Case 4a</u> $x_1 \in [l_r, l_r + l_{\max}]$, $x_2 \leq l_r$, $x_c = x_2 + n_s l_{\max}$, $x_c \leq x_1$	Reach the (triangular) queue between x_1 and x_c	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_1 - x_c}{l_{\max}}$	Unif.	$[n_s R + \delta^c(x_1),$ $n_s R + \delta^c(x_c)]$
	Reach the (triangular) queue between x_c and l_r	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_c - l_r}{l_{\max}}$	Unif.	$[(n_s-1)R + \delta^c(x_c),$ $n_s R]$
	Reach the (remaining) queue between l_r and $x_1 - l_{\max}$	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{l_r - x_1 + l_{\max}}{l_{\max}}$	Mass	$\{n_s R\}$
	Stop at x_1	ζ_{x_1}	Unif.	$[n_s R,\, n_s R + \delta^c(x_1)]$
	Stop at x_2	ζ_{x_2}	Unif.	$[(n_s-1)R + \delta^c(x_c),$ $n_s R + \delta^c(x_c)]$
<u>Case 4b</u> $x_1 \in [l_r, l_r + l_{\max}]$, $x_2 \leq l_r$, $x_c = x_2 + n_s l_{\max}$, $x_c \geq x_1$	Reach the (triangular) queue between x_1 and l_r	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_1 - l_r}{l_{\max}}$	Unif.	$[(n_s-1)R + \delta^c(x_1),$ $n_s R]$
	Reach the (remaining) queue between l_r and $x_c - l_{\max}$	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{l_r - x_c + l_{\max}}{l_{\max}}$	Mass	$\{n_s R\}$
	Reach the (remaining) queue between $x_c - l_{\max}$ and $x_1 - l_{\max}$	$\mathcal{P}(\bar{s}_{x_1}, \bar{s}_{x_2})\times$ $\dfrac{x_c - x_1}{l_{\max}}$	Mass	$\{(n_s-1)R\}$
	Stop at x_1	ζ_{x_1}	Unif.	$[(n_s-1)R,$ $(n_s-1)R + \delta^c(x_1)]$
	Stop at x_2	ζ_{x_2}	Unif.	$[(n_s-1)R,\, n_s R]$

Table 1: The pdf of measured delay is a mixture distribution. The different components and their associated weight depend on the location of stops of the vehicles with respect to the queue length and sampling locations.

5 Probability distributions of travel times

On a path between x_1 and x_2, the travel time y_{x_1,x_2} is a random variable. It is the sum of two random variables: the delay δ_{x_1,x_2} experienced between x_1 and x_2 and the free flow travel time of the vehicles $y_{f;\,x_1,x_2}$. The free flow travel time is proportional to the distance of the path and the free flow pace p_f such that $y_{f;\,x_1,x_2} = p_f(x_1 - x_2)$. We have $y_{x_1,x_2} = \delta_{x_1,x_2} + y_{f;\,x_1,x_2}$.

In the following, we assume that the delay and the free flow pace are independent random variables, thus so are the delay and the free flow travel time.

We model the differences in traffic behavior by assuming a prior distribution on the free flow pace p_f. The free flow pace is modeled as a random variable with distribution φ^p and support $\mathcal{D}_{\varphi^p}$. For convenience, we define for a pdf φ with support $\mathcal{D}_\varphi$, its prolongation by zero of out of $\mathcal{D}_\varphi$. With a slight abuse of notation, we call this new function φ.

Using a linear change of variables, we derive the probability distribution $\varphi^y_{x_1,x_2}$ of free flow travel time $y_{f;\,x_1,x_2}$ between x_1 and x_2:

$$p_f \sim \varphi^p(p_f) \Rightarrow \varphi^y_{x_1,x_2}(y_{f;\,x_1,x_2}) = \varphi^p\left(\frac{y_{f;\,x_1,x_2}}{x_1 - x_2}\right)\frac{1}{x_1 - x_2}$$

To derive the pdf of travel times we use the following fact:

Fact 1 (Sum of independent random variables)**.** *If X and Y are two independent random variables with respective pdf f_X and f_Y, then the pdf f_Z of the random variable $Z = X + Y$ is given by $f_Z(z) = f_X * f_Y(z)$*

This classical result in probability is derived by computing the conditional pdf of Z given X and then integrating over the values of X according to the total probability law.

For each regime s, the probability distribution of travel times reads:

$$g^s(y_{x_1,x_2}) = \left(h^s * \varphi^y_{x_1,x_2}\right)(y_{x_1,x_2})$$

We notice that the delay distributions are mixtures of mass probabilities and uniform distributions. We derive the general expression of the travel time distributions when vehicles experience a delay with mass probability in Δ and when vehicles experience a delay with uniform distribution on $[\delta_{\min}, \delta_{\max}]$.

5.1 Travel time distributions

Travel time distribution when the delay has a mass probability in Δ

The stopping time is Δ. This corresponds to trajectories with n_s stops ($n_s \geq 0$) in the remaining queue. This includes the non stopping vehicle in the undersaturated regime, when the remaining queue has length zero. The travel time distribution is derived as

$$\begin{aligned} g(y_{x_1,x_2}) &= \mathrm{Dir}_{\{\Delta\}} * \varphi^y_{x_1,x_2}(y_{x_1,x_2}) \\ &= \varphi^y_{x_1,x_2}(y_{x_1,x_2} - \Delta). \end{aligned} \tag{8}$$

Travel time distribution when the delay is uniformly distributed on $[\delta_{\min}, \delta_{\max}]$.
Vehicles experience a uniform delay between a minimum and maximum delay respectively denoted $\delta_{\min}$ and $\delta_{\max}$. The probability of observing a travel time y_{x_1,x_2} is given by

$$g(y_{x_1,x_2}) \quad = \quad \frac{1}{\delta_{\max} - \delta_{\min}} \int_{-\infty}^{+\infty} \mathbf{1}_{[\delta_{\min},\delta_{\max}]}(y_{x_1,x_2} - z)\ \varphi^y_{x_1,x_2}(z)\, dz. \tag{9}$$

The integrand is not null if and only if $y_{x_1,x_2} - z \in [\delta_{\min}, \delta_{\max}]$, *i.e.* if and only if $z \in [y_{x_1,x_2} - \delta_{\max},\ y_{x_1,x_2} - \delta_{\min}]$. Since $\varphi^y_{x_1,x_2}(z)$ is equal to zero for $z \in \mathbb{R} \setminus \mathcal{D}_\varphi$, the integrand is not null if and only if $z \in [y_{x_1,x_2} - \delta_{\max},\ y_{x_1,x_2} - \delta_{\min}] \bigcap \mathcal{D}_\varphi$.

As an illustration, we derive the probability distribution of travel times on an entire link in the undersaturated regime, for a pace distribution with support on $\mathbb{R}^+$ (Figure 7 (left)). The length of the link is denoted L. A fraction $1 - \eta^u_{L,0}$ of the vehicles entering the link in a cycle has a delay with mass probability in 0 (vehicles do not stop on the link). The probability distribution of travel times of these vehicles is computed via Equation (8) with $\Delta = 0$. The reminder of the vehicles (fraction $\eta^u_{L,0}$) experiences a delay that is uniformly distributed on $[0, R]$. The probability distribution of travel times of these vehicles is computed via Equation (9) with $\delta_{\min} = 0$ and $\delta_{\max} = R$. The probability distribution of travel times on an undersaturated arterial link reads:

$$g^u(y_{L,0}) = \begin{cases} 0 & \text{if} \quad y_{L,0} \leq 0 \\ (1 - \eta^u_{L,0})\varphi^y_{L,0}(y_{L,0}) + \dfrac{\eta^u_{L,0}}{R} \displaystyle\int_0^{y_{L,0}} \varphi^y_{L,0}(z)\, dz & \text{if} \quad y_{L,0} \in [0, R] \\ (1 - \eta^u_{L,0})\varphi^y_{L,0}(y_{L,0}) + \dfrac{\eta^u_{L,0}}{R} \displaystyle\int_{y_{L,0}-R}^{y_{L,0}} \varphi^y_{L,0}(z)\, dz & \text{if} \quad y_{L,0} \geq R \end{cases}, \tag{10}$$

In the more general case of a travel time distribution on an undersaturated partial link between locations x_1 and x_2, we write the delay distribution as a mixture of mass probabilities and uniform distributions. We use the linearity of the convolution to treat each component of the mixture separately and sum them with their respective weights to derive the probability distribution of travel times.

The derivations are similar in the congested regime. For the different cases described in Section 4.3, the delay is a mixture of mass probabilities and uniform distributions. For example, the probability distribution of link travel times (Case 1) is illustrated in Figure 7 (right)). When the delay is uniformly distributed on $[\delta_{\min}, \delta_{\max}]$, the probability distribution of travel times is computed via Equation (9) and reads

$$g^c(y_{L,0}) = \begin{cases} 0 & \text{if} \quad y_{L,0} \leq \delta_{\min} \\ \frac{1}{\delta_{\max}-\delta_{\min}} \int_0^{y_{L,0}-\delta_{\min}} \varphi^y_{L,0}(z) dz & \text{if} \quad y_{L,0} \in [\delta_{\min}, \delta_{\max}] \\ \frac{1}{\delta_{\max}-\delta_{\min}} \int_{y_{L,0}-\delta_{\max}}^{y_{L,0}-\delta_{\min}} \varphi^y_{L,0}(z)\, dz & \text{if} \quad y_{L,0} \geq \delta_{\max} \end{cases} \tag{11}$$

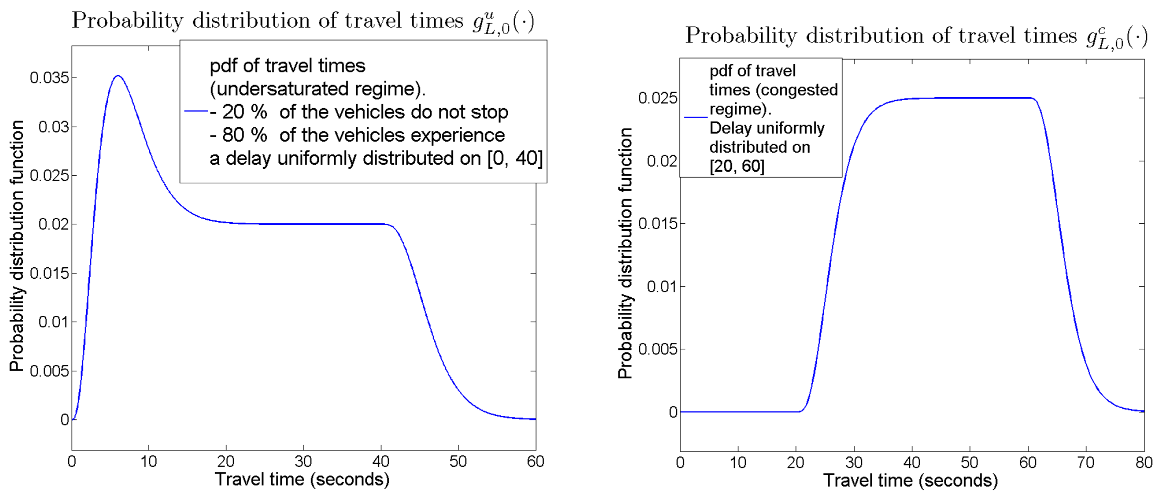

Figure 7: Probability distributions of link travel times. **Left:** Undersaturated regime. The figure represents pdf for a traffic light of duration 40 seconds when 80% of the vehicles stop at the light ($\eta^u_{L,0} = .8$). **Right:** Congested regime. The figure represents the pdf for a traffic light of duration 40 seconds when all the vehicles stop in the triangular queue and 50% of the vehicles stop once in the remaining queue. Both figures are produced for a link of length 100 meters. The free flow pace is a random variable with Gamma distribution. The mean free flow pace is 1/15 s/m and the standard deviation is 1/30 s/m. We recall that the probability distribution γ of a Gamma random variable $x \in \mathbb{R}^+$ with shape α and inverse scale parameter β is given by $\gamma(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$, where Γ is the Gamma function defined on $\mathbb{R}^+$ and with integral expression $\Gamma(x) = \int_0^{+\infty} t^{x-1} e^{-t}\, dt$.

5.2 Quasi-concavity properties of the probability distributions of link travel times

The pdf of travel times depend on a set of parameters that must be estimated to fully determine the statistical distribution of the travel times. A parameter with true value θ_0 is estimated via an estimator $\hat{\theta}$. We require this estimator to have some optimality properties—extremum point based on an objective function, *e.g.* least square estimator, maximum likelihood estimator. In particular, the maximum likelihood estimator is widely used in statistics for its convergence properties. Its computation requires the maximization of the likelihood (or log-likelihood) function, which represents the probability of observing a set of data points, given the value of a parameter.

In this work, the function to maximize is the probability distribution of travel times. Properties on the concavity of this function are important for designing efficient maximization algorithms with guaranties of global optimality. In this section, we present the proof of the quasi-concavity of the link travel time distributions in both the undersaturated and the congested regimes. We also prove the log-concavity of the different components of the distribution of travel times, considered as mixture distributions.

Definition 1 (Quasi-concavity (Boyd)). *[5] A function $f : \mathbb{R}^n \to \mathbb{R}$ is called quasi-concave if its domain is convex and if $\forall \alpha \in \mathbb{R}$, the superlevel set Sf_α ($Sf_\alpha = \{x \in \mathcal{D}_f | f(x) \geq \alpha\}$) is convex.*

From this definition, one can derive equivalent characterization of quasi-concavity, when f has first (and second) order derivatives. The reader should refer to [5] for further references on quasi-concavity. In particular, we use the characterization of continuous quasi-concave functions on $\mathbb{R}$ (Lemma and the second order characterization:

Lemma 1 (Characterization of continuous quasi-concave functions on $\mathbb{R}$). *[5] A continuous function $f : \mathcal{D}_f \to \mathbb{R}$ is quasi-concave if and only if at least one of the following conditions holds:*

- *f is nondecreasing*
- *f is nonincreasing*
- *there is a point $x_c \in \mathcal{D}_f$ such that for $x \leq x_c$ (and $x \in \mathcal{D}_f$), f is nonincreasing, and for $x \geq x_c$ (and $x \in \mathcal{D}_f$), f is nondecreasing*

Lemma 2 (Second order characterization of quasi-concave functions). *[5] $f \in \mathcal{C}^2$ is quasi-concave if and only if $\forall (x, y) \in {\mathcal{D}_f}^2,\ y^T \nabla f(x) = 0 \Rightarrow y^T \nabla^2 f(x) y \leq 0$. If f is unidimensional, f is quasi-concave if and only if $f'(x) = 0 \Rightarrow f''(x) \leq 0$.*

Note that for probability distributions, we are interested in the properties of the log of the probability function.

Definition 2 (Log-concavity (Boyd)). *[5] A function $f : \mathbb{R}^n \to \mathbb{R}^+_*$ is log-concave if and only if $\ln(f)$ is concave. The second order characterization is as follows:*
$f \in \mathcal{C}^2$ is log-concave is and only if $\forall x\, f(x)f''(x) - (f'(x))^2 \leq 0$.

Fact 2. *For f a twice differentiable function taking values in $\mathbb{R}_*^+$, f is quasi-concave $\Leftrightarrow \ln(f)$ is quasi-concave.*

Proof. We have $\nabla \ln f(x) = \dfrac{\nabla f(x)}{f(x)}$ and $\nabla^2 \ln f(x) = \dfrac{f(x)\nabla^2 f(x) - \nabla f(x)\nabla f(x)^T}{f(x)^2}$.

- We assume that f is quasi-concave, we want to show that $ln(f)$ is quasi-concave:
  We assume $\forall(x,y),\ y^T\nabla f(x) = 0 \Rightarrow y^T\nabla^2 f(x)y \leq 0$.
  Let x and y be such that $y^T\nabla\ln(f(x)) = 0$, *i.e.* $y^T\nabla f(x) = 0$. From the quasi-concavity of f, we have $y^T\nabla^2 f(x)y \leq 0$

$$\begin{aligned} y^T\nabla^2\ln(f(x))y &= \frac{f(x)\,y^T\nabla^2 f(x)y - y^T\nabla f(x)\nabla f(x)^T y}{f(x)^2} \\ &= \frac{f(x)\,y^T\nabla^2 f(x)y}{f(x)^2} \quad \text{since } y^T\nabla f(x) = 0 \\ &\leq 0 \qquad\qquad\qquad \text{using the quasi-concavity of } f \end{aligned}$$

  So $y^T\nabla\ln(f(x)) = 0 \Rightarrow y^T\nabla^2\ln(f(x))y \leq 0$ and $\ln(f)$ is quasi-concave.
- We assume that $\ln(f)$ is quasi-concave, we want to show that f is quasi-concave:
  We assume $\forall(x,y),\ y^T\nabla\ln(f(x)) = 0 \Rightarrow y^T\nabla^2\ln(f(x))y \leq 0$. Using the expression of $\nabla\ln(f(x))$ and $\nabla^2\ln(f(x))$, this condition can be rewritten as follows:

$$\forall(x,y),\ y^T\nabla f(x) = 0 \Rightarrow y^T\nabla^2 f(x)y \leq 0$$

  And this proves that f is quasi-concave.

□

In the following, we assume that the pdf φ^p of the free flow pace is strictly log-concave, and thus so is the pdf $\varphi^y_{x_1,x_2}$ of the free flow travel time between location x_1 and x_2. Note that most common probability distributions (*e.g.* Gaussian or Gamma with shape greater than 1) are log-concave.

Fact 3. *In one dimension, a strictly log-concave probability distribution function φ defined on $\mathcal{D}_\varphi \subset \mathbb{R}$ has a unique critical point $y_c \in \overline{\mathcal{D}_\varphi}$. On its domain, φ is strictly increasing for $y \leq y_c$ and strictly decreasing for $y \geq y_c$*

This result comes from the fact that log-concavity implies quasi-concavity (see [5]). Note that we allow y_c to be at the bounds of the domain $\mathcal{D}_f$. If $\exists a$ such that $\forall x \in (a, +\infty), \varphi(x) > 0$, then φ is either strictly decreasing or has a unique critical point (reasoning by contradiction and using the integrability of φ). Similarly, if $\exists b$ such that $\forall x \in (-\infty, b), \varphi(x) > 0$, then φ is either strictly increasing or has a unique critical point.

5.2.1 Proof of the quasi-concavity of the travel time probability distribution in the undersaturated regime

The goal of this section is to prove that the undersaturated travel time probability distribution function is a quasi-concave function. Let Δ denote the maximum delay experienced (i.e. the red time R) and η the fraction of delayed vehicles (previously denoted $\eta^u_{L,0}$). The length of the link L is a scale parameter that does not change the concavity properties of the function. For notational simplicity, we denote φ the pdf of travel times and omit the locations $x_1 = L$ and $x_2 = 0$ in this section. Recall the travel time distribution on an undersaturated link:

$$g^u(y) = (1-\eta)\varphi(y) + \frac{\eta}{\Delta}\int_{y-\Delta}^{y} \varphi(z)\,dz$$

with the convention $\varphi(z) = 0$ for $z \leq 0$.

The function g^u is continuously differentiable on $\mathbb{R}^+$ and $\forall\, y \in \mathbb{R}^+$ we have:

$$(g^u)'(y) = (1-\eta)\varphi'(y) + \frac{\eta}{\Delta}(\varphi(y) - \varphi(y-\Delta)). \tag{12}$$

The function $(g^u)'$ is continuously differentiable on $\mathbb{R}^+$ and $\forall\, y \in \mathbb{R}^+$ we have:

$$(g^u)''(y) = (1-\eta)\varphi''(y) + \frac{\eta}{\Delta}(\varphi'(y) - \varphi'(y-\Delta)) \tag{13}$$

Using the expression of $(g^u)'(y)$, we have

$$(g^u)'(y) = 0 \Leftrightarrow (1-\eta)\varphi'(y) = \frac{\eta}{\Delta}(\varphi(y-\Delta) - \varphi(y)). \tag{14}$$

Our goal is to prove that $(g^u)'(y) = 0 \Rightarrow (g^u)''(y) \leq 0$, so let y be such that $(g^u)'(y) = 0$.

- <u>Case 1: $\varphi'(y) > 0$</u>
  Using Fact 3, we know that φ is strictly increasing on $(-\infty, y]$. Thus $\varphi(y-\Delta) < \varphi(y)$. Plugging back into (12), we prove that $\varphi'(y) > 0 \Rightarrow (g^u)'(y) > 0$ which contradicts the hypothesis $(g^u)'(y) = 0$.
- <u>Case 2: $\varphi'(y) \leq 0$</u>
  From (13) and the log concavity of φ we have

$$\begin{aligned}
(g^u)''(y) &\leq (1-\eta)\frac{(\varphi'(y))^2}{\varphi(y)} + \frac{\eta}{\Delta}(\varphi'(y) - \varphi'(y-\Delta)) \\
&\quad \text{Using (14), we replace } (1-\eta)\varphi'(y) \text{ by } \frac{\eta}{\Delta}(\varphi(y-\Delta) - \varphi(y)) \\
&= \frac{\eta}{\Delta}\left(\frac{(\varphi'(y))}{\varphi(y)}\left(\varphi(y-\Delta) - \varphi(y)\right) + \varphi'(y) - \varphi'(y-\Delta)\right) \\
&= \frac{\eta}{\Delta}\left(\frac{(\varphi'(y))}{\varphi(y)}\varphi(y-\Delta) - \varphi'(y-\Delta)\right)
\end{aligned}$$

  Moreover, equation (14) and the condition $\varphi'(y) \leq 0$ imply that $\varphi(y-\Delta) \leq \varphi(y)$. Reasoning by contradiction, we assume that $\varphi'(y-\Delta) \leq 0$. From Fact 3, we know that $\varphi'(y-\Delta) \leq 0$ implies $\varphi(y-\Delta) > \varphi(y)$, which contradicts the assumption of Case 2. Thus necessarily, we have $\varphi'(y-\Delta) \geq 0$ and plugging into (13), $(g^u)''(y) \leq 0$.

We conclude that $(g^u)'(y) = 0 \Rightarrow (g^u)''(y) \leq 0$. From the definition of quasi-concavity (Definition 1), we conclude that $g^u(y)$ is quasi-concave.

5.2.2 Proof of the quasi-concavity of the travel time probability distribution in the congested regime

The goal of this section is to prove that the congested travel time probability distribution function is a quasi-concave function[1]. Let $\delta_{\min}$ (resp. $\delta_{\max}$) denote the minimum (resp. maximum) delay experienced. Recall the travel time probability distribution on a congested link:

$$g_c(y) = \frac{1}{\delta_{\max} - \delta_{\min}} \int_{y-\delta_{\max}}^{y-\delta_{\min}} \varphi(y)\, dy$$

The function g_c is continuously differentiable on $\mathbb{R}^+$ and $\forall\, y \in \mathbb{R}^+$ we have:

$$g_c'(y) = \frac{1}{\delta_{\max} - \delta_{\min}} \left(\varphi(y - \delta_{\min}) - \varphi(y - \delta_{\max})\right).$$

We prove that there exists an interval I such that $y \notin I \Rightarrow g_c'(y) \neq 0$. From the characterization of quasi-concave function given in Lemma 1, we conclude that g_c is quasi-concave.

Referring to Fact 3, we note y_c the critical point of the pace distribution φ. We have:

- For $y \in [0,\, y_c + \delta_{\min}]$, we have $y - \delta_{\max} < y - \delta_{\min} \leq y_c$. Thus $\varphi(y - \delta_{\max}) < \varphi(y - \delta_{\min})$ and $g_c'(y) > 0$.
- For $y \in [y_c + \delta_{\max},\, +\infty]$, we have $y_c \leq y - \delta_{\max} < y - \delta_{\min}$. Thus $\varphi(y - \delta_{\max}) > \varphi(y - \delta_{\min})$ and $g_c'(y) < 0$.
- For $y \in [y_c + \delta_{\min},\, y_c + \delta_{\max}]$, we have $y - \delta_{\max} \leq y_c \leq y - \delta_{\min}$. For all $y \in [y_c + \delta_{\min},\, y_c + \delta_{\max}]$, $y - \delta_{\max} \leq y_c$ and thus the function $y \mapsto \varphi(y - \delta_{\max})$ is strictly increasing on $[y_c + \delta_{\min},\, y_c + \delta_{\max}]$. Similarly, for all $y \in [y_c + \delta_{\min},\, y_c + \delta_{\max}]$, $y - \delta_{\max} \geq y_c$ and the function $y \mapsto \varphi(y - \delta_{\min})$ is strictly decreasing on $[y_c + \delta_{\min},\, y_c + \delta_{\max}]$. The function g'^c is strictly decreasing on $[y_c + \delta_{\min},\, y_c + \delta_{\max}]$. Moreover $g'^c(y_c + \delta_{\min}) > 0$ and $g'^c(y_c + \delta_{\max}) < 0$. Using the monotonicity of g'^c on $[y_c + \delta_{\min},\, y_c + \delta_{\max}]$ and the theorem of intermediate values, we show that g_c' is equal to zero in a unique point on $[y_c + \delta_{\min},\, y_c + \delta_{\max}]$.

The function g_c has a unique critical point (unique point where g'^c equals zero). From the characterization of quasi-concavity given in Lemma 1, we conclude that $g_c(y)$ is quasi-concave. Note that we have also proven *strict* quasi-concavity, the critical point of g_c is unique.

5.3 Log-concavity properties of the different components of the mixture model

For any locations x_1 and x_2 and any regime (undersaturated or congested) the probability distribution of travel times is a mixture distribution. Each component of the mixture, denoted $\psi_i(y_{x_1,x_2})$, is the probability distribution of travel times associated with a delay with either a mass or a uniform probability. In this section, we prove that each of the component is log-concave. For each of the component ψ_i, one of the following statement is true:

- $\exists \Delta \geq 0$ such that $\psi_i(y_{x_1,x_2}) = \mathbf{1}_{\{\Delta\}} * \varphi^y_{x_1,x_2}(y_{x_1,x_2})$,
- $\exists \delta_{\min} \geq 0$, $\delta_{\max} > \delta_{\min}$ such that $\psi_i(y_{x_1,x_2}) = \frac{1}{\delta_{\max}-\delta_{\min}} \mathbf{1}_{[\delta_{\min},\, \delta_{\max}]} * \varphi^y_{x_1,x_2}(y_{x_1,x_2})$,

[1] We will show in Section 5.3 that the pdf is log-concave, but this involves results on log-concave functions that are not trivial to prove.

To conclude on the log-concavity of each component, we use the following facts:

Fact 4 (Integration of log-concave functions [19, 20]). *If $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a log-concave function, then $g(x) = \int_{\mathbb{R}^m} f(x, y)\, dy$ is a log-concave function of x on $\mathbb{R}^n$.*

In particular, log-concavity is closed under convolution.

Fact 5 (Log-concavity is closed under convolution [5]). *If f and g are log-concave on $\mathbb{R}^n$, then so is the convolution h of f and g, $h(x) = \int_{\mathbb{R}^n} f(x - y)g(y)\, dy$. Indeed the function $(x, y) \mapsto f(x-y)g(y)$ is log-concave as the product of two log-concave functions (log-concavity is closed under the multiplication since concavity is closed under summation). The result follows from Fact 4.*

We have written each component of the mixture distribution as a convolution between (i) a Dirac distribution and the pdf of free flow travel time or (ii) a Uniform distribution (constant function on a convex interval) and the pdf of free flow travel times. Under the assumption that the probability distribution of pace is log-normal, so is each component of the mixture.

Remark that when the delay distribution has a mass probability, this result can also be derived by noticing that the probability distribution of travel times is a translation of the probability distribution of free flow travel times.

For any locations x_1 and x_2 on a link and any congestion regime (undersaturated or congested), we have derived an expression for the probability distribution of travel times. These probability distributions are mixture distributions. Each of the component is the convolution between the probability distribution of free flow travel time and either a mass probability or a uniform probability distribution. We have proven that the link travel times are quasi-concave for both the undersaturated and the congested regime. Moreover, for any locations x_1 and x_2 on a link and any congestion regime, the probability distribution of travel times is a mixture of log-concave probability distributions.

6 Conclusion

This report presents the application of traffic flow theory to the construction of a statistical model of arterial traffic conditions based on standard assumptions in transportation engineering. In particular, we assume that time can be discretized into periods of stationary conditions and we then study the traffic dynamics for the duration of one period.

The model validates the intuition that the average density of vehicles is higher close to the end of the links because of the presence of traffic signals. We provide analytical derivations for the average density and spatial distribution of vehicles on a link, parameterized by traffic parameters. When probe vehicles send their location periodically in time, this model is used to learn traffic conditions via the estimation of queue length.

Under similar traffic conditions, the delay experienced by vehicles depends on the time at which they enter the link. Assuming uniform arrivals, we compute the probability distribution of delays among the vehicles entering the link in a cycle to model the differences in delay experienced by the vehicles. We show that the delay distribution is a finite mixture distribution, where each mixture corresponds to an interval of arrival times. Each mixture distribution corresponds to a delay with either a mass or a uniform distribution.

Under free-flow conditions, vehicles may have different driving behavior. We model this fact by considering the free flow pace as a random variable with a specific distribution. We use the model of driving behavior and the probability distribution of delays to derive the probability distribution of travel times between any two locations on an arterial link. We show that the probability density functions of link travel times are quasi-concave and that the probability distributions of travel times between any two arbitrary location on the link are mixtures of log-concave distributions. The probability distributions of travel times between arbitrary locations are parameterized by the traffic signal parameters (red time and cycle time), the driving behavior, the queue length and the queue length at saturation. When probe vehicles send pairs of successive locations, we can compute their travel times to go from one location to the next. As we receive data from different probes, we can estimate the parameters that maximize the probability of receiving the observations. This modeling approach is used and developed further in subsequent work to produce accurate arterial traffic estimates and short-term forecast [14]. It can also be used with historical data to estimate the parameters of the network (parameters of the traffic signals, queue length at saturation). The concavity properties of the travel time distributions are used to guarantee global optimality of a sub-problem of the estimation algorithm.

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A Derivation of the probability distribution of total delay between arbitrary locations in the congested regime

We derive the probability distribution of travel times for vehicles traveling from a location x_1 to a location x_2 on the link. As in the previous notations, x represents the distance to the intersection.

We call n_s the maximum number of stops in the remaining queue experienced by the vehicles between the locations x_1 and x_2, and omit the indices x_1 and x_2 for notational simplicity. In the duration of a light cycle, the distance traveled by vehicles stopped in the queue is $l_{\max}$. Thus, the maximum number of stops in the remaining queue, between x_1 and x_2,

$$n_s = \left\lceil \frac{\min(x_1, l_r) - \min(x_2, l_r)}{l_{\max}} \right\rceil .$$

The delay experienced when reaching the triangular queue is readily derived from the expression of the delay in the undersaturated regime. The delay experienced when reaching the remaining queue is the duration of the red time R. The expression of the delay at location x is then

$$\delta^c(x) = \begin{cases} R & \text{if } x \leq l_r \\ R\,\frac{l_r + l_{\max} - x}{l_{\max}} & \text{if } x \in [l_r, l_r + l_{\max}] \\ 0 & \text{if } x \geq l_r + l_{\max} \end{cases}$$

Case 1: x_1 is upstream of the total queue and x_2 is in the remaining queue (Figure 8)

Condition 1: $x_1 \geq l_r + l_{\max} \quad x_2 \leq l_r$

Since x_1 is upstream of the total queue and x_2 is in the remaining queue, all the vehicles stop once in the triangular queue between x_1 and x_2. We define the critical location x_c as the location in the triangular queue such that

- Vehicles reaching the triangular queue upstream of x_c stop n_s times in the remaining queue. They represent a fraction $\frac{l_r + l_{\max} - x_c}{l_{\max}}$ of the vehicles entering the link in a cycle.
- Vehicles reaching the triangular queue downstream of x_c stop $n_s - 1$ times in the remaining queue. They represent a fraction $\frac{x_c - l_r}{l_{\max}} = 1 - \frac{l_r + l_{\max} - x_c}{l_{\max}}$ of the vehicles entering the link in a cycle.

The location x_c is given by $x_c = x_2 + n_s l_{\max}$.

The values of the minimum and maximum delays are given by $\delta_{\min} = (n_s - 1)R + \delta^c(x_c)$ and $\delta_{\max} = n_s R + \delta^c(x_c)$. The delay experienced by the vehicles is uniformly distributed on $[\delta_{\min}, \delta_{\max}]$. We note that $n_s \geq 1$ since $x_2 \leq l_r$.

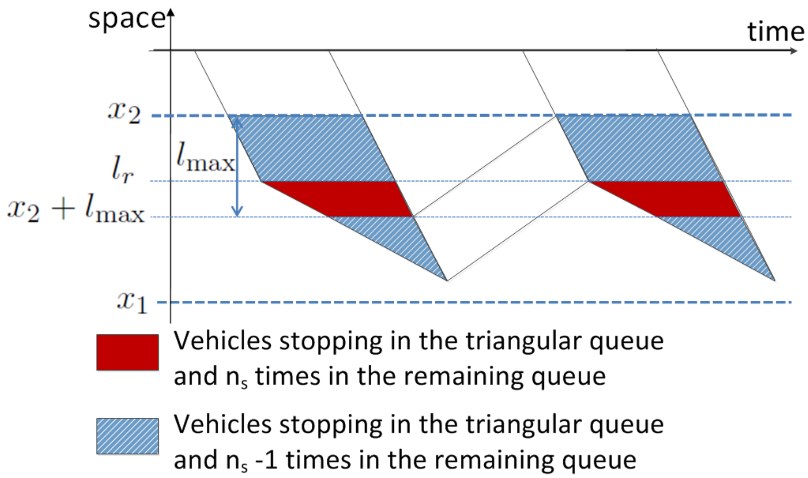

Figure 8: Case 1: All the vehicles stop in the triangular queue. A fraction stops n_s times in the remaining queue, the other ones stop $n_s - 1$ times.

Case 2: x_1 and x_2 are upstream of the remaining queue (Figure 9)

<u>Condition 2:</u> $x_1 \geq l_r \quad x_2 \geq l_r$

Given that x_2 is upstream of the remaining queue, this case is similar to the undersaturated regime. A fraction of the vehicles does not experience delay between x_1 and x_2. The vehicles reaching the queue between x_1 and x_2 experience a delay in the triangular queue. This delay is a random variable, uniformly distributed on $[\delta^c(x_1), \delta^c(x_2)]$.

The fraction of vehicles experiencing delay is $\eta^c_{x_1,x_2} = \frac{\min(l_{\max}+l_r,x_1)-\min(l_{\max}+l_r,x_2)}{l_{\max}}$

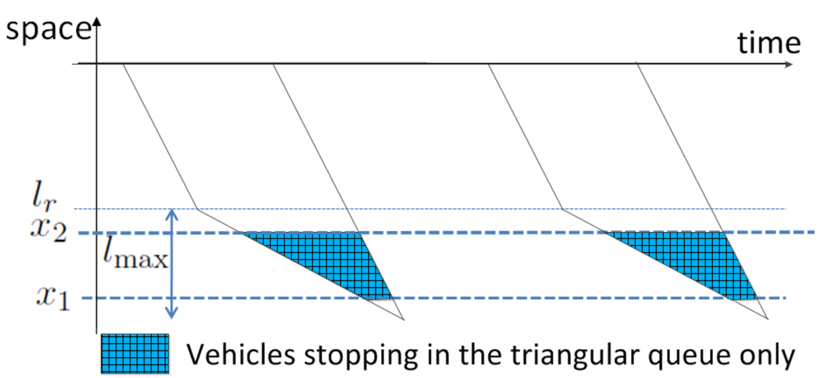

Figure 9: Case 2: Some vehicles stop in the triangular queue. The others do not experience delay.

Case 3: x_1 is in the remaining queue, and thus so is x_2 (Figure 10)

<u>Condition 3:</u> $x_1 \leq l_r$ (which implies $x_2 \leq l_r$)

The path starts downstream of the triangular queue. Some vehicles stop n_s times and experience a delay n_sR and the other vehicles stop $n_s - 1$ times and experience a delay $(n_s - 1)R$.

We define the critical location x_c as the location in the remaining queue such that

- Vehicles joining the queue between x_1 and x_c stop n_s times between x_1 and x_2. Their stopping time is n_sR and they represent a fraction $(x_1 - x_c)/l_{\max}$ of the vehicles entering the link in a cycle.
- Vehicles joining the queue between x_c and $x_c - l_{\max}$ stop $n_s - 1$ times between x_1 and x_2. Their stopping time is $(n_s - 1)R$ and they represent a fraction $1 - (x_1 - x_c)/l_{\max}$ of the vehicles entering the link in a cycle.

The critical location x_c is given by $x_c = x_2 + (n_s - 1)l_{\max}$.

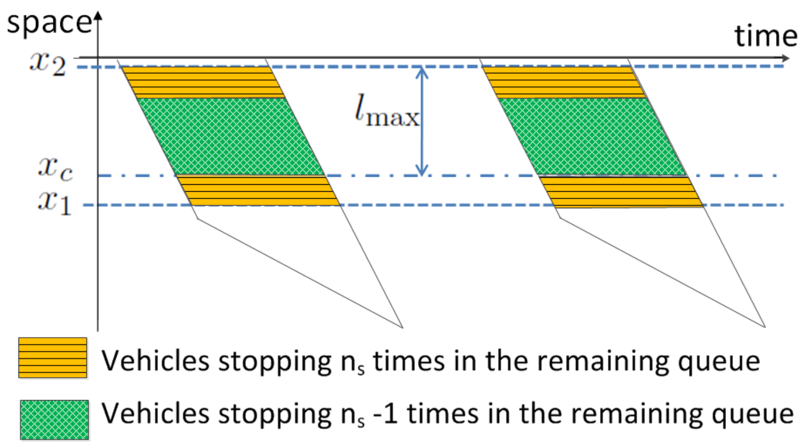

Figure 10: Case 3: A fraction of the vehicles stop n_s times in the remaining queue. The rest stop $n_s - 1$ times in the remaining queue.

Case 4: x_1 is in the triangular queue, x_2 is in the remaining queue

We distinguish two different cases to derive the probability distribution of travel times. We define the critical location x_c as $x_c = x_2 + n_sl_{\max}$ and derive probability distributions of travel times for the two subcases 4a ($x_c \leq x_1$, Figure 11 (top)) and 4b ($x_c \geq x_1$, Figure 11 (bottom)).

Case 4a. $x_c \leq x_1$. The delay patterns are the following:

- One stop in the triangular queue and n_s stops in the remaining queue. The queue is first reached between x_1 and x_c. The delay is a random variable with uniform distribution with support $[\delta^c(x_1) + n_sR, \, \delta^c(x_c) + n_sR]$. The vehicles following this pattern represent a fraction $\frac{x_1 - x_c}{l_{\max}}$ of the vehicles entering the link in a cycle.
- One stop in the triangular queue and $n_s - 1$ stops in the remaining queue. The queue is first reached between x_c and l_r. The delay is a random variable with uniform distribution with support $[\delta^c(x_c) + (n_s - 1)R, \, \delta^c(l_r) + (n_s - 1)R]$. Noticing that $\delta^c(l_r) = R$, we derive that the support of the delay distribution is $[\delta^c(x_c) + (n_s - 1)R, \, n_sR]$. The vehicles

following this pattern represent a fraction $\frac{x_c - l_r}{l_{\max}}$ of the vehicles entering the link in a cycle.

– No stop in the triangular queue and n_s stops in the remaining queue. The queue is first reached between l_r and $x_1 - l_{\max}$. The delay is $n_s R$. The vehicles following this pattern represent a fraction $\frac{l_r - (x_1 - l_{\max})}{l_{\max}}$ of the vehicles entering the link in a cycle.

We can check that the weights of the different components sum to 1:

$$\frac{x_1 - x_c}{l_{\max}} + \frac{x_c - l_r}{l_{\max}} + \frac{l_r - (x_1 - l_{\max})}{l_{\max}} = 1$$

We remark that, $x_2 \leq l_r$ implies that $n_s \geq 1$. Then using the definition of x_c, $x_c = x_2 + n_s l_{\max}$ and the fact that $x_1 \geq x_c$, we prove that $x_1 - l_{\max} \geq x_2$ and all vehicles reach the queue between x_1 and $x_1 - l_{\max}$.

Case 4b. $x_c \geq x_1$. The delay patterns are the following:

– One stop in the triangular queue and $n_s - 1$ stops in the remaining queue. The queue is first reached between x_1 and l_r. The delay is a random variable with uniform distribution on $[\delta^c(x_1) + (n_s - 1)R,\ \delta^c(l_r) + (n_s - 1)R]$, *i.e.* uniform distribution on $[\delta^c(x_1) + (n_s - 1)R,\ n_s R]$. The vehicles following this pattern represent a fraction $\frac{x_1 - l_r}{l_{\max}}$ of the vehicles entering the link in a cycle.

– No stop in the triangular queue and n_s stops in the remaining queue. The queue is first joined between l_r and $x_c - l_{\max}$. The delay is $n_s R$. The vehicles following this pattern represent a fraction $\frac{l_r - (x_c - l_{\max})}{l_{\max}}$ of the vehicles entering the link in a cycle.

– No stop in the triangular queue and $n_s - 1$ stops in the remaining queue. The queue is first joined between $x_c - l_r$ and $x_1 - l_{\max}$. The delay is $(n_s - 1)R$. The vehicles following this pattern represent a fraction $\frac{x_c - x_1}{l_{\max}}$ of the vehicles entering the link in a cycle.

We can check that the weights of the different components sum to 1:

$$\frac{l_r - (x_c - l_{\max})}{l_{\max}} + \frac{x_1 - l_r}{l_{\max}} + \frac{x_c - x_1}{l_{\max}} = 1.$$

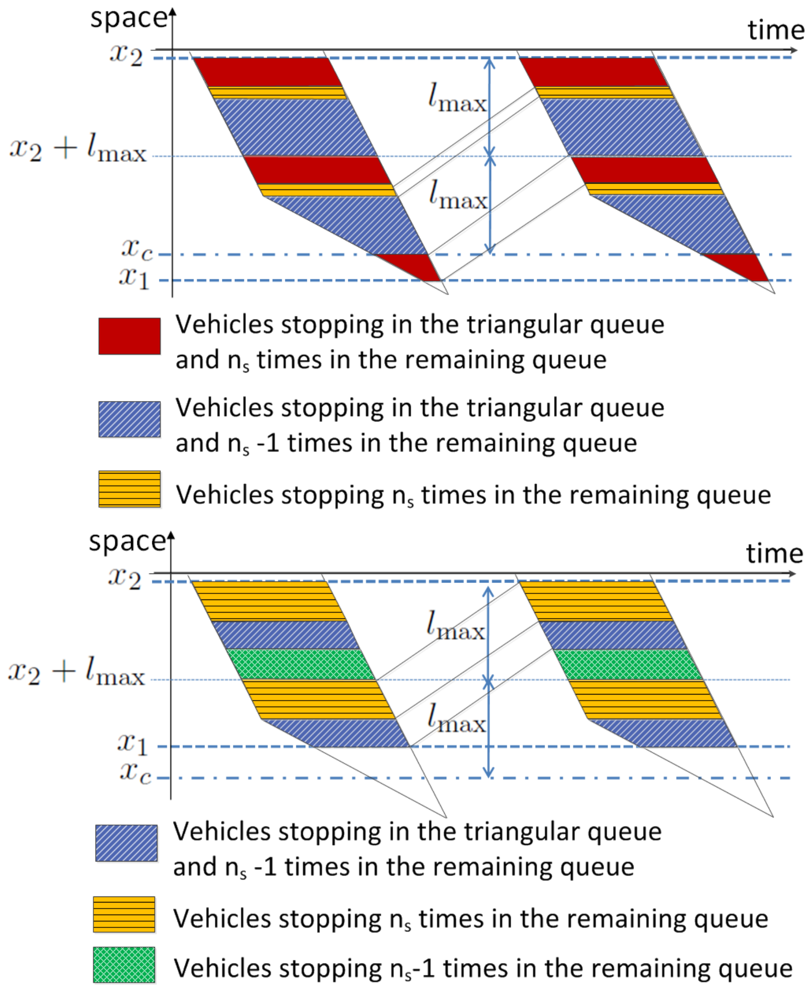

Figure 11: Case 4: (**Top**) Case 4a: a fraction of the vehicles stop in the triangular queue and n_s times in the remaining queue, a fraction of the vehicles stop in the triangular queue and n_s times in the remaining queue, the rest stop n_s times in the remaining queue. (**Bottom**) Case 4b: a fraction of the vehicles stop in the triangular queue and $n_s - 1$ times in the remaining queue, a fraction of the vehicles stop n_s times in the remaining queue, the rest stop $n_s - 1$ times in the remaining queue.