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Time-Continuous Instantaneous and Past Memory Routing on Traffic Networks: A Mathematical Analysis on the Basis of the Link-Delay Model*

Alexandre Bayen†, Alexander Keimer‡, Emily Porter‡, and Michele Spinola§

Abstract. This article presents an extensive theoretical framework to mathematically defined and information-based routing operators, applied to the continuous-time *dynamic traffic assignment* problem. Because of the difficulty of the mathematical framework required to provide existence and uniqueness proofs of the solution to the problem in the presence of a routing operator at nodes, the approach is instantiated with a link model, consisting of a system of *ordinary delay differential equations* and modeling traffic flow macroscopically. The routing operators distributing the incoming flow can encompass a wide range of information patterns, which can include past knowledge of the network state (statistical, or past deterministic information) up to real time and thus satisfying a nonanticipative character. We show, for a rather broad class of routing operators, the existence and uniqueness of solutions on the full network. This framework can be extended to more advanced traffic flow models such as *partial differential equation* models.

Key words. link-delay model, traffic flow on networks, routing, delay differential equation, dynamic traffic assignment, instantaneous routing, existence of a solution on the network, uniqueness of a solution on the network

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1. Introduction.

1.1. General context of this work.

In recent years, due to the increased usage of routing applications, modeling the dynamical process of routing in road networks has become a necessity, as information patterns provided to motorists by smartphones and connected devices/vehicles have progressively induced large scale macroscopic changes in mobility patterns [14, 13]. These impacts are directly felt by numerous residential neighborhoods that have noticed a drastic increase in through traffic generated by routing apps in recent years [42, 8, 29, 1, 2]. These have in turn taken defensive actions, which involve (1) spoofing apps [60, 59], (2) reporting fake car accidents on roads [41], (3) reprogramming signal timing plans to actively prevent traffic through the corresponding neighborhoods (thereby increasing queues

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†Department of Civil and Environmental Engineering, University of California at Berkeley, Berkeley, CA, 94720-1710 (bayen@berkeley.edu).
‡ITS, UC Berkeley, Berkeley, CA (keimer@berkeley.edu, emilyporter@berkeley.edu).
§Mathematik, Friedrich-Alexander Universität Erlangen-Nürnberg (FAU), Erlangen, 91058, Germany (michele.spinola@fau.de).

at off ramps of freeways) [48], (4) issuing tickets to nonresident thru-traffic [37], (5) building infrastructure to slow through traffic (street bumps, turn restrictions, stop signs) [48], or (6) directly suing companies held responsible for these issues [66].

Numerous publications present *dynamic traffic assignment* (DTA) models or extensions of these, integrating dynamical routing. Many of them address the problem over the full time horizon as an optimal control problem or via *variational inequalities* implementing a time-dependent Wardrop's principle for the routing at every instant of time [7]. The specific emphasis of the present work is to define a general mathematical framework capable of encoding information patterns generated by mobile routing apps and to instantiate them for the specific context of selfish routing, as mentioned in the references above.

In the present work, we will therefore present a mathematical model based on an already existing macroscopic traffic flow *ordinary differential equation* (ODE) model with delay, dependent on the density [63]. Since we aim for a network description with many different destinations and different types/groups of drivers, the model consists indeed of a system of ODEs on every link in the network, coupled by a summarized density of all the different density functions on the specific link (compare also [24]). The flow allocation at the junctions of the network is realized by routing operators. We define a broad class of these operators which guarantee existence and uniqueness of solutions on the network. Thereby, the routing operators depend on the status of the network at real time or past time (or a combination of both). Due to their structure, they can encompass point evaluations of the network status as well as averaged/aggregated information, etc. The generality of the proposed approach also enables it to use the results in more realistic and complex macroscopic traffic flow models. To the best of our knowledge such a full mathematical approach which incorporates a wide class of routing realizations has not been published yet in the literature.

1.2. DTA and routing in the literature.

In this section, we give a short survey of already existing modeling and routing approaches. Due to the vast number of publications in this area, the presented list of publications cannot be complete. However, we aim for describing the different approaches specifically and at a technical level for each class. We thus start this overview with the famous books [15, 45, 54], which have a more extensive bibliography of this field.

1.2.1. Time-continuous modeling.

Generally, in a time-continuous setting, the link dynamics can be described via ODEs or partial differential equations (PDEs). The articles [43, 44, 56, 55, 20] consider DTA as a time-optimal control problem. The link dynamics are represented by a (system) of ODEs with in- and outflow functions. A proper objective function is specified, and optimality conditions are deduced. However, there is no natural delay, and the optimization is considered on the full time horizon, so that one has to know information about inflow over the full time horizon to solve the problem.

In [52, 7, 18], more general ODE models (sometimes also with delay using a model which is also used in this work) are considered, and variational inequalities describe the routing at the junctions. Depending on if the variational inequality is considered over the full time horizon (with an L^2 scalar product) or at every time (with the ordinary scalar product in $\mathbb{R}^n$ and the variational inequality to hold at every time), the computation of a solution again requires full information of the input datum over the full time horizon considered. A rigorous analysis under which conditions existence (and uniqueness) of solutions is guaranteed is not demonstrated in detail. However, in [54] a more detailed analysis is provided. Another ODE model is considered in [50], but the routing is realized by a specific routing function taking into account the status of the network. It also proves some stability estimates w.r.t. the routing considered.

Another modeling approach which is also considered in [39, 3] consists of using the Vickrey or point-queue model (or a modified version) [27, 28, 26], again aiming for a routing (and departure choice time) based on variational inequalities.

These modeling approaches can be generalized to dynamics which are prescribed by PDEs. We refer for these model types to [10, 9, 21, 22, 31, 11]. Usually, the underlying link dynamics are modeled by the Lighthill–Whitham–Richards PDE, a hyperbolic conservation law, allowing spillback and congestion. However, how to define the boundary terms at the junctions subject to the proposed routing is due to those phenomena nontrivial. The exhaustive books [21, 22] treat this problem by defining specific Riemann solvers at the junctions, while [10, 9, 19, 4] use a buffer (limited or unlimited) model at the junctions dealing with the boundary datum. Another approach uses nonlocal PDE models to simulate traffic flow at junctions [35, 24, 36].

1.2.2. Time-discretized modeling.

There is a significant number of articles which deal with already discretized time dynamical models. Part of the appeal for such models is the easiness to define solutions as there is no need for proofs of existence or uniqueness. For the sake of an exhaustive overview, see [23, 49, 58, 61, 16, 38]. The articles [34, 33, 30] consider a discretized traffic flow model based on ODEs and time-optimal control problem to obtain the routing. It is worth mentioning that [38] considers another discretized ODE model, but this time the routing is implemented by a probabilistic approach depending on the status of the network.

1.2.3. Modeling with a simulator and route choice.

For a general overview we refer to [51], and for a simulator approach to [5] and [6]. For general route choice models we refer to [16, 40].

1.3. Modeling approach considered in the present article.

One way of justifying many of the above prescribed modeling approaches following time-optimal control problems or variational inequalities over the full time horizon considered might be a repeating character of traffic scenarios where the participating drivers might adjust in the long term to a type of time-dependent Wardrop equilibrium [62, 47, 54]. With more routing applications, such behavior might change and—to some extent—has already changed the observed traffic patterns. Drivers rely more and more on routing information which might be real time or slightly delayed or involve simplified forecasts. In addition, different drivers using different applications or no applications for routing will—based on their information and the used routing—behave quite differently and thus have to be modeled differently. The aforementioned trends in the change of traffic behavior make it necessary to present a mathematically rigorous and precise framework which can capture time dynamical effects and is capable of modeling these mentioned challenges.

This is the aim of the present work and the reason why we study the named problem by a control approach where the control is located in the routing level. One of the major contributions of the present work is the proper mathematical definition of routing operators to encompass new traffic information patterns generated by traffic apps. It is also a rigorous and precise continuation of [35], where we introduced and prescribed the emerging challenges and performed some first steps for such an approach: A percentage of drivers might get rerouted instantaneously, and in the near future real time information about the road network might be used by almost all drivers. However, there is also a layer between this information and actual traffic suggestions from applications which might not only incorporate this real time information but also past time information, forecast.

We present a mathematically rigorous framework of routing for multicommodity traffic flow on a network. The flow dynamics are described exemplary by a classical link-delay model [63, 21, 53] (a system of *ordinary delay differential equations* (ODDEs)); the routing at the nodes is modeled by a very general routing operator, which depends on the status of the network up to real time and on externalities. We give weak conditions on the routing operator under which the solution on the network exists and is unique (or only exists, depending on the prescribed assumptions of the operator). The coupling of the different ODDEs is due to the fact that the roads, i.e., the links, in the network are connected to each other following the specified network topology. They are also coupled due to the fact that the routing operator might consider information of the traffic on the network and in particular on the considered links to adjust the flow ratio accordingly in real time. The coupling at the junctions, which is realized by the previously mentioned routing operators, requires a mathematical analysis for existence and well-posedness of solutions.

We summarize the generality of the routing operators by proposing the modeling assumptions as follows.

Assumption 1.1 (modeling assumptions on the routing operators).

- The routing operator allows the routing at a specific intersection at time $t \in [0, T]$ to depend on the status of the network at time t or before time t, satisfying a non-anticipative modeling.
- Different groups of drivers, commodities, can be routed differently.
- The routing operator is kept as general as possible, incorporating a variety of routings as long as they do not use information on the network in future times.
- A simplified forecast (as long as it uses at time $t \in [0, T]$ only information about the status of the network at time $t \in [0, T]$ or before) can be incorporated as well as a stochastic modeling of routings by the use of the mentioned externalities.

1.4. Structure of the article. In section 2, we introduce the fundamental notational framework and present the considered problem. In subsection 2.1, the network notation as well as the considered network are introduced. Time-dependent source flows, origin-destination pairs, all possible routes between an origin and a destination, etc. are presented in a notationally consistent way. Subsection 2.2 presents the dynamical model used on the links in the network. It is a system of ODDEs, which is chosen for the aim of simplicity required to construct the mathematical framework but possessing inherently a notion of travel time as well as a delay which depends on the traffic density at the specific link. Although this model does not possesses the capability of modeling traffic jams, it has the required properties needed for a time dynamical traffic flow model. We have to consider a system of ODDEs as we have different destinations and groups of drivers which we have to keep track of separately. We conclude this section with an existence and uniqueness result from the literature, which we will also use in this work. Subsection 2.3 is one of the key pieces as it introduces the routing operators which we will use in the network. We distinguish Lipschitz-continuous, continuous, and delayed routing, always w.r.t. the traffic status on the network. The network dynamics subject to the routing operators are then formulated in subsection 2.4. Fundamental results for the then presented network dynamics are studied in detail in section 3. We analyze Lipschitz and delay routing and prove the existence and uniqueness of solutions on the network with upper bounds on the solution. For the previously mentioned continuous routing we show that there exists a solution on the network. In section 4 we show how our theory can be applied by instantiating several routing operators which have to some extent been considered in the literature. We also look deeply into a shortest path real time traffic routing and show that this will in general produce quite challenging troubles of missing stability. Finally, we conclude our work in section 5.

2. Problem statement and network formulation and routing.

In this section we give some basic notations, introduce the network formulation, and pose the considered problem class.

Definition 2.1 (generic notation). *We use the following general notation consistently throughout this article:*

1. ***Bold** letters for vectors $(\boldsymbol{x},\dots)$ and "mathcal" for sets $(\mathcal{V},\dots)$.*
2. *For a set $\mathcal{U}$ and $\mathcal{M}\subset\mathcal{U}$ the characteristic function $\chi_{\mathcal{M}}:\mathcal{U}\to\{0,1\},\ u\mapsto\{\begin{smallmatrix}1 & \text{if } u\in\mathcal{M}\\ 0 & \text{else.}\end{smallmatrix}$*
3. *For $\boldsymbol{x}\in\mathbb{R}^n$ define $|\boldsymbol{x}|_p:=(\sum_{i=1}^n|\boldsymbol{x}_i|^p)^{\frac{1}{p}}$ for $p\in[1,\infty)$ and $|\boldsymbol{x}|_\infty:=\max_{i\in\{1,\dots,n\}}|\boldsymbol{x}_i|$.*
4. *For $p\in[1,\infty]$, $n\in\mathbb{N}_{\geq 1}$ and $\boldsymbol{x}\in L^p((0,T);\mathbb{R}^n)$: $\|\boldsymbol{x}\|_{L^p((0,T);\mathbb{R}^n)}:=\||\boldsymbol{x}|_p(\cdot)\|_{L^p((0,T))}$ where $|\cdot|_p$ denotes the usual p-norm in a finite dimensional vector space. For the vector valued continuous functions $\boldsymbol{x}\in C([0,T];\mathbb{R}^n)$ we define $\|\boldsymbol{x}\|_{C([0,T];\mathbb{R}^n)}:=\sum_{j=1}^n\|\boldsymbol{x}_j\|_{C([0,T])}$.*

2.1. Network.

We consider a transportation network represented as a directed graph on which we describe link dynamics and routing behavior at nodes (i.e., node dynamics) as follows.

Definition 2.2 (network and paths). *We call a directed graph $G=(\mathcal{V},\mathcal{A})$ where $\mathcal{V}$ is a finite set with $|\mathcal{V}|\in\mathbb{N}_{\geq 1}$ with vertices $v\in\mathcal{V}$ and arcs $a\in\mathcal{A}\subset\mathcal{V}\times\mathcal{V}$ a* network. *Furthermore, we define for $v\in\mathcal{V}$ the sets of* incoming arcs *and* outgoing arcs

$$\mathcal{A}_{\text{in}}(v):=\{(\tilde{v},v)\in\mathcal{A}:\tilde{v}\in\mathcal{V}\},\qquad \mathcal{A}_{\text{out}}(v):=\{(v,\tilde{v})\in\mathcal{A}:\tilde{v}\in\mathcal{V}\}.$$

We define the set of paths $\mathcal{P}^{v,d}$ between two nodes $(v,d)\in\mathcal{V}^2$ *without cycles as*

$$\mathcal{P}^{v,d}:=\bigcup_{k=1}^{|\mathcal{V}|^2}\Big\{\boldsymbol{p}\in\mathcal{A}^k:\ \forall i,j\in\{1,\dots,k+1\},i\neq j:v_i,v_j\in\mathcal{V}:v_i\neq v_j,$$
$$v_1=v,\ v_{k+1}=d,\boldsymbol{p}=((v_i,v_{i+1}))_{i\in\{1,\dots,k\}}\Big\}$$

For every path $\boldsymbol{p} := (\boldsymbol{p}_1, \dots, \boldsymbol{p}_k)^T \in \mathcal{P}^{v,d}$ *with* $k \in \mathbb{N}_{\geq 1}$ *we define its* length *by* $\operatorname{len}(\boldsymbol{p}) := k$*. In the network we specify* source nodes *as*

$$\mathcal{O} \subset \mathcal{V}$$

and destination nodes *as*

$$\mathcal{D} \subset \Big\{ d \in \mathcal{V} : \ \exists v \in \mathcal{O} \ s.t. \ \mathcal{P}^{v,d} \neq \emptyset \Big\}.$$

We define $\mathcal{OD}$ *as the set of* origin/source-destination pairs *by*

$$\mathcal{OD} := \Big\{ (v,d) \in \mathcal{O} \times \mathcal{D} : \mathcal{P}^{v,d} \neq \emptyset \Big\}.$$

Finally, we define for $d \in \mathcal{D}$ *the set* $\mathcal{OP}^d$ *of arcs which are part of paths starting from an arbitrary node* $\tilde{v} \in \mathcal{O}$ *and ending in* d *if* $(\tilde{v}, d) \in \mathcal{OD}$*:*

$$\mathcal{OP}^d := \bigcup_{\ell=1}^{|\mathcal{A}|} \Big\{ \boldsymbol{p}_\ell \in \mathcal{A} : \exists \tilde{v} \in \mathcal{O} \ s.t. \ (\tilde{v}, d) \in \mathcal{OD}, \ \boldsymbol{p} \in \mathcal{P}^{\tilde{v},d}, \ \ell \leq \operatorname{len}(\boldsymbol{p}) \Big\} \tag{2.1}$$

and with that the outgoing arcs *for a node* $v \in \mathcal{V}$ *from which* destination $d \in \mathcal{D}$ *is* reachable *as*

$$\begin{aligned} \mathcal{A}^d_{\text{out}}(v) &:= \Big\{ a \in \mathcal{A} : \exists \tilde{v} \in \mathcal{V} \ with \ a = (v, \tilde{v}) \ \wedge a \in \mathcal{OP}^d \Big\} \\ &= \mathcal{A}_{\text{out}}(v) \cap \mathcal{OP}^d. \end{aligned}$$

Assumption 2.3 (feasible network). The network G in Definition 2.2 contains at least one $\mathcal{OD}$ pair, i.e., $|\mathcal{OD}| > 0$.

2.2. Introduction into the considered flow model. To describe flow dynamics on the full network we first define the dynamics on a specific link and then the node dynamics. Note that we use the terms link and arc interchangeably. The modeling of the dynamics on a specific link is realized by the following link-delay model (first described in the seminal work of [18] and used extensively afterwards such as in [46] and [63]). Before presenting the specific modeling we require another definition.

Definition 2.4 (multicommodity). *For a given* $n \in \mathbb{N}_{\geq 1}$ *we define the* multicommodity set *as* $\mathcal{C} := \{1, \dots, n\}$*.*

Remark 2.5 (interpretation of commodities). We consider multiple commodities to distinguish specific flows within the overall flow. This concept is illustrated more precisely in Definition 2.10 where we see that commodity type directly influences flow behavior at nodes. By using commodities we can allow subsets of the flow to behave independently at nodes.

Remark 2.6 (order in $\mathcal{A}$). Definition 2.7 requires that $\mathcal{A}$ is ordered. Thus, we take an arbitrary ordering of $\mathcal{A}$ as its fixed ordering which is due to the finiteness of the considered set easily possible.

The dynamics on the arc are represented by the link-delay model as in Definition 2.7. Recall that we have still to consider a multicommodity link-delay model where we distinguish several commodities and also destinations (even though they are not reasonable for only one given link, they become crucial in the network setting later). Thus, we have a system of delay ODE equations, presented in Definition 2.7.

Definition 2.7 (link-delay model for multicommodity with affine linear delay on a single link). *Denote by the superscript $(c,d) \in \mathcal{C} \times \mathcal{D}$ a tuple containing commodity type and destination node as given in Definition 2.4. Assume that $\boldsymbol{x} : [0,T] \to \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}$ represents the density, and let $\boldsymbol{u} : [0,T] \to \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}$ be the inflow. Then, for a* free flow travel *time $b \in \mathbb{R}_{>0}$ and* congestion factor *$h \in \mathbb{R}_{\geq 0}$ we consider the following* system of delay ODE*:*

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{u}(t) - \boldsymbol{g}[\boldsymbol{u}, x, \dot{x}](t), \qquad t \in [0,T], \tag{2.2}$$

$$\boldsymbol{x}(0) = \boldsymbol{0}, \tag{2.3}$$

$$x(t) := \sum_{c \in \mathcal{C}} \sum_{d \in \mathcal{D}} \boldsymbol{x}^{c,d}(t), \qquad t \in [0,T], \tag{2.4}$$

$$\tau[x](t) := b + hx(t), \qquad t \in [0,T], \tag{2.5}$$

$$\rho[x](t) := t + \tau[x](t), \qquad t \in [0,T], \tag{2.6}$$

$$\boldsymbol{g}[\boldsymbol{u}, x, \dot{x}](t) := \begin{cases} \boldsymbol{0}, & a.e.\ t \in [0,b), \\ \frac{\boldsymbol{u}(\rho[x]^{-1}(t))}{1 + h\dot{x}(\rho[x]^{-1}(t))}, & a.e.\ t \in [b, T + \tau[x](T)], \end{cases} \tag{2.7}$$

where $\rho[x]^{-1}$ is the inverse function of $[0,T] \ni t \mapsto \rho[x](t)$.

In Definition 2.7 we use the vectors $\boldsymbol{x}$ to describe attributes which represent commodities and different destinations. We see that (2.2) describes the change in link volume as the difference between the inflow and outflow at any given time where the outflow is a function of inflow, link volume, and the time derivative of link volume. To ensure well-posedness, (2.3) prescribes that the link is initialized with zero volume (i.e., the link is empty prior to the time horizon of interest $[0,T]$). As described, Definition 2.7 distinguishes density on links by commodity $c \in \mathcal{C}$ so that routing decisions can be made separately for each commodity. However, the evolution of density on links is still influenced by the total traffic volume (i.e., the combined total of all commodity types). We see this in (2.4) in which the total volume on a link is determined by summing the density over all commodities $c \in \mathcal{C}$ and all destinations $d \in \mathcal{D}$. The total link volume from (2.4) is then used in (2.5) to determine the travel time on each link. Equation (2.6) states that inflow entering the link at time $t \in [0,T]$ will leave the link at time $\rho[x](t)$. Finally, (2.7) shows how link outflow is calculated. We see that in the first case, when $t < b$ the outflow is 0 because the incoming flow on the link requires at least b (i.e., the free flow travel time) to traverse the link. When $b < t < T$ we see that the outflow is a scaled version of inflow at a previous time $\rho[x]^{-1}(t)$ (i.e., the time at which the exiting flow entered the link). The scaling factor $\frac{1}{1+h\dot{x}(\rho[x]^{-1}(t))}$ serves to spread or concentrate flow.

In order to maintain conservation of mass on every link, we require the total inflow in $[0,T]$ to correspond to the total outflow in $[b, T + \tau[x](T)]$. By taking (2.7), we see that the therein defined outflow $\boldsymbol{g}$ fulfills this property since

$$\int_{b}^{T^*+\tau[x](T^*)} \boldsymbol{g}[\boldsymbol{u},x,\dot{x}](t)\,\mathrm{d}t = \int_{0}^{T^*} \boldsymbol{g}[\boldsymbol{u},x,\dot{x}](t+\tau[x](t))(1+h\dot{x}(t))\,\mathrm{d}t = \int_{0}^{T^*} \boldsymbol{u}(t)\,\mathrm{d}t \quad \forall T^* \in [0,T], \tag{2.8}$$

where the integrals are taken componentwise and the first equation is due to the change of variables rule. Together, (2.2)–(2.7) describe the evolution of flow on arcs in a network according to the link-delay model.

Remark 2.8 (flows and unreachable destinations). In Definition 2.7 we do not distinguish whether a destination node $d \in \mathcal{D}$ is "reachable" by an arc $a \in \mathcal{A}$ or not. If the latter is true, i.e., there are no $v \in \mathcal{V}$ such that $\mathcal{P}^{v,d} \neq \emptyset$, we set for every $c \in \mathcal{C}$ $\boldsymbol{u}_a^{c,d} \equiv 0$ such that $\boldsymbol{x}_a^{c,d} \equiv 0$.

Due to the fact that $\boldsymbol{g}$ is a function of $\boldsymbol{u}, x, \dot{x}$ and the inverse of ρ is involved, it is unclear if the previously stated dynamical system admits a solution, and in case it does, it is unclear what regularity $\boldsymbol{g}$ would possess. These questions are answered in Theorem 2.9. The proof has been provided in [63].

Theorem 2.9 (existence and uniqueness of the link-delay model with affine linear delay as presented in Definition 2.7). *Let $T \in \mathbb{R}_{>0}$ and the link-delay model with affine linear delay as in Definition 2.7 be given, and assume that $\boldsymbol{u} \in L^\infty((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$ is given as well as $b \in \mathbb{R}_{>0}$ and $h \in \mathbb{R}_{\geq 0}$. Then, the delay ODE system in (2.2)–(2.7) is well-posed and admits a unique Carathéodory solution $\boldsymbol{x} \in W^{1,\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$. Moreover, $\rho[x], \rho[x]^{-1}$ are well-defined with regularity $\rho[x], \rho[x]^{-1} \in W^{1,\infty}((0,T);\mathbb{R})$. In addition, $\frac{\mathrm{d}}{\mathrm{d}t}\rho[x]^{-1}(t),\ t \in [0,T]$, is bounded away from zero, and the outflow $\boldsymbol{g}$ as given in (2.7) is nonnegative and satisfies the estimate*

$$\|\boldsymbol{g}\|_{L^\infty((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})} \leq \|\boldsymbol{u}\|_{L^\infty((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})}.$$

The derivative $\rho[x]'$ satisfies for $h\|\boldsymbol{u}\|_{L^\infty((0,T);\mathbb{R}^n)} \neq 0$ the lower bound for every $t \in [0,T]$

$$\rho[x]'(t) \geq h|\boldsymbol{u}(t)|_1 + \left(\sum_{n=0}^{\left\lceil \frac{T}{b} \right\rceil} \left(h\||\boldsymbol{u}|_1\|_{L^\infty((0,T))}\right)^n\right)^{-1}.$$

Proof. The proof can be found in [63, Theorem 3.1, Theorem 3.2]. The basic ingredient in the proof consists of the idea that due to the free flow travel time b the system can be solved stepwise in time, first on time horizon $(0,b)$ by a simple integration of the inflow since in this time period the outflow is considered to be zero. Then, the outflow for the next time interval $(b,2b)$ is due to the delay character given and can be used to integrate the solution again. This can be iterated until one reaches the final time horizon T. ∎

[63] also presents some results for nonlinear travel times, i.e., travel times $\tau[x](t) := f(x(t)),\ t \in [0,T]$ with a nonaffine linear function $f: \mathbb{R} \to \mathbb{R}_{\geq 0}$; however, in that generality there can only be shown the existence of a solution for sufficiently small inflow. Since the inflow in the middle of the network is a function of the outflow of other links, we cannot necessarily guarantee these bounds and would need to prescribe box constraints on inflow and outflow as well, complicating things significantly, while the goal of this publication is to

consider routing made dependent on the status of the network. Thus, we keep the dynamics on the links as simple as possible for the sake of obtaining nicer mathematical results which already give some insights into what is reasonable in the continuous modeling of the routing choices.

2.3. Routing operators. In this section we present the routing which describes how flow moves from one link to another within the network. We already anticipate Example 2.17 where notation concerning the network and the states is introduced. In the case under consideration the routing will actually depend on the state of the network (i.e., the volume and ensuing travel time on each link). More precisely, at time $t \in [0,T]$ flow at node $v \in \mathcal{V}$ is directed based on the available information of the travel time in the network at time $t \in [0,T]$, at a previous time, for a combination of both or at the full time horizon until t using, for instance, an averaging over time (nonlocal measuring) (or even at future time). We will also assume that the operator can be dependent on **ext**, externalities caused, for instance, by weather, time of the day, or other impacts.

Definition 2.10 (general routing operator). *Let $T \in \mathbb{R}_{>0}$, and let for $v \in \mathcal{V}$ be $d \in \mathcal{D}, c \in \mathcal{C}$, and $a \in \mathcal{A}_{\text{out}}^d(v)$ be given as in Definition 2.2. Let $\mathbf{ext} \in L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}})$. Then, we call $\mathscr{R}_a^{c,d}$* routing operator *iff*

$$\mathscr{R}_a^{c,d} : L^\infty\left((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) \times L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}}) \to L^\infty((0,T);[0,1]) \tag{2.9}$$

such that for every $(\boldsymbol{x},\mathbf{ext}) \in L^\infty((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}) \times L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}})$

$$\sum_{\tilde{a} \in \mathcal{A}_{\text{out}}^d(v)} \mathscr{R}_{\tilde{a}}^{c,d}[\boldsymbol{x},\mathbf{ext}](t) \equiv 1, \qquad t \in [0,T] \text{ a.e.} \tag{2.10}$$

The presented routing operator in this very general form allows a forecasting as well as very general and rather arbitrary information of the network situation used for routing. Although this generality is nice, the well-posedness of the network system in not necessarily guaranteed anymore when using future information (since this means that routing at time $t \in [0,T]$ will depend on status of the network at $\tilde{t} \in (t,T)$ while the network solution at $\tilde{t}$ will also depend on routing choices at $t \in [0,T]$). A solution approach for this situation might consist of reformulating this problem into a fixed-point problem over the entire network and time horizon and applying Schauder's fixed-point theorem. Equation (2.10) guarantees conservation of flow, while (2.9) gives a regularity assumption in particular for the image being measurable and essentially bounded between 0 and 1.

In Definition 2.12 we will instantiate this routing operator in case of routing which uses for routing at time $t \in [0,T]$ the network status from times $\tilde{t} \in [0,t]$, i.e., we consider only non-anticipative routing operators. For the dynamical evolution of the system and the simulation of real traffic networks in real time, this assumption is reasonable since there is no information available for future times. However, the routing operator still allows the implementation of a simplified forecasting (or similar approaches) as long as the information of the network used for the forecast is not in the future.

For the following analysis it makes sense to distinguish routing operators which behave continuously based on the network flow from routing operators which do not or might only use for the routing at time $t \in [0,T]$ information of the network density before time $\delta_a^{c,d}(t) \in [0,t]$ for $\delta_a^{c,d}(\cdot)$ given, a kind of delayed routing. Of course, all these operators have still to satisfy the previous routing properties in Definition 2.10. However, first we need to introduce a projection mapping which will allow us to track and handle information of the density for previous times, thus also delayed information.

Definition 2.11 (projection mapping). *Define for $\alpha \in \mathbb{R}$ the projection mapping $\mathrm{P}_{[0,\alpha]} : \mathbb{R} \to \mathbb{R}$ on $[0,\alpha]$ by* $\forall t \in \mathbb{R}: \qquad \mathrm{P}_{[0,\alpha]}(t) := \begin{cases} \min\{\max\{0,t\},\alpha\} & \text{if } \alpha \in \mathbb{R}_{\geq 0} \\ 0 & \text{else.} \end{cases}$

As pointed out before, the routing operator in Definition 2.10 is too general to obtain uniqueness or existence of solutions on the entire network. Thus, we restrict the routing operator more. This is detailed in Definition 2.12.

Definition 2.12 (routing operators—nonanticipative). *Let the assumptions of Definition 2.10 be given. Then, we define the following under the assumption that all routing operators introduced in the following satisfy (2.9) and (2.10):*

(A) L-Continuous routing: *We call $\mathscr{RL}_a^{c,d}$* Lipschitz-continuous routing operator *iff it is Lipschitz-continuous in the following sense: For a $p \in (1,\infty]$ it holds that*

$$\forall t \in [0,T]\ \forall \mathbf{ext} \in L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}})\ \exists L \in \mathbb{R}_{\geq 0}\ \forall \boldsymbol{x}, \tilde{\boldsymbol{x}} \in C\left([0,T];\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right):$$
$$\left\|\mathscr{RL}_a^{c,d}[\boldsymbol{x},\mathbf{ext}] - \mathscr{RL}_a^{c,d}[\tilde{\boldsymbol{x}},\mathbf{ext}]\right\|_{L^p((0,t))} \leq L\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)}.$$

(B) Delay-type routing: *We call $\mathscr{RD}_a^{c,d}$* a delay-type routing operator *iff there exist for $N_a^{c,d} \in \mathbb{N}_{\geq 1}$ a time vector $\boldsymbol{t}_a^{c,d} := ((\boldsymbol{t}_a^{c,d})_1, \dots, (\boldsymbol{t}_a^{c,d})_{N_a^{c,d}})^T \in (0,T]^{N_a^{c,d}}$ with $(\boldsymbol{t}_a^{c,d})_{N_a^{c,d}} := T$ and*

$$\left(\boldsymbol{t}_a^{c,d}\right)_i < \left(\boldsymbol{t}_a^{c,d}\right)_{i+1} \qquad \forall i \in \left\{1,\dots,N_a^{c,d}-1\right\},$$

a delay function $\delta_a^{c,d} \in L^\infty((0,T);[0,T])$ such that for every $i \in \{1,\dots,N_a^{c,d}-1\}$ the following regularity and estimate on the delay $\delta_a^{c,d}$ hold:

$$\delta_a^{c,d}\big|_{\left[0,\left(\boldsymbol{t}_a^{c,d}\right)_1\right)} :\equiv 0,$$
$$\delta_a^{c,d}\big|_{\left[\left(\boldsymbol{t}_a^{c,d}\right)_i,\left(\boldsymbol{t}_a^{c,d}\right)_{i+1}\right)} \in C\left(\left[\left(\boldsymbol{t}_a^{c,d}\right)_i,\left(\boldsymbol{t}_a^{c,d}\right)_{i+1}\right)\right), \tag{2.11}$$
$$\left\|\delta_a^{c,d}\right\|_{C\left(\left[\left(\boldsymbol{t}_a^{c,d}\right)_i,\left(\boldsymbol{t}_a^{c,d}\right)_{i+1}\right)\right)} \leq \left(\boldsymbol{t}_a^{c,d}\right)_i$$

and, finally, for a.e. $t \in [0,T]$, $\forall \boldsymbol{x} \in C([0,T];\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$

$$\mathscr{RD}_a^{c,d}[\boldsymbol{x},\mathbf{ext}](t) = \mathscr{RD}_a^{c,d}\left[\boldsymbol{x}\circ \mathrm{P}_{\left[0,\delta_a^{c,d}(t)\right]},\mathbf{ext}\right](t). \tag{2.12}$$

(C) Continuous routing: *We call $\mathscr{RC}_a^{c,d}$* continuous routing operator *if it is continuous in the following sense: For given $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in C([0,T];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ it holds that*

$$\begin{aligned}(2.13)\qquad &\forall t \in [0,T]\ \forall \varepsilon \in \mathbb{R}_{\geq 0}\ \exists \delta \in \mathbb{R}_{>0}: \quad \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C([0,t];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})} \leq \delta \\ &\implies \left\| \mathscr{RC}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}] - \mathscr{RC}_a^{c,d}[\tilde{\boldsymbol{x}}, \mathbf{ext}] \right\|_{L^1((0,t))} \leq \varepsilon.\end{aligned}$$

Remark 2.13 (functional (L)-continuity vs. pointwise L-continuity). The previous Definition 2.12 and also the following proofs of well-posedness on the network subject to the routing operators in Theorems 3.1, 3.4, and 3.8 can be interpreted as operator versions of the famous Peano existence theorem for item (C) (compare, for instance, [64, Theorem 3.B]) and of the Picard–Lindelöf existence (and uniqueness) theorem for item (A) (compare [64, Theorem 3.A]). Delay routing in item (B) finds its correspondence in delay ODEs (note that the delay here is not meant for the dynamics on the links which is a delay ODE, but for the routing which uses only delayed information) (see [17, 57]).

However, all results are significantly different from these three theorems due to the operator dependency, the not pointwise dependency as usual for ODEs, and the network situation. For instance, item (A) is defined in a way that the routing operator at time $t \in [0,T]$ might take into account information on the network on the full time horizon $[0,t]$ and not only at time t.

Remark 2.14 (applicability of the routing operators). At first, we present two possible instantiations of delay functions which show the range of applicability of the used routing operators. We omit for a better readability the indices a, c, d. Now, on the one hand, we can simulate, up to a small time, a constant delay $\varepsilon \in \mathbb{R}_{>0}$. There, one defines

$$N := \left\lceil \tfrac{T}{\varepsilon} \right\rceil, \qquad \boldsymbol{t}_i := \tfrac{i}{N} T \quad \forall i \in \{1, \dots, N-1\}, \qquad \delta(t) := \max\{t - \varepsilon, 0\}.$$

On the other hand, if $\boldsymbol{t} := (\boldsymbol{t}_1, \dots, \boldsymbol{t}_N)^T \in [0,T]^N$ is a given vector with $\boldsymbol{t}_N := T$ and $\boldsymbol{t}_i < \boldsymbol{t}_{i+1}$, $i \in \{0, \dots, N-1\}$, by setting $\delta|_{[\boldsymbol{t}_i, \boldsymbol{t}_{i+1})} :\equiv \boldsymbol{t}_i$, we obtain, with $\boldsymbol{t}_0 := 0$, a case where the delay vanishes when approaching $\boldsymbol{t}_i$ from the right. For a visualization of this see Figures 1 and 2. The mentioned figure also shows the basic properties of the delay routing as in Definition 2.12 item (B). The most characteristic one is that within any interval $[\boldsymbol{t}_i, \boldsymbol{t}_{i+1})$, $i \in \{0, \dots, N-1\}$, the delay function does not attend values greater than $\boldsymbol{t}_i$. This allows for a.e. $t \in [0,T]$ to

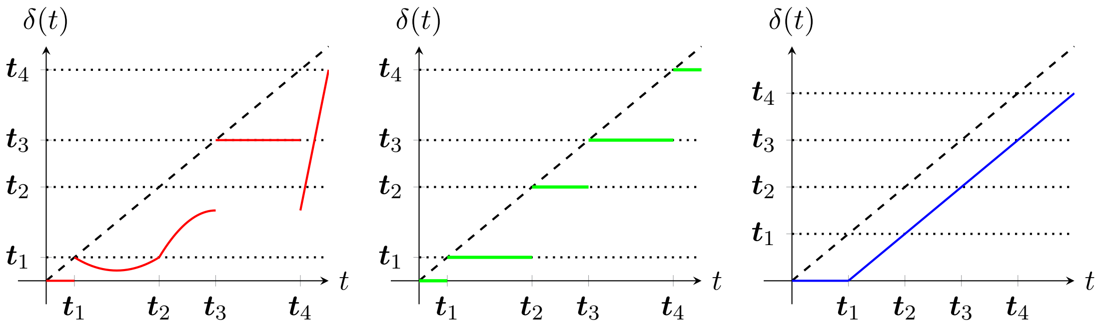

Figure 1. *Illustration of a delay function. The thick diagonal line represents the identity; the red graph shows an example of a general feasible delay function δ. Furthermore, the green graph shows the case when a piecewise vanishing delay and the blue graph when a constant delay $\varepsilon \in \mathbb{R}_{>0}$ occurs.*

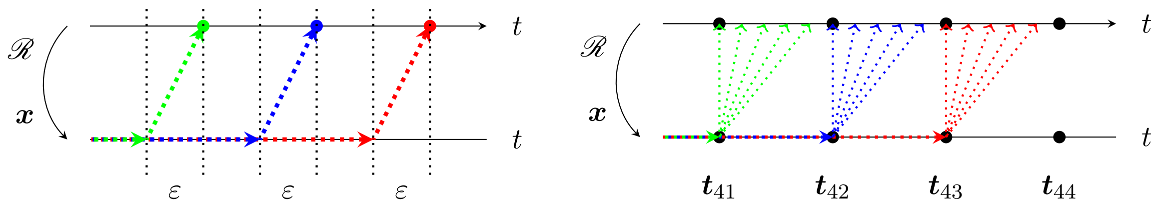

Figure 2. *Illustration of the density information used by the routing operator. Every horizontal line represents the time line. The delay-type routing operator $\mathscr{R}$ maps time points—which are represented on the upper horizontal lines—to time periods that do not occur later in time. This produced time periods—which are part of the lower horizontal lines—build the possible evaluation time periods for the density $\boldsymbol{x}$. Depicted are the constant ε-delay (left) and the piecewise vanishing delay (right).*

obtain a sufficiently large delay such that the routing only relies on density information up to time $\boldsymbol{t}_i$. Another, quite reasonable routing suggestion could consist of a piecewise constant routing where routing choice is made—dependent on the status of the network—at every $\boldsymbol{t}_i$ but not changed in between.

Remark 2.15 (an invariance property of delay-type routing operators). Consider a routing operator $\mathscr{R}_a^{c,d}$ as in Definition 2.10 with the following property. For an arbitrary, but fixed $t \in [0,T]$ assume there exists $\alpha \in [0,T]$ such that for every $(\boldsymbol{x}, \mathbf{ext}) \in C([0,T]; \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}) \times L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}})$

$$\mathscr{R}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}](t) = \mathscr{R}_a^{c,d}[\boldsymbol{x} \circ \mathrm{P}_{[0,\alpha]}, \mathbf{ext}](t)$$

holds. Then for any $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in C([0,T]; \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ with $\boldsymbol{x}|_{[0,\alpha]} = \tilde{\boldsymbol{x}}|_{[0,\alpha]}$, we obtain

$$\mathscr{R}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}](t) = \mathscr{R}_a^{c,d}[\boldsymbol{x} \circ \mathrm{P}_{[0,\alpha]}, \mathbf{ext}](t) = \mathscr{R}_a^{c,d}[\tilde{\boldsymbol{x}} \circ \mathrm{P}_{[0,\alpha]}, \mathbf{ext}](t) = \mathscr{R}_a^{c,d}[\tilde{\boldsymbol{x}}, \mathbf{ext}](t). \tag{2.14}$$

Thus, information of the input flow after time α does not influence the routing such that the routing is invariant w.r.t. continuous extensions of a continuous function defined on $[0,\alpha]$ to a superset.

This also states that there is some degree of freedom of delay-type routing operators as in Definition 2.12, item (B) w.r.t. the argument $\boldsymbol{x}$ if the delay satisfies $\delta_a^{c,d}(t) < t$ for $t \in [0,T]$. This property will be crucial in the proof of Theorem 3.1 where we show the existence and uniqueness of the network dynamics when routing with this class of operators. It allows us to construct the unique solution successively over time.

Remark 2.16 (topological vs. flow-based sources and destinations). The definition of source and destination nodes in Definition 2.2 does not state that "topological" sources, i.e., nodes $v \in \mathcal{V}$ with $\mathcal{A}_{\text{in}}(v) = \emptyset$, have to be in $\mathcal{O}$ even if there exist paths from v to a given $d \in \mathcal{D}$. This means that these v do not send out any flow. Similarly, "topological" destination nodes, i.e., $w \in \mathcal{V}$ with $\mathcal{A}_{\text{out}}(w) = \emptyset$, do not have to be part of $\mathcal{D}$ such that they do not receive any inflow.

However, even if they could be excluded from the network, for reasons of simplicity we leave these "dead" nodes in the network.

The next (network) Example 2.17 will serve as an illustration of introduced notation and network topology as well as all the later for the needed routing operators.

Example 2.17 (topology of networks, notation, etc.). On the benchmark network as shown in Figure 3, we choose two different commodities, i.e., $\mathcal{C} = \{1, 2\}$. Then, we present the previous introduced notations and definitions and write accordingly

$$\mathcal{V} = \{v_1, \dots, v_8\}, \qquad \mathcal{A} = \{a_1, \dots, a_{11}\}, \qquad \mathcal{O} = \{v_2, v_3, v_5\}, \qquad \mathcal{D} = \{v_5, v_6, v_7\}$$

and

$$\begin{array}{lll}
\mathcal{A}_{\text{in}}(v_1) = \emptyset, & \mathcal{A}_{\text{in}}(v_2) = \emptyset, & \mathcal{A}_{\text{out}}(v_1) = \{a_1\}, \\
\mathcal{A}_{\text{out}}(v_2) = \{a_2, a_3, a_4\}, & \mathcal{A}_{\text{in}}(v_3) = \{a_1, a_2, a_5\}, & \mathcal{A}_{\text{in}}(v_4) = \{a_6\}, \\
\mathcal{A}_{\text{out}}(v_3) = \{a_9\}, & \mathcal{A}_{\text{out}}(v_4) = \{a_5\}, & \mathcal{A}_{\text{in}}(v_5) = \{a_3, a_8\}, \\
\mathcal{A}_{\text{in}}(v_6) = \{a_4, a_7\}, & \mathcal{A}_{\text{out}}(v_5) = \{a_6, a_7, a_{10}\}, & \mathcal{A}_{\text{out}}(v_6) = \{a_8, a_{11}\}, \\
\mathcal{A}_{\text{in}}(v_7) = \{a_9, a_{10}\}, & \mathcal{A}_{\text{in}}(v_8) = \{a_{11}\}, & \mathcal{A}_{\text{out}}(v_7) = \emptyset, \\
\mathcal{A}_{\text{out}}(v_8) = \emptyset, & \mathcal{A}^{v_7}_{\text{out}}(v_5) = \{a_6, a_{10}\}, & \mathcal{A}^{v_8}_{\text{out}}(v_2) = \{a_3, a_4\}, \\
\mathcal{A}^{v_7}_{\text{out}}(v_2) = \{a_2, a_3\}, & \mathcal{A}^{v_6}_{\text{out}}(v_5) = \{a_7\}, & \dots\dots
\end{array}$$

and the vector of the density on every arc satisfies $\boldsymbol{x} \in C\left([0,T]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)$ with $|\mathcal{A}||\mathcal{C}||\mathcal{D}| = 11 \cdot 2 \cdot 3$. For reasons of simplicity we only show its components $\boldsymbol{x}_{a_3}$ and $\boldsymbol{x}_{a_6}$ and obtain for $t \in [0,T]$

$$\begin{aligned}
\boldsymbol{x}_{a_3}(t) &= \left(\boldsymbol{x}^{1,v_5}_{a_3}(t), \boldsymbol{x}^{1,v_6}_{a_3}(t), \boldsymbol{x}^{1,v_7}_{a_3}(t), \boldsymbol{x}^{2,v_5}_{a_3}(t), \boldsymbol{x}^{2,v_6}_{a_3}(t), \boldsymbol{x}^{2,v_7}_{a_3}(t)\right)^T, \\
\boldsymbol{x}_{a_6}(t) &= \left(0, 0, \boldsymbol{x}^{1,v_7}_{a_6}(t), 0, 0, \boldsymbol{x}^{2,v_7}_{a_6}(t)\right)^T.
\end{aligned}$$

The zero components in $\boldsymbol{x}_{a_6}$ reflect that there is no way to reach the destinations v_5, v_6 when on link a_6.

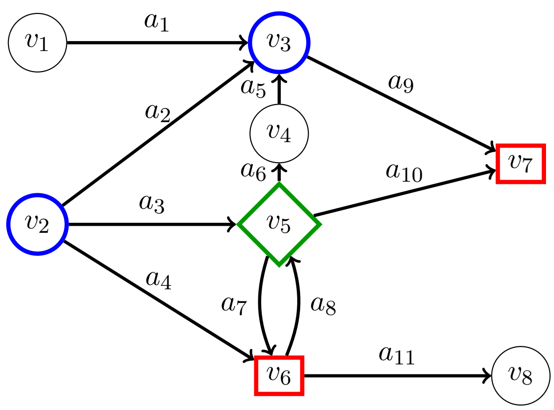

Figure 3. *A network illustration of the imposed basic attributes: blue circles illustrate nodes which are source nodes but not destination nodes (v_2, v_3), red rectangles show the nodes which are destination nodes but not source nodes (v_6, v_7), green diamonds illustrate nodes which are both sources and destinations (v_5), and simple circles describe nodes which are neither sources nor destinations (v_1, v_4, v_8).*

2.4. The dynamical model on the network subject to specific routing. In this section, we present for the considered routing operators in Definition 2.12 the dynamics subject to the ODDE model introduced in Definition 2.7 and a network as in Definition 2.2.

Definition 2.18 (network formulation of the link-delay model with routing operators: DTA). *Consider for a time horizon $T \in \mathbb{R}_{>0}$ a network $G = (\mathcal{V}, \mathcal{A})$ as in Definition 2.2 with origins $\mathcal{O} \subseteq \mathcal{V}$ and destinations $\mathcal{D} \subseteq \mathcal{V}$ as in Definition 2.2. Let the inflow into the network for $\tilde{v} \in \mathcal{O}$ be given as $\boldsymbol{s}_{\tilde{v}} \in L^{\infty}((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$. Then, we pose the dynamics subject to the link-delay model for multidestination with routing as proposed in Definition 2.10 for all $t \in [0,T]$ and $a \in \mathcal{A}$ with $b_a \in \mathbb{R}_{>0}$ and $h_a \in \mathbb{R}_{\geq 0}$ as*

$$\dot{\boldsymbol{x}}_a(t) = \boldsymbol{u}_a(t) - \boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](t), \qquad t \in [0,T], \tag{2.15}$$

$$\boldsymbol{x}_a(0) = \boldsymbol{0}, \tag{2.16}$$

$$x_a(t) := \sum_{c \in \mathcal{C}} \sum_{d \in \mathcal{D}} \boldsymbol{x}_a^{(c,d)}(t), \qquad t \in [0,T], \tag{2.17}$$

$$\tau_a[x_a](t) := b_a + h_a x_a(t), \qquad t \in [0,T], \tag{2.18}$$

$$\rho_a[x_a](t) := t + \tau_a[x_a](t), \qquad t \in [0,T], \tag{2.19}$$

$$\boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](t) := \begin{cases} 0, & a.e.\ t \in [0, b_a), \\ \frac{\boldsymbol{u}_a(\rho_a[x_a]^{-1}(t))}{1 + h_a \dot{x}_a(\rho_a[x_a]^{-1}(t))}, & a.e.\ t \in [b_a, T + \tau_a[x_a](T)] \end{cases} \tag{2.20}$$

for a.e. $t \in [0,T]$. In addition, we define the summarized load at a junction—dependent on if there is an external source or not—for $(c,d,v) \in \mathcal{C} \times \mathcal{D} \times \mathcal{V}$

$$\boldsymbol{r}_v^{c,d} :\equiv \sum_{\tilde{a} \in \mathcal{A}_{\text{in}}(v)} \boldsymbol{g}_{\tilde{a}}^{c,d}[\boldsymbol{u}_{\tilde{a}}, x_{\tilde{a}}, \dot{x}_{\tilde{a}}] + \begin{cases} \boldsymbol{s}_v^{c,d} & (v,d) \in \mathcal{OD}, c \in \mathcal{C}, \\ 0 & v \in \mathcal{V} \setminus \mathcal{O}, c \in \mathcal{C} \end{cases} \quad on\ [0,T]. \tag{2.21}$$

Finally, recalling the routing operator $\mathscr{R}_a^{c,d}$ given in Definition 2.10 with $d \in \mathcal{D}$, the coupling condition for the connecting nodes is given for $(v,c,d) \in \mathcal{V} \times \mathcal{C} \times \mathcal{D}, a \in \mathcal{A}_{\text{out}}(v)$

$$\boldsymbol{u}_a^{c,d} :\equiv \begin{cases} \boldsymbol{r}_v^{c,d} \cdot \mathscr{R}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}] & if\ a \in \mathcal{A}_{\text{out}}^d(v), \\ 0, & else \end{cases} \qquad on\ [0,T] \tag{2.22}$$

with $\mathcal{A}_{\text{out}}^d(\cdot)$ as in Definition 2.2. The presented set of equations will be called the DTA considered.

The network formulation in Definition 2.18 combines link dynamics from Definition 2.7 and routing on a network from Definition 2.10 in order to describe how vehicular flow is moved from origin node $v \in \mathcal{O}$ to destination node $d \in \mathcal{D}$ in a full network. The coupling via the routing operators is described in Figure 4. In the above Definition 2.18, (2.15)–(2.20) have been previously described following the initial introduction of the link-delay model in Definition 2.7. Here, the equations are introduced for each link in the network as shown by the subscript a. Equation (2.21) denotes the sum of the inflow into a node from entering links and the—possible—origin flow and is defined for keeping notation simple. (2.22) describes

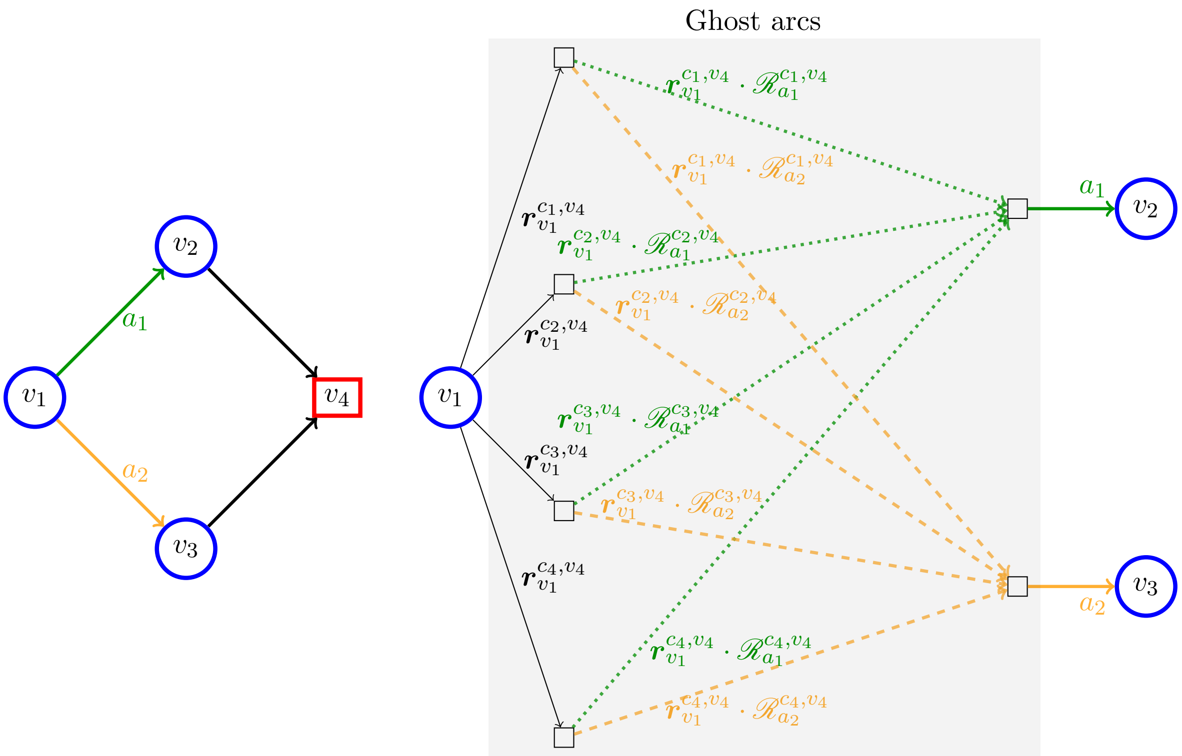

Figure 4. *An illustration of the application of routing operators. (Left) A simple network with an $\mathcal{OD}$ pair (v_1, v_4). (Right) An extraction of the left network extended by "ghost arcs" which serve to illustrate how the inflow is split into its compounds $\boldsymbol{r}_{v_1}^{c_1,v_4}, \boldsymbol{r}_{v_1}^{c_2,v_4}$ and then the split ratio realized by the routing operators applied. The gray area is only for illustration and not needed for the theory.*

how the routing operator directs flow from one link to another. The idea behind the definition of $\boldsymbol{u}_a^{c,d}$ in (2.22) is as follows. First, if (v,d) is an $\mathcal{OD}$ pair and $a \in \mathcal{A}_{\text{out}}^d(v)$, we prescribe the (independent) inflow $\boldsymbol{s}_v^{c,d}$ and add the network inflow $\boldsymbol{g}_a^{c,d}$ scaled by the routing operator. Second, we consider the case where v is not a source node (i.e., $v \notin \mathcal{O}$). However, since it can lie on a path between an $\mathcal{OD}$ pair, where d is the destination component, we prescribe flow given by the network and again scaled by the routing operator. Finally, we consider the remaining case, i.e., when there is no path starting at a source node $\tilde{v} \in \mathcal{O}$ with $(\tilde{v}, d) \in \mathcal{OD}$ that contains v and ends at $d \in \mathcal{D}$, i.e., $a \notin \mathcal{OP}^d$. Thus, we prescribe zero inflow. We see that the routing operator $\mathscr{R}_a^{c,d}$ is calculated separately based on the node $v \in \mathcal{V}$ at which flow is located at time t as well as the commodity type $c \in \mathcal{C}$ and the destination node $d \in \mathcal{D}$ of the flow.

Given this information the routing operator $\mathscr{R}_a^{c,d}$ determines how much flow to route onto available arcs a at time $t \in [0,T]$ which then becomes the inflow $\boldsymbol{u}_a^{c,d}(t)$ on these arcs.

Remark 2.19 (Dependency of the routing operator). For reasons of generality, we have assumed that the routing operator $\mathscr{R}$ might depend at time $t \in [0,T]$ on the status of the network $\boldsymbol{x}$ up to time $t \in [0,T]$. However, usually, it would depend on travel time (which is a function of $\boldsymbol{x}$) and only take into account links between the origin and the destination. This is particularly in the considered case no problem as travel time is for the ODDE model as in Definition 2.7 an affine linear function in $\boldsymbol{x}_a$ for $a \in \mathcal{A}$. The generality will also be shown in section 4.

3. Fundamental results.

In this section, we tackle the task of showing the solvability of the previously presented network model (with instantaneous routing) as in Definition 2.18. Thereby, we provide for each of the three routing operator classes in Definition 2.12 its own proof. Even if they share the same fundamental idea of a sequential determination of a solution as sketched in Theorem 2.9, the large discrepancy in the regularity of the routing operators demands the mentioned split of the proofs. We will start analyzing the case of delay-type operators in item (B) since, as it will be shown in the following Theorem 3.1, the delay only provides some further technical extension of the previously mentioned theorem.

Theorem 3.1 (existence/uniqueness of the network model for routings with delay property). *Recall the setting described in Definition 2.18 where every routing operator $\mathscr{R}_a^{c,d}$ fulfills the conditions according to item (B) in Definition 2.12 with time vectors $\boldsymbol{t}_a^{c,d} \in [0,T]^{N_a^{c,d}}$. Then there is a unique solution of the system presented in Definition 2.18, and the solution satisfies*

$$\boldsymbol{x}_a \in W^{1,\infty}\left((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}\right) \qquad \forall a \in \mathcal{A}.$$

Proof. This proof consists of merging the strategy used in [63, pages 344–347] together with the delay properties. At first we define

$$b_{\min} := \min\{b_a : a \in \mathcal{A}\}, \quad t_{\min} := \min_{\substack{d\in\mathcal{D},c\in\mathcal{C},\\ a\in\mathcal{A}_{\text{out}}^d(v)}} \min\left\{\left(\boldsymbol{t}_a^{c,d}\right)_1, \min_{i\in\left\{1,\dots,N_a^{c,d}-1\right\}} \left(\boldsymbol{t}_a^{c,d}\right)_{i+1} - \left(\boldsymbol{t}_a^{c,d}\right)_i\right\} > 0$$

with, without loss of generality, $\min\{b_{\min}, t_{\min}\} < T$; otherwise, we can exactly use the solution theory proposed in [63, pages 344–347] since also, due to the initial condition $\boldsymbol{x}(0) = \boldsymbol{0} \in \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}$, every routing operator would only depend on the initial state so that for the considered small time horizon one would have

$$\mathscr{R}\mathscr{D}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}] = \mathscr{R}\mathscr{D}_a^{c,d}[\mathbf{0}_{\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}}, \mathbf{ext}].$$

Now we are going to construct a sequence $(\boldsymbol{x}^{[k]})_{k\in\{0,\dots,k_{t_{\min}}\}} \subset C([0,T];\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$, $k_{t_{\min}} \in \mathbb{N}_{\geq 1}$, iteratively and show that these will yield the unique solution on the network. To this end, we construct Lipschitz-continuous functions on subintervals of $[0,T]$ and extend them Lipschitz-continuously on $[0,T]$. We only need this extension to obtain feasible arguments for the routing operator, and due to Remark 2.15 the Lipschitz-continuous extension can be set arbitrarily. For reasons of simplicity, we choose a constant extension defined as follows: The constant extension $\hat{\boldsymbol{y}} \in W^{1,\infty}((0,T);\mathbb{R}^m)$, $m \in \mathbb{N}_{\geq 1}$, of a Lipschitz-continuous function $\boldsymbol{y} \in W^{1,\infty}((0,s);\mathbb{R}^m)$, $s \in (0,T]$, is

$$\hat{\boldsymbol{y}}(\theta) := \begin{cases} \boldsymbol{y}(\theta), & \theta \in [0,s], \\ \boldsymbol{y}(s), & \theta \in (s,T]. \end{cases}$$

In the upper setting, the evaluations of $\boldsymbol{y}, \hat{\boldsymbol{y}}$ are well-defined since $W^{1,\infty}((0,s);\mathbb{R}^m)$ is continuously embedded in $C([0,s];\mathbb{R}^m)$. At first, initialize

$$\boldsymbol{x}^{[0]} := \mathbf{0} \in \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}, \qquad x_a^{[0]} := 0 \in \mathbb{R},\ \forall a \in \mathcal{A}, \qquad \ell_0 := 0, \qquad t_0 := 0.$$

With [63, pages 344–347] proceed with the following iterations for $k \in \{1, \dots, \lceil \frac{T}{b_{\min}} \rceil\}$. Define

$$\ell_k := \ell_{k-1} + \min_{a\in\mathcal{A}} \left\lceil \frac{\rho_a\left[x_a^{[\ell_{k-1}]}\right](t_{\ell_{k-1}}) - t_{\ell_{k-1}}}{t_{\min}} \right\rceil,$$

$$t_i := \min\left\{T, t_{\ell_{k-1}} + \min_{a\in\mathcal{A}} \frac{i-\ell_{k-1}}{\ell_k - \ell_{k-1}} \left(\rho_a\left[x_a^{[\ell_{k-1}]}\right](t_{\ell_{k-1}}) - t_{\ell_{k-1}}\right)\right\}, \quad i \in \{\ell_{k-1}+1, \dots, \ell_k\},$$

and in the following $x_a^{[\ell_{k-1}]} \in W^{1,\infty}((0,T))$ will also be constructed iteratively. Note that $\ell_k > \ell_{k-1}$ for every $k \in \{1, \dots, \lceil \frac{T}{b_{\min}} \rceil\}$ since $\rho_a[x_a^{[\ell_{k-1}]}](t_{\ell_{k-1}}) \geq t_{\ell_{k-1}} + b_{\min}$. Then, for $i \in \{\ell_{k-1}+1, \dots, \ell_k\}$ we obtain the following system for $a \in \mathcal{A}$:

$$\boldsymbol{x}_a(t) = \boldsymbol{x}_a^{[i-1]}(t), \qquad \text{a.e. } t \in [0, t_{i-1}), \tag{3.1}$$

$$\dot{\boldsymbol{x}}_a(t) = \boldsymbol{u}_a(t) - \boldsymbol{g}_a\left[\boldsymbol{u}_a, x_a^{[i-1]}, \dot{x}_a^{[i-1]}\right](t), \qquad \text{a.e. } t \in [t_{i-1}, t_i), \tag{3.2}$$

where for every $(c,d,v) \in \mathcal{C} \times \mathcal{D} \times \mathcal{V}$ and for a.e. $t \in [0,T]$ we have—following (2.21)—the load at a considered node as

$$\boldsymbol{r}_v^{c,d[i-1]} :\equiv \sum_{\tilde{a}\in\mathcal{A}_{\text{in}}(v)} \boldsymbol{g}_{\tilde{a}}^{c,d}\left[\boldsymbol{u}_{\tilde{a}}, x_{\tilde{a}}^{[i-1]}, \dot{x}_{\tilde{a}}^{[i-1]}\right] + \begin{cases} \boldsymbol{s}_v^{c,d} & (v,d) \in \mathcal{OD}, c \in \mathcal{C}, \\ 0 & v \in \mathcal{V} \setminus \mathcal{O}, c \in \mathcal{C} \end{cases} \qquad \text{on } [0,T]$$

and for $a \in \mathcal{A}_{\text{out}}(v)$ according to (2.22) the inflow into the following links:

$$\boldsymbol{u}_a^{c,d} :\equiv \begin{cases} \boldsymbol{r}_v^{c,d[i-1]} \cdot \mathscr{RD}_a^{c,d}\left[\boldsymbol{x}^{[i-1]} \circ \mathrm{P}_{\left[0,\delta_a^{c,d}(\cdot)\right]}, \mathbf{ext}\right] & \text{if } a \in \mathcal{A}_{\text{out}}^d(v), \\ 0 & \text{else.} \end{cases} \qquad \text{on } [0,T].$$

Now, the aim is to show that the routing operator only uses already known information of $\boldsymbol{x}$ so that it would result into an explicitly known routing term. The upper system is uniquely solvable in $[0, t_1)$. For $i \in \{2, \dots\}$ we distinguish two cases. The first case is $\delta_a^{c,d}(t_{i-1}) < t_{i-1}$ which is trivial to solve. The second one is that $\delta_a^{c,d}(t_{i-1}) = t_{i-1}$. Here we use again that we already have a Lipschitz-continuous solution in $[0, t_{i-1})$ such that we obtain by extension the unique solution in $[0, t_{i-1}]$, and the right-hand side of (3.2) is evaluable in t_{i-1}. In $[t_{i-1}, t_i)$ only explicitly known information of $\boldsymbol{x}$ is used such that we can solve (3.2) uniquely.

Altogether, we obtain a unique solution of (3.1) and (3.2) in $W^{1,\infty}((0,t_i);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$, and we denote its constant extension on $[0,T]$ by $\boldsymbol{x}_a^{[i]}$ and define $x_a^{[i]} := \sum_{c\in\mathcal{C}, d\in\mathcal{D}} (\boldsymbol{x}_a^{[i]})^{c,d}$ and $\boldsymbol{x}^{[i]} := (\boldsymbol{x}_{\tilde{a}}^{[i]})_{\tilde{a}\in\mathcal{A}}^T$ where the order in $\mathcal{A}$ is given by Remark 2.6.

Moreover, for every $i \in \{\ell_{k-1}+1,\dots,\ell_k\}$ it holds that $\rho_a[x_a^{[i]}](t_{\ell_{k-1}}) = \rho_a[x_a^{[\ell_{k-1}]}](t_{\ell_{k-1}})$, and since $\rho_a[x_a^{[i]}]^{-1}([t_{\ell_{k-1}},t_i]) \subset [t_{\ell_{k-2}},t_{\ell_{k-1}}]$, we obtain

$$\left\|\boldsymbol{x}_a^{[i]}\right\|_{C\left([0,t_i];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \|\boldsymbol{s}_v\|_{L^\infty\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + \sum_{\tilde{a}\in\mathcal{A}_{\text{in}}(v)} \|\boldsymbol{u}_{\tilde{a}}\|_{L^\infty\left(\left(0,t_{\ell_{k-1}}\right);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}.$$

The upper method terminates after finitely many iterations $k_{t_{\min}} \in \mathbb{N}_{\geq 1}$ with

$$\begin{aligned} k_{t_{\min}} &\leq \sum_{k=1}^{\left\lceil \frac{T}{b_{\min}} \right\rceil} \min_{a\in\mathcal{A}} \left\lceil \frac{\rho_a\left[x_a^{[\ell_{k-1}]}\right](t_{\ell_{k-1}})-t_{\ell_{k-1}}}{t_{\min}} \right\rceil \leq \left\lceil \frac{b_{\min}}{t_{\min}} \right\rceil \sum_{k=1}^{\left\lceil \frac{T}{b_{\min}} \right\rceil} \min_{a\in\mathcal{A}} \left\lceil \frac{\rho_a\left[x_a^{[\ell_{k-1}]}\right](t_{\ell_{k-1}})-t_{\ell_{k-1}}}{b_{\min}} \right\rceil \\ &\leq 2\left\lceil \frac{T}{b_{\min}} \right\rceil \left\lceil \frac{b_{\min}}{t_{\min}} \right\rceil. \end{aligned}$$

Thanks to the structure of the delay functions the case $\delta_a^{c,d}(t) = t$ for some delay functions $\delta_a^{c,d}$ and $t \in [t_{i-1},t_i)$, for some $i \in \{1,\dots,\ell_{k_{t_{\min}}}\}$, can only happen for $t = t_{i-1}$. Since $\boldsymbol{x}^{[i-1]} \in W^{1,\infty}((0,t_{i-1});\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$, it possesses a unique Lipschitz-continuous extension on $[0,t_{i-1}]$ (see [12, Theorem 8.8]). For every $t \in (t_{i-1},t_i)$ we obtain $\delta_a^{c,d}(t) \leq t_{i-1}$ such that, all in all, we obtain together with Remark 2.15 and per constructionem that $\boldsymbol{x}^{[k_{t_{\min}}]}$ is the unique solution in $W^{1,\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ of the link-delay system with routing operators described as in Definition 2.12, item (B). ∎

Next, we will show also in the (L-)continuous case existence and uniqueness of a solution of the network model. Before we come to the main result w.r.t. these operators—namely, Theorem 3.4—we require a method how to sequentially use Banach's fixed-point theorem.

Remark 3.2 (global vs. local (L-)continuity). According to the very Definition 2.12, item (A), the considered routing operators fulfill a global Lipschitz-continuity w.r.t. the state $\boldsymbol{x}$, i.e., the Lipschitz-constant $L := L[\mathbf{ext}]$ depends only on $\mathbf{ext}$. However, it is sufficient for obtaining a unique, global (on $[0,T]$) solution to assume that the Lipschitz-constant is only "local" w.r.t. $\boldsymbol{x}$, i.e., $L := \sup_{\boldsymbol{x}\in\mathcal{B}_\kappa} L[\mathbf{ext},\boldsymbol{x}]$ with

$$\mathcal{B}_\kappa := \left\{\tilde{\boldsymbol{x}} \in W^{1,\infty}\left((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) :\ \|\tilde{\boldsymbol{x}}\|_{W^{1,\infty}\left((0,T);\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)} \leq \kappa\right\}$$

for a $\kappa \in \mathbb{R}_{>0}$.

Thus, for every fixed $\mathbf{ext} \in L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}})$ the Lipschitz-constant of the Lipschitz-continuous routing operator only needs to satisfy $\sup_{\boldsymbol{x}\in\mathcal{B}_\kappa} L[\boldsymbol{x},\mathbf{ext}] < \infty$, and Theorem 3.4 still holds. This is due to the fact that all incoming flow of a considered node is a priori essentially bounded by the assumptions on $\boldsymbol{s}$ in Definition 2.18.

Lemma 3.3 (clustering argument with Banach's fixed-point theorem). *Let $n \in \mathbb{N}_{\geq 1}, L,T \in \mathbb{R}_{>0}$, and $\Omega \subset C([0,T];\mathbb{R}^n)$ closed in the induced topology be given. Furthermore, let $\boldsymbol{\Psi} : \Omega \to \Omega$ be a Lipschitz-continuous self-mapping with*

$$\|\boldsymbol{\Psi}[\boldsymbol{x}] - \boldsymbol{\Psi}[\tilde{\boldsymbol{x}}]\|_{C([0,t];\mathbb{R}^n)} \leq Lt^\alpha \|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C([0,t];\mathbb{R}^n)} \quad \forall t \in [0,T]\ \forall \boldsymbol{x},\tilde{\boldsymbol{x}} \in \Omega$$

for some $\alpha \in \mathbb{R}_{>0}$. Then, there exists a unique fixed-point $\boldsymbol{x}^* \in \Omega$ of $\boldsymbol{\Psi}$.

Proof. Let $T' \in \mathbb{R}_{>0}$ be given with $T'^{\alpha} < \frac{1}{L}$. Then, the set $\Omega_1 := \{\boldsymbol{x}|_{[0,T']} : \boldsymbol{x} \in \Omega\} \subset C([0,T'];\mathbb{R}^n)$ is closed in the induced topology, $\boldsymbol{\Psi}|_{\Omega_1}$ is a contraction on Ω_1, and due to Banach's fixed-point theorem (see [65, page 19]) we obtain a unique fixed-point $\boldsymbol{x}_1 \in \Omega_1$ of $\boldsymbol{\Psi}|_{\Omega_1}$. If $T' = T$, we are done. If $T' < T$, consider

$$\Omega_2 := \left\{ \boldsymbol{x}\big|_{[0,2T']} : \boldsymbol{x}\big|_{[0,T']} = \boldsymbol{x}_1, \boldsymbol{x} \in \Omega \right\} \subset C\left(\left[0,2T'\right]; \mathbb{R}^n \right)$$

which is closed in the induced topology. Again we obtain a unique fixed-point $\boldsymbol{x}_2 \in \Omega_2$. Since T is finite, this procedure terminates after finitely many iterations and we obtain a unique fixed-point $\boldsymbol{x}^* \in \Omega$ of $\boldsymbol{\Psi}$. ■

Theorem 3.4 (existence/uniqueness of the network model for (L-)continuous routing operator). *Let the network with the link-delay dynamics as in Definition 2.18 be given, and assume that $\mathscr{R}_a^{c,d}$ is an (L-)continuous routing operator, i.e., satisfies Item (A) in Definition 2.12. Then there exists a unique solution $\boldsymbol{x}_a$, $a \in \mathcal{A}$, of link-delay ODEs on the network, and the solution satisfies*

$$\boldsymbol{x}_a \in W^{1,\infty}\left((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}\right) \qquad \forall a \in \mathcal{A}.$$

Proof. Again, we want to construct a solution iteratively where we combine the strategy used in [63, pages 344–347] together with Lemma 3.3. Define

$$b_{\min} := \min_{a\in\mathcal{A}} b_a, \qquad K := \left\lceil \frac{T}{b_{\min}} \right\rceil,$$

$$M := K \sum_{v\in\mathcal{O}} \max_{c\in\mathcal{C},d\in\mathcal{D}} \left\| \boldsymbol{s}_v^{c,d} \right\|_{L^1((0,T))}, \qquad M' := K \sum_{v\in\mathcal{O}} \|\boldsymbol{s}_v\|_{L^\infty\left((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}.$$

Then, M defines an upper bound when measuring the inflow which can occur at any node in the network in a L^1 sense, while M' gives an upper bound of the inflow in the network in the L^∞ sense. In the following let $\mathbf{ext} \in \mathbb{R}^{n_{\text{ext}}}$ be arbitrary but given, and use the Lipschitz-constant L as described in Remark 3.2 with $\kappa := \max\{M, M'\}$. Now, initialize $t_0 := 0$ and $\boldsymbol{x}^{[0]} := \left([0,T] \ni t \mapsto \mathbf{0} \in \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)$. For $i \in \{1,\dots,K\}$ define iteratively

$$t_i := \min_{a\in\mathcal{A}} \min \left\{ \rho_a \left[x_a^{[i-1]} \right](t_{i-1}), T \right\}$$

and the following space:

$$\Omega_{M'}^{M,[i]} := \Bigg\{ \boldsymbol{y} \in W^{1,\infty}\left((0,t_i);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) : \boldsymbol{y}\Big|_{[0,t_{i-1}]} = \boldsymbol{x}^{[i-1]}\Big|_{[0,t_{i-1}]},$$
$$\|\boldsymbol{y}\|_{W^{1,\infty}\left((0,t_i);\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)} \leq \max\left\{ \tfrac{iM}{K}, \tfrac{iM'}{K} \right\} \Bigg\},$$

where $\boldsymbol{x}^{[i-1]}, x_a^{[i-1]}$ are given by the previous iteration. Next we set the operators $\boldsymbol{\Psi}^{[i]} := (\boldsymbol{\Psi}_a^{[i],c,d})_{a\in\mathcal{A},c\in\mathcal{C},d\in\mathcal{D}}^T : \Omega_{M'}^{M,[i]} \to C([0,t_i];\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ such that for every $a \in \mathcal{A}$ and $\boldsymbol{x} \in \Omega_{M'}^{M,[i]}$

$$\boldsymbol{\Psi}_a^{[i]}[\boldsymbol{x}](t) := \begin{cases} \boldsymbol{x}_a^{[i-1]}(t), & t \in [0,t_{i-1}), \\ \int_{t_{i-1}}^{t} \boldsymbol{u}_a(\theta) - \boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](\theta)\,\mathrm{d}\theta, & t \in [t_{i-1},t_i]. \end{cases}$$

Thereby, we used $x_a :\equiv \sum_{c\in\mathcal{C}}\sum_{d\in\mathcal{D}} \boldsymbol{x}_a^{c,d}$, $\boldsymbol{g}_a$ defined according to Definition 2.18 with $c \in \mathcal{C}, d \in \mathcal{D}$ and

$$\boldsymbol{u}_a^{c,d} := \left(\hat{\boldsymbol{s}}_v^{c,d} + \sum_{\bar{a}\in\mathcal{A}_{\text{in}}(v)} \boldsymbol{g}_{\bar{a}}^{c,d}[\boldsymbol{u}_{\bar{a}}, x_{\bar{a}}, \dot{x}_{\bar{a}}]\right) \widehat{\mathscr{R}\mathscr{L}}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}]$$

with

$$\hat{\boldsymbol{s}}_v^{c,d} := \begin{cases} \boldsymbol{s}_v^{c,d}, & (v,d) \in \mathcal{OD}, c \in \mathcal{C}, \\ 0, & v \in \mathcal{V}\setminus\mathcal{O}, c \in \mathcal{C}, \end{cases}$$
$$\widehat{\mathscr{R}\mathscr{L}}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}] := \begin{cases} \mathscr{R}\mathscr{L}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}], & a \in \mathcal{A}_{\text{out}}^d(v), \\ 0 & \text{else.} \end{cases}$$

Then, the following holds for every $a \in \mathcal{A}$ and $i \in \{1,\dots,K\}$ with $t_i \le T$:

1. $t_i - t_{i-1} \ge b_{\min}$.
2. $\boldsymbol{u}_a \in L^\infty((0,t_i);\mathbb{R}_{\ge 0}^{|\mathcal{C}||\mathcal{D}|})$, $\sum_{\hat{a}\in\mathcal{A}} \|\boldsymbol{u}_{\hat{a}}\|_{L^1\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \le \frac{iM}{K}$, $\sum_{\hat{a}\in\mathcal{A}} \|\boldsymbol{u}_{\hat{a}}\|_{L^\infty\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \le \frac{iM'}{K}$.
3. $\boldsymbol{\Psi}^{[i]}$ admits a unique fixed-point $\boldsymbol{x}^{[i]}$ in $\Omega_{M'}^{M,[i]}$.
4. $\frac{\mathrm{d}}{\mathrm{d}t}\rho_a[x_a^{[i]}](t) \ge 1\ \forall t \in [0,t_i]$ and $\|\boldsymbol{g}_a[\boldsymbol{u}_a, \boldsymbol{x}_a^{[i]}, \dot{\boldsymbol{x}}_a^{[i]}]\|_{L^\infty\left((0,t_{i+1});\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \le \|\boldsymbol{u}_{\hat{a}}\|_{L^\infty\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}$.

For $i = 1$ obviously items 1 and 2 hold, and for every $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \Omega_{M'}^{M,[1]}$ and for every $t \in [0,t_1]$ we obtain together with Hölder's inequality

$$\left\|\boldsymbol{\Psi}^{[1]}[\boldsymbol{x}]\right\|_{C\left([0,t_1];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)} \le \tfrac{M}{K}, \qquad \left\|\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}^{[1]}[\boldsymbol{x}]\right\|_{L^\infty\left((0,t_1);\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)} \le \tfrac{M'}{K}$$

$$\begin{aligned} \left\|\boldsymbol{\Psi}_a^{[1],c,d}[\boldsymbol{x}] - \boldsymbol{\Psi}_a^{[1],c,d}[\tilde{\boldsymbol{x}}]\right\|_{C([0,t])} &\le \left\|\hat{\boldsymbol{s}}_v^{c,d}\right\|_{L^\infty((0,t_1))} \left\|\widehat{\mathscr{R}\mathscr{L}}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}] - \widehat{\mathscr{R}\mathscr{L}}_a^{c,d}[\tilde{\boldsymbol{x}}, \mathbf{ext}]\right\|_{L^1((0,t))} \\ &\le \tfrac{1}{K} t^{1-\frac{1}{p}} LM' \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)} \end{aligned} \tag{3.3}$$

such that $\boldsymbol{\Psi}^{[1]}$ is Lipschitz-continuous with Lipschitz-constant $\frac{1}{K} t_1^{1-\frac{1}{p}} LM'$. Furthermore, $\Omega_{M'}^{M,[1]}$ is closed in the $W^{1,\infty}$-topology. Together with Lemma 3.3 and $\alpha := 1 - \frac{1}{p}$ we obtain a fixed-point in $\boldsymbol{x}^{[1]} \in \Omega_{M'}^{M,[1]}$. Thus, we have shown item 3 for $i = 1$ for which also item 4 holds. Now, we assume for an $i \in \{1,\dots,K\}$ the upper setting is well-defined and items 1 to 3 are fulfilled. Let in the following $a := (v, v^*) \in \mathcal{A}$ with $(v, v^*) \in \mathcal{V}^2$ be arbitrary. By the very definition of ρ_a and since $\boldsymbol{x}^{[i]} \ge 0$, item 1 is fulfilled for $i + 1$. Next, because of the definition of $\Omega_{M'}^{M,[i+1]}$ we obtain for $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in \Omega_{M'}^{M,[i+1]}$ that $\boldsymbol{x}_{|[0,t_i]} = \tilde{\boldsymbol{x}}_{|[0,t_i]} = \boldsymbol{x}^{[i]}_{|[0,t_i]} = \tilde{\boldsymbol{x}}_{|[0,t_i]}$ such that

$$\rho_a[x_a]^{-1}([t_i, t_{i+1}]) = \rho_a[\tilde{x}_a]^{-1}([t_i, t_{i+1}]) = \rho_a[x_a^{[i]}]^{-1}([t_i, t_{i+1}]) \subset [0, t_i]. \tag{3.4}$$

Thus for a.e. $t \in [0, t_{i+1}]$ it holds that

$$\boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](t) = \boldsymbol{g}_a[\boldsymbol{u}_a, \tilde{x}_a, \dot{\tilde{x}}_a](t) = \boldsymbol{g}_a\left[\boldsymbol{u}_a, x_a^{[i]}, \dot{x}_a^{[i]}\right](t). \tag{3.5}$$

Since item 2 is true for i and by (3.4), we obtain for every $a \in \mathcal{A}$ that $\boldsymbol{u}_a \geq \mathbf{0} \in \mathbb{R}^{|\mathcal{C}||\mathcal{D}|}$ a.e. in $[0, t_{i+1}]$. Now, from (3.4) and (3.5) it follows for $t \in [t_i, t_{i+i}]$ that

$$\boldsymbol{\Psi}_a^{[i+1]}[\boldsymbol{x}](t) = \int_0^t \boldsymbol{u}_a(\theta)\,\mathrm{d}\theta - \int_{b_{\min}}^{\rho_a\left[x_a^{[i]}\right]^{-1}(t)} \boldsymbol{u}_a(\theta)\,\mathrm{d}\theta \geq \mathbf{0} \in \mathbb{R}^{|\mathcal{C}||\mathcal{D}|}$$

and therefore $\boldsymbol{\Psi}^{[i+1]}[\boldsymbol{x}] \geq \mathbf{0} \in \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}$. Furthermore, we obtain

$$\|\hat{\boldsymbol{s}}_v\|_{L^1\left((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + \sum_{\bar{a}\in\mathcal{A}_{\text{in}}(v)} \|\boldsymbol{u}_{\bar{a}}\|_{L^1\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + \|\boldsymbol{u}_a\|_{L^1\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \tfrac{i+1}{K} M,$$
$$\|\hat{\boldsymbol{s}}_v\|_{L^\infty\left((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + \sum_{\bar{a}\in\mathcal{A}_{\text{in}}(v)} \|\boldsymbol{u}_{\bar{a}}\|_{L^\infty\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + \|\boldsymbol{u}_a\|_{L^\infty\left((0,t_i);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \tfrac{i+1}{K} M';$$

thus

$$\left\|\boldsymbol{\Psi}_a^{[i+1]}[\boldsymbol{x}]\right\|_{C([0,t_{i+1}];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})} \leq \tfrac{i+1}{K} M \qquad \left\|\tfrac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\Psi}_a^{[i+1]}[\boldsymbol{x}]\right\|_{L^\infty\left((0,t_{i+1});\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \tfrac{i+1}{K} M'.$$

The upper estimates show that also for $i+1$ item 2 is fulfilled. Next, by using (3.5) we obtain for every $t \in [0, t_{i+1}]$

$$\begin{aligned}
&\left\|\boldsymbol{\Psi}_a^{[i+1],c,d}[\boldsymbol{x}] - \boldsymbol{\Psi}_a^{[i+1],c,d}[\tilde{\boldsymbol{x}}]\right\|_{C([0,t])} \\
&\leq \int_0^t \left(\hat{\boldsymbol{s}}_v^{c,d}(\theta) + \sum_{\bar{a}\in\mathcal{A}_{\text{in}}(v)} \boldsymbol{g}_{\bar{a}}^{c,d}\left[\boldsymbol{u}_{\bar{a}}, x_{\bar{a}}^{[i]}, \dot{x}_{\bar{a}}^{[i]}\right](\theta)\right) \left|\widehat{\mathscr{RL}}_a^{c,d}[\boldsymbol{x}, \mathbf{ext}](\theta) - \widehat{\mathscr{RL}}_a^{c,d}[\tilde{\boldsymbol{x}}, \mathbf{ext}](\theta)\right| \mathrm{d}\theta \\
&\leq \tfrac{i+1}{K} t^{1-\frac{1}{p}} L M' \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)}.
\end{aligned} \tag{3.6}$$

Thus, $\boldsymbol{\Psi}^{[i+1]}(\Omega_{M'}^{M,[i+1]}) \subset \Omega_{M'}^{M,[i+1]}$ and $\boldsymbol{\Psi}^{[i+1]}$ is globally Lipschitz-continuous when measuring in the $C([0, t_{i+1}]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$-topology such that we obtain with the closedness of $\Omega_{M'}^{M,[i+1]}$ and Lemma 3.3 with $\alpha := 1-\frac{1}{p}$ a unique solution $\boldsymbol{x}^{[i+1]} \in \Omega_{M'}^{M,[i+1]}$ for every $\mathbf{ext} \in L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}})$ which shows item 3 for $i+1$. Following the strategy as in [63, Theorem 3.2] we obtain for a.e. $t \in [t_{i+1}, t_{i+2}]$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\rho_a\left[x_a^{[i+1]}\right](t) \geq 1 \quad \text{and} \quad \left\|\boldsymbol{g}_a[\boldsymbol{u}_a, \boldsymbol{x}_a^{[i+1]}, \dot{\boldsymbol{x}}_a^{[i+1]}]\right\|_{L^\infty\left((0,t_{i+2});\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \|\boldsymbol{u}_{\hat{a}}\|_{L^\infty\left((0,t_{i+1});\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}$$

which shows item 4 for $i+1$.

The upper procedure terminates after finitely many iterations $k^* \in \mathbb{N}$, with $k^* \leq K$ due to item 1. Furthermore, the constructed function $\boldsymbol{x}^{[k^*]} \in W^{1,\infty}((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ is the unique solution of the network formulation of the link-delay model with (L-)continuous routing operators which concludes the proof. ∎

Corollary 3.5 (existence and uniqueness of the network model for combinations of delay-type and (L-)continuous routing operators). *Let the network with the link-delay dynamics as in Definition 2.18 be given, and assume that the routing operators $\mathscr{R}$ satisfy item (A) or item (B) of Definition 2.12. Then, there exists a unique solution on the network $\boldsymbol{x}_a \in W^{1,\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$ for every $a \in \mathcal{A}$.*

Proof. The proof consists of combining the proofs of Theorem 3.4 and Theorem 3.1. ∎

Remark 3.6 (further combinations of the different types of routing operators). Of course, the previously defined routing operators, Lipschitz-continuous and delay routing, can also be combined for more precise modeling. There, we do not only mean that we have both types of routings distributed over the network but also that each routing operator itself can exemplarily be chosen as a linear combination of the both types mentioned. Again, the solution theory would be based on the combination of the proofs used for the Lipschitz and in the delay case.

For the proof of the existence of a solution on the network in case of continuous routing operators (Theorem 3.8) we require the following stability result for the link-delay model.

Lemma 3.7 (stability of the outflow w.r.t. the inflow). *Let only one single link as in Definition 2.7 be given with $T \in \mathbb{R}_{>0}$, and assume two inflows $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in L^{\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$ with $\|\boldsymbol{u}\|_{L^{\infty}((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})} \leq K \geq \|\tilde{\boldsymbol{u}}\|_{L^{\infty}((0,T);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})}$ with $K \in \mathbb{R}_{>0}$ are given. Let $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in W^{1,\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$ denote the corresponding solutions. Then, we have for all $t \in [0,T]$*

$$\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq (2 + |\mathcal{C}||\mathcal{D}|hC(K,h))^{\left\lceil \frac{t-b}{b} \right\rceil} \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^1\left((0,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}$$

and for $t \in (b,T]$

$$\begin{aligned}\Big| \|\boldsymbol{g}[\boldsymbol{u},x,\dot{x}]\|_{L^1\left((b,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} - \|\boldsymbol{g}[\tilde{\boldsymbol{u}},\tilde{x},\dot{\tilde{x}}]\|_{L^1\left((b,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \Big| &\leq \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^1\left((0,t-b);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} + |\mathcal{C}||\mathcal{D}|K \\ &\quad \cdot C(K,h)\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C\left([0,t-b];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\end{aligned}$$

with $C(K,h) := \sum_{n=0}^{\lceil \frac{T}{b} \rceil} (h|\mathcal{C}||\mathcal{D}|K)^n$.

Proof. We begin with the time interval $t \in [0,b]$, where the outflow is zero. Then, by the assumptions, we obviously obtain

$$\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) = \int_0^t \boldsymbol{u}(s) - \tilde{\boldsymbol{u}}(s)\,\mathrm{d}s \implies \|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)} \leq \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{L^1\left((0,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}.$$

Now, iterating over the time horizons $t \in [kb,(k+1)b]$ for $k \in \mathbb{N}_{\geq 1}$ will provide us the claim. First, let be $k = 1$ such that we obtain for the outflow the estimate for $t \in [b,2b]$,

$$\boldsymbol{x}(t) - \tilde{\boldsymbol{x}}(t) = \boldsymbol{x}(b) - \tilde{\boldsymbol{x}}(b) + \int_b^t \boldsymbol{u}(s) - \tilde{\boldsymbol{u}}(s)\,\mathrm{d}s - \int_0^t \boldsymbol{g}[\boldsymbol{u},x,\dot{x}](s) - \boldsymbol{g}[\tilde{\boldsymbol{u}},\tilde{x},\dot{\tilde{x}}](s)\,\mathrm{d}s,$$

and using integration by parts we obtain for the outflow terms—as also performed in (2.8)—

$$\int_0^t \boldsymbol{g}[\boldsymbol{u},x,\dot{x}](s) - \boldsymbol{g}[\tilde{\boldsymbol{u}},\tilde{x},\dot{\tilde{x}}](s)\,\mathrm{d}s = \int_0^{\rho[x]^{-1}(t)} \boldsymbol{u}(s)\,\mathrm{d}s - \int_0^{\rho[\tilde{x}]^{-1}(t)} \tilde{\boldsymbol{u}}(s)\,\mathrm{d}s.$$

Now, estimating the exit flows, we obtain

$$\sum_{(c,d)\in\mathcal{C}\times\mathcal{D}}\left|\int_0^t \boldsymbol{g}^{c,d}[\boldsymbol{u}^{c,d},x,\dot{x}](s)-\boldsymbol{g}^{c,d}[\tilde{\boldsymbol{u}}^{c,d},\tilde{x},\dot{\tilde{x}}](s)\,\mathrm{d}s\right|$$
$$\leq \|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((0,b);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}+|\mathcal{C}||\mathcal{D}|\|\boldsymbol{u}\|_{L^\infty\left((0,b);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\left|\rho[x]^{-1}(t)-\rho[\tilde{x}]^{-1}(t)\right|.$$

Estimating the latter term, we have, by picking $s\in[0,T]$ such that $t=\rho[x](s)$,

$$\left|\rho[x]^{-1}(t)-\rho[\tilde{x}]^{-1}(t)\right|=\left|s-\rho[\tilde{x}]^{-1}(\rho[x](s))\right|\leq C(K,h)|x(s)-\tilde{x}(s)|,$$

where $C(K,h):=\sum_{n=0}^{\lceil\frac{T}{b}\rceil}(h|\mathcal{C}||\mathcal{D}|K)^n$ is defined according to Theorem 2.9. Since $t\in[b,2b]$ and $\rho[x](\cdot)$ monotonically increasing, we directly have by the nonnegativity of x that $2b\geq t=\rho[x](s)\geq s+b \implies s\leq b$ so that

$$\left\|\rho[x]^{-1}-\rho[\tilde{x}]^{-1}\right\|_{C([b,2b])}\leq C(K,h)\|x-\tilde{x}\|_{C([0,b])}.$$

Altogether, we obtain for $t\in[b,2b]$

$$\begin{aligned}\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\leq{}&|\boldsymbol{x}(b)-\tilde{\boldsymbol{x}}(b)|_1+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((b,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((0,b);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\\&+|\mathcal{C}||\mathcal{D}|KC(K,h)\|x-\tilde{x}\|_{C([0,b])},\end{aligned}$$

and since $\|x-\tilde{x}\|_{C([0,t])}\leq\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|})}\forall t\in[0,T]$, recalling item 4 of Definition 2.1, we have

$$\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\leq\left(2+|\mathcal{C}||\mathcal{D}|KC(K,h)\right)\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((0,b);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}+\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((b,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}$$

which we can estimate roughly by

$$\|\boldsymbol{x}-\tilde{\boldsymbol{x}}\|_{C\left([0,t];\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}\leq\left(2+|\mathcal{C}||\mathcal{D}|KC(K,h)\right)\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{L^1\left((0,t);\mathbb{R}^{|\mathcal{C}||\mathcal{D}|}\right)}.$$

This satisfies the stated estimate. Extending it to the full time horizon gives the claim. ■

Theorem 3.8 (existence of the network model for (only) continuous routing operators). *For $T\in\mathbb{R}_{>0}$ let a network with link-delay dynamics as in Definition 2.18 be given, and let the routing operators satisfy $\mathscr{R}\equiv\mathscr{RC}$, i.e., let them be continuous in the sense of Definition 2.10, item (C). Then, there exists a continuous solution $\boldsymbol{x}\in W^{1,\infty}((0,T);\mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ on the entire network with $\boldsymbol{x}_a^{c,d}(t)\leq\sum_{v\in\mathcal{O}}\|\boldsymbol{s}_v^{c,d}\|_{L^1((0,t))}$ $\forall t\in[0,T]$ $\forall a\in\mathcal{A},\ (c,d)\in\mathcal{C}\times\mathcal{D}$.*

Proof. The proof consists of two different parts. In the first part, we use the fact that the network is for $t=0$ empty and initialize it. In the second part, we prove subsequently in time the existence of a solution. The proof uses Schauder's fixed-point theorem to guarantee the existence of a solution by means of a compactness argument.

First, we define some necessary constants, such as the minimal free flow travel time on the entire network and the maximal cycle number (i.e., the number telling how often a flow can in a worst case estimate flow through a cycle):

$$b_{\min}:=\min_{a\in\mathcal{A}}b_a,\ \ K:=\left\lceil\tfrac{T}{b_{\min}}\right\rceil.$$

Since the routing at time $t \in [0, T]$ is a function of the solution on the network $\boldsymbol{x}(s), s \in [0, t]$, it becomes necessary to solve the entire network simultaneously. However, due to the delay character of the considered dynamical system (link-*delay* model), for $t \in [0, b_{\min}]$ the exit flow is according to (2.20) zero on every link, decoupling and thus simplifying the problem significantly. We have in this case only to study the following system of integral equations—assuming for clarity a network with numbered links $\{a_1, \ldots, a_n\} = \mathcal{A}$, for $n \coloneqq |\mathcal{A}|$—

$$\begin{pmatrix} \boldsymbol{x}_{a_1}(t) \\ \boldsymbol{x}_{a_2}(t) \\ \vdots \\ \boldsymbol{x}_{a_n}(t) \end{pmatrix} = \int_0^t \begin{pmatrix} \boldsymbol{s}_{\mathrm{tail}(a_1)}(y) \odot \mathscr{RC}_{a_1}^{\mathrm{tail}(a_1)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \boldsymbol{s}_{\mathrm{tail}(a_2)}(y) \odot \mathscr{RC}_{a_2}^{\mathrm{tail}(a_2)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \vdots \\ \boldsymbol{s}_{\mathrm{tail}(a_n)}(y) \odot \mathscr{RC}_{a_n}^{\mathrm{tail}(a_n)}[\boldsymbol{x}, \mathbf{ext}](y) \end{pmatrix} \mathrm{d}y, \tag{3.7}$$

where $\mathrm{tail}(a) \coloneqq \{v \in \mathcal{V} : \ a = (v, \tilde{v})\}$ for $a \in \mathcal{A}$ and $\odot$ defines the componentwise multiplication of two vectors and allows us to formulate (3.7) in a rather compact way by intrinsically containing different commodities and destination pairs.

Defining the operator

$$\begin{aligned} \boldsymbol{F} : C\left([0, b_{\min}]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) &\to C\left([0, b_{\min}]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right), \\ \boldsymbol{x} &\mapsto \left(t \mapsto \int_0^t \begin{pmatrix} \boldsymbol{s}_{\mathrm{tail}(a_1)}(y) \odot \mathscr{RC}_{a_1}^{\mathrm{tail}(a_1)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \boldsymbol{s}_{\mathrm{tail}(a_2)}(y) \odot \mathscr{RC}_{a_2}^{\mathrm{tail}(a_2)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \vdots \\ \boldsymbol{s}_{\mathrm{tail}(a_n)}(y) \odot \mathscr{RC}_{a_n}^{\mathrm{tail}(a_n)}[\boldsymbol{x}, \mathbf{ext}](y) \end{pmatrix} \mathrm{d}y \right), \end{aligned}$$

this operator is a self-mapping on the set

$$\begin{aligned} \boldsymbol{\Omega}_{b_{\min}} \coloneqq \Bigg\{ &\boldsymbol{x} \in W^{1,\infty}\left((0, b_{\min}); \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) : \\ &\left\| \boldsymbol{x}_a^{c,d} \right\|_{W^{1,\infty}((0,b_{\min}))} \leq \max\{1, b_{\min}\} \left\| \boldsymbol{s}_{\mathrm{tail}(a)}^{c,d} \right\|_{L^\infty((0,b_{\min}))} \forall (a, c, d) \in \mathcal{A} \times \mathcal{C} \times \mathcal{D} \Bigg\} \end{aligned}$$

due to the essential bound on the routing operator $\mathscr{RC}$ by 1 as postulated in Definition 2.10 (this is a property which is independent of the specific routing.)

Now, we claim that the operator is continuous in the uniform topology. For this, let $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in C([0, T]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$ be given; we obtain for $t \in [0, b_{\min}]$ by the assumption on the continuous routing as in Definition 2.12, item (C),

$$\begin{aligned} |\boldsymbol{F}[\boldsymbol{x}](t) - \boldsymbol{F}[\tilde{\boldsymbol{x}}](t)| \leq & \max_{(c,d,v) \in \mathcal{C} \times \mathcal{D} \times \mathcal{V}} \left\| \boldsymbol{s}_v^{c,d} \right\|_{L^\infty((0,b_{\min}))} \\ & \cdot \max_{(a,c,d) \times \mathcal{A} \times \mathcal{C} \times \mathcal{D}} \left\| \mathscr{RC}_a^{\mathrm{tail}(a),c,d}[\boldsymbol{x}, \mathbf{ext}] - \mathscr{RC}_a^{\mathrm{tail}(a),c,d}[\tilde{\boldsymbol{x}}, \mathbf{ext}] \right\|_{L^1((0,b_{\min}))}, \end{aligned}$$

the continuity of $\boldsymbol{F}$ on $\boldsymbol{\Omega}_{b_{\min}}$.

As the right-hand side of the previous estimate is uniform in $t \in [0, b_{\min}]$, the fixed-point mapping $\boldsymbol{F} : \boldsymbol{\Omega}_{b_{\min}} \to \boldsymbol{\Omega}_{b_{\min}}$ is continuous in the uniform topology. In addition, $\boldsymbol{\Omega}_{b_{\min}}$ is convex and closed in the uniform topology, i.e., $\overline{\boldsymbol{\Omega}_{b_{\min}}}^{\|\cdot\|_{C([0,b_{\min}];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})}} = \boldsymbol{\Omega}_{b_{\min}}$. In addition, by Ascoli–Arzelà [12, Theorem 4.25] $\boldsymbol{\Omega}_{b_{\min}} \overset{c}{\hookrightarrow} C([0,T];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$, i.e., $\boldsymbol{\Omega}_{b_{\min}}$ is compactly embedded in $C([0,T];\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$. Thus, we can apply on the fixed-point problem as posed in (3.7) Schauder's fixed-point theorem [64, Corollary 2.13] and obtain a solution $\boldsymbol{x}^* \in \boldsymbol{\Omega}_{b_{\min}}$ on $[0, b_{\min}]$.

Having this existence result, we pass to the next time intervals $[kb, (k+1)b]$ with $k \in \{1, \dots, K-1\}$. As pointed out in the previous proofs, now the outflow is not necessarily zero anymore but was determined at the time interval $[kb, (k+1)b]$ from the solution at $[0, kb]$. Thus, the outflow is given, and there is no need to consider it in the fixed-point equation as variable value even though it goes in as a function.

We can thus define the fixed-point operator for

$$\boldsymbol{\Omega}_{k\cdot b_{\min}} := \Big\{ \boldsymbol{x} \in W^{1,1}\left((0,(k+1)b_{\min});\mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) : \boldsymbol{x}\big|_{[0,kb_{\min}]} \equiv \boldsymbol{x}^* \wedge,$$
$$\left\|\boldsymbol{x}_a^{c,d}\right\|_{W^{1,\infty}((0,(k+1)\cdot b_{\min}))} \leq \max\{1, b_{\min}\}\, K \left\|\boldsymbol{s}_{\mathrm{tail}(a)}^{c,d}\right\|_{L^\infty((0,(k+1)b_{\min}))} \ \forall (a,c,d) \in \mathcal{A}\times\mathcal{C}\times\mathcal{D}\Big\},$$

$$\boldsymbol{F}^{(k)} : \begin{cases} \boldsymbol{\Omega}_{k\cdot b_{\min}} & \to \boldsymbol{\Omega}_{k\cdot b_{\min}} \\ \boldsymbol{x} & \mapsto t \mapsto \boldsymbol{x}(k\cdot b_{\min}) + \displaystyle\int_{k\cdot b_{\min}}^{t} \begin{pmatrix} \boldsymbol{r}^*_{\mathrm{tail}(a_1)}(y) \odot \mathscr{RC}_{a_1}^{\mathrm{tail}(a_1)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \boldsymbol{r}^*_{\mathrm{tail}(a_2)}(y) \odot \mathscr{RC}_{a_2}^{\mathrm{tail}(a_2)}[\boldsymbol{x}, \mathbf{ext}](y) \\ \vdots \\ \boldsymbol{r}^*_{\mathrm{tail}(a_n)}(y) \odot \mathscr{RC}_{a_n}^{\mathrm{tail}(a_n)}[\boldsymbol{x}, \mathbf{ext}](y) \end{pmatrix} \mathrm{d}y \\ & \qquad - \int_{b_{\min}}^{t} \boldsymbol{G}\left[\boldsymbol{x}^{(k-1)}\right](s)\,\mathrm{d}s \end{cases}$$

with $\boldsymbol{G}[\boldsymbol{x}^{(k-1)}](s) := (\boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](s))_{a\in\mathcal{A}}$ only dependent on the solution $\boldsymbol{x}$ and $\boldsymbol{u}$ on time horizon $[0, k\cdot b_{\min}]$ (for this delay character see Theorem 2.9 and Lemma 3.7). As in the first step this fixed-point mapping is continuous in the uniform topology and also compact. Again, using the same argumentation and applying Schauder's fixed-point mapping we obtain the existence of a solution. This can be continued iteratively w.r.t. $k \in \{2, \dots, K\}$ until one approaches the finite time horizon $T \in \mathbb{R}_{>0}$. Thus, a solution of the ODE delay system has been constructed. ■

So far we have shown for (L-)continuous routing operators and for delay-type routing existence and uniqueness of a solution on the network model (Definition 2.18). In the following Remark 3.9, by usage of the Example 2.17, we show not only that item (C) in Definition 2.12 is not sufficient to obtain uniqueness but also that the delay property in (2.11) of item (B) in Definition 2.12 cannot be substantially weakened to obtain uniqueness of the solution via delay-type routing operators.

Remark 3.9 (vanishing-type delay and nonuniqueness of solution). In Definition 2.12, item (B), we allow the operator $\mathscr{RD}$ to be discontinuous w.r.t. $\boldsymbol{x}$ as long as we have the delay function fulfilling (2.11). We postulate for every $i \in \{1, \dots, N\}$ the existence of a $\tilde{t}_i \in (\boldsymbol{t}_i, T]$ such that $\delta_a^{c,d}(t) \leq \boldsymbol{t}_i$ for every $t \in [\boldsymbol{t}_i, \tilde{\boldsymbol{t}}_i]$. Now, we drop this condition by defining

$$\boldsymbol{t}_1 := 0, \quad \boldsymbol{t}_2 := T, \qquad \delta_{a_1}^{1,v_4}(t) := \tfrac{t}{42}\ \forall t \in [0,T],$$

and, moreover, we take an explicit, continuous—but not (L-)continuous—routing operator. Consider now the network illustrated in Figure 4 (left), and define $T := 5$, $\mathcal{C} := \{1\}$, $\mathcal{O} := \{v_1\}$, $\mathcal{D} := \{v_4\}$, $n_{\text{ext}} \in \mathbb{N}_{\geq 1}$, and for $t \in [0,T]$

$$b_{a_1} := b_{a_2} := b \in \mathbb{R}_{>0},\ \ h_{a_1}, h_{a_2} \in \mathbb{R}_{>0}, \qquad (b_{a_3}, b_{a_4}) := (2,2),\ (h_{a_3}, h_{a_4}) := (0,0).$$

Furthermore, let $\mathbf{ext} \in L^\infty((0,T);\mathbb{R}^{n_{\text{ext}}})$ be arbitrary but fixed and $\mathscr{R}^{1,v_4}_{a_1}[\boldsymbol{x},\mathbf{ext}](t) := \min\{42\sqrt{\boldsymbol{x}_{a_1}(\tfrac{t}{42})}, 1\}$. Due to the partition of the unity as in (2.10) we obtain for every $t \in [0,T]$

$$\mathscr{R}^{1,v_4}_{a_2}[\boldsymbol{x},\mathbf{ext}](t) := 1 - \min\left\{42\sqrt{\boldsymbol{x}_{a_1}\left(\tfrac{t}{42}\right)}, 1\right\}, \quad \mathscr{R}^{1,v_4}_{a_3} :\equiv \mathscr{R}^{1,v_4}_{a_4} :\equiv 1.$$

This defines a continuous routing operator since for every $\boldsymbol{x}, \tilde{\boldsymbol{x}} \in C([0,T];\mathbb{R}^4)$ we obtain for every $t \in [0,T]$

$$\begin{aligned}|\mathscr{R}^{1,v_4}_{a_1}[\boldsymbol{x},\mathbf{ext}](t) - \mathscr{R}^{1,v_4}_{a_1}[\tilde{\boldsymbol{x}},\mathbf{ext}](t)| &= \left|\min\left\{42\sqrt{\boldsymbol{x}_{a_1}(\tfrac{t}{42})}, 1\right\} - \min\left\{42\sqrt{\tilde{\boldsymbol{x}}_{a_1}(\tfrac{t}{42})}, 1\right\}\right| \\ &\leq 42\left\|\sqrt{\boldsymbol{x}_{a_1}(\cdot)} - \sqrt{\tilde{\boldsymbol{x}}_{a_1}(\cdot)}\right\|_{C([0,T])} \\ &\leq 42\sqrt{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|_{C([0,T];\mathbb{R}^4)}}.\end{aligned}$$

For $t \in [0, \min\{2,b\})$ and $\boldsymbol{s}_{v_1} :\equiv 1$ solving the following initial value problem

$$\dot{\boldsymbol{x}}_{a_1}(t) = 42\sqrt{\boldsymbol{x}_{a_1}(\tfrac{t}{42})}, \quad \boldsymbol{x}_{a_1}(0) = 0$$

yields infinitely many different solutions, and in the following two of those are considered, namely, $\boldsymbol{x}^*_{a_1}$ and $\boldsymbol{x}^{**}_{a_1}$ defined as $\boldsymbol{x}^*_{a_1} :\equiv 0$ and $\boldsymbol{x}^{**}_{a_1}(t) := \frac{t^2}{4}$. Thus, for $t \in [0, \min\{2,b\})$ solutions $\boldsymbol{x}_{a_2}$ on a_2 must fulfill $\boldsymbol{x}_{a_2}(t) = t - \boldsymbol{x}_{a_1}(t)$, and we also obtain $\boldsymbol{x}_{a_3} := \boldsymbol{x}_{a_4} := 0$. Therefore, there exist infinitely many solutions on the network following the proposed dynamic. This shows the necessity of postulating a Lipschitz-continuous routing or a delay routing to obtain uniqueness. Of course, the presented counterexample is one of the "famous" counterexamples of uniqueness in the theory of ODEs (for instance, compare [25, Example 1.2]).

4. Common routing rules applied.

In this section, we will use the developed framework to instantiate common routing operators. We will consider routing which satisfies one of the conditions in Definition 2.12 but also routing operators which do not satisfy these conditions.

Definition 4.1 (order in $\mathcal{P}^{v,d}$). *Let $\mathcal{P}^{v,d}$ for $(v,d) \in \mathcal{V} \times \mathcal{D}$ be given as in Definition 2.2. Then, with a slight abuse of notation we assume now that the elements in $\mathcal{P}^{v,d}$ are ordered and define this as $\boldsymbol{P}^{v,d}_o := (\boldsymbol{p}^{o^{v,d}(1)}, \ldots, \boldsymbol{p}^{o^{v,d}(\ell^{v,d})})$ for the given ordering $o^{v,d} : \{1, \ldots, l^{v,d}\} \to \mathbb{N}_{\geq 1}$ and $\ell^{v,d} := |\mathcal{P}^{v,d}|$.*

The specified order in Definition 4.1 is not important, and the ordering itself is from a mathematical point of view no problem as the set is finite. We only require to be able to distinguish all different routes as distinct routes might have the same travel time so that the following Definition 4.2 would otherwise not make sense.

Definition 4.2 (path travel time). *Let $(v,d) \in \mathcal{V} \times \mathcal{D}$ as in Definition 2.2 be given. Then, we define for $t \in [0,T]$ the set $\boldsymbol{\mathcal{X}}^{v,d}(t)$ of the involved* path flows *from node v to destination d, where $\boldsymbol{x}_a \in C([0,T];\mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$, $a \in \mathcal{A}$, as*

$$\boldsymbol{\mathcal{X}}^{v,d}(t) := \bigcup_{\boldsymbol{p} \in \mathcal{P}^{v,d}} \left\{ \left(\boldsymbol{x}_{\boldsymbol{p}_1}^T(t), \dots, \boldsymbol{x}_{\boldsymbol{p}_{\mathrm{len}(\boldsymbol{p})}}^T(t) \right)^T \right\}, \qquad t \in [0,T].$$

Let the travel time $\tau_a[x_a]$ on every arc $a \in \mathcal{A}$, $\ell := |\mathcal{P}^{v,d}|$ and $\boldsymbol{P}_o^{v,d}$ from Definition 4.1 be given. After redefining its components—for a better readability—respectively, to $\tilde{\boldsymbol{p}}^1, \dots, \tilde{\boldsymbol{p}}^\ell$, we define the vector of possible travel times *$\boldsymbol{\tau}^{v,d}$ from node v to destination d by*

$$\begin{aligned} \boldsymbol{\tau}^{v,d}\left[\boldsymbol{\mathcal{X}}^{v,d}\right](t) &:= \left(\sum_{i=1}^{\mathrm{len}(\tilde{\boldsymbol{p}}^1)} \tau_{\tilde{\boldsymbol{p}}_i^1}\left[x_{\tilde{\boldsymbol{p}}_i^1}\right](t), \dots, \sum_{i=1}^{\mathrm{len}(\tilde{\boldsymbol{p}}^\ell)} \tau_{\tilde{\boldsymbol{p}}_i^\ell}\left[x_{\tilde{\boldsymbol{p}}_i^\ell}\right](t) \right), & t \in [0,T], \\ &= \left(\sum_{i=1}^{\mathrm{len}(\tilde{\boldsymbol{p}}^j)} \tau_{\tilde{\boldsymbol{p}}_i^j}\left[x_{\tilde{\boldsymbol{p}}_i^j}\right](t) \right)_{j \in \{1,\dots,\ell\}}, & t \in [0,T]. \end{aligned}$$

Remark 4.3 (notation of vector of possible travel times). To obtain a similar structure between $\boldsymbol{\mathcal{X}}$ and the vector of possible travel times in Definition 4.2, we would have preferred to also define $\boldsymbol{\tau}^{v,d}$ as a set over $\boldsymbol{p} \in \mathcal{P}^{v,d}$. However, since we will need an ordered structure and some path travel times in Definition 2.18 could coincide, we accept this apparent "inconsistency."

Using Definition 4.2 we obtain a vector that contains the travel time on all possible paths $\mathcal{P}^{v,d}$. The travel time on a path is equivalent to the summation of the travel time on all arcs contained in the path.

Example 4.4 (topology of networks continued). Considering again the network illustration in Figure 3 with the network in Example 2.17, we now explain the meaning of $\boldsymbol{\mathcal{X}}^{v,d}$ and $\boldsymbol{P}_o^{v,d}$. For simplicity we only show $\boldsymbol{\mathcal{X}}^{v_2,v_6}$, $\boldsymbol{P}_o^{v_2,v_6}$, and $\boldsymbol{P}_o^{v_2,v_7}$. We have for $t \in [0,T]$

$$\begin{aligned} \boldsymbol{\mathcal{X}}^{v_2,v_6}(t) = \Big\{ \Big(&\left(\boldsymbol{x}_{a_3}^{1,v_5}(t), \boldsymbol{x}_{a_3}^{1,v_6}(t), \boldsymbol{x}_{a_3}^{1,v_7}(t), \boldsymbol{x}_{a_3}^{2,v_5}(t), \boldsymbol{x}_{a_3}^{2,v_6}(t), \boldsymbol{x}_{a_3}^{2,v_7}(t)\right), \\ &\left(\boldsymbol{x}_{a_7}^{1,v_5}(t), \boldsymbol{x}_{a_7}^{1,v_6}(t), \boldsymbol{x}_{a_7}^{1,v_7}(t), \boldsymbol{x}_{a_7}^{2,v_5}(t), \boldsymbol{x}_{a_7}^{2,v_6}(t), \boldsymbol{x}_{a_7}^{2,v_7}(t)\right) \Big)^T, \\ &\left(\boldsymbol{x}_{a_4}^{1,v_5}(t), \boldsymbol{x}_{a_4}^{1,v_6}(t), \boldsymbol{x}_{a_4}^{1,v_7}(t), \boldsymbol{x}_{a_4}^{2,v_5}(t), \boldsymbol{x}_{a_4}^{2,v_6}(t), \boldsymbol{x}_{a_4}^{2,v_7}(t)\right)^T \Big\} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{P}_o^{v_2,v_6} &= \Big((a_3, a_7), (a_4) \Big), \\ \boldsymbol{P}_o^{v_2,v_7} &= \Big((a_2, a_9), (a_3, a_6, a_5, a_9), (a_3, a_{10}), (a_4, a_8, a_6, a_5, a_9), (a_4, a_8, a_{10}) \Big). \end{aligned}$$

For the following routing definitions we heavily rely on the arg min function, which we define as follows.

Definition 4.5 (arg min function). *Let a set of elements $\{y_i \in \mathbb{R} : \ i \in \mathcal{M}\}$ where $\mathcal{M}$ defines some set be given. Then, we define*

$$\arg\min_{i\in\mathcal{M}} y_i := \{m \in \mathcal{M} : y_m \le y_i \ \forall i \in \mathcal{M}\}. \tag{4.1}$$

Definition 4.5 allows us to find paths $\mathcal{P}^{v,d}$ ordered by travel time, which we use in Routings 4.6–4.11.

Routing 4.6 (shortest next link). *Given $v \in \mathcal{V}, d \in \mathcal{D}$, and $a \in \mathcal{A}^d_{\text{out}}(v)$ we define* shortest next link *routing as*

$$\mathscr{R}^{c,d}_a[\boldsymbol{x}, \mathbf{ext}](t) := \begin{cases} |\mathcal{M}(t)|^{-1} & \text{if } a \in \mathcal{M}(t), \\ 0 & \text{else} \end{cases} \quad \text{with } \mathcal{M}(t) := \arg\min_{\tilde{a}\in\mathcal{A}^d_{\text{out}}(v)} \tau_{\tilde{a}}[x_{\tilde{a}}](t) \tag{4.2}$$

for every $t \in [0, T]$ and $\mathcal{A}^d_{\text{out}}(v)$ as in Definition 2.2.

Remark 4.7 (shortest next link Routing 4.6). This routing assigns all the flow to the roads contained in $\mathcal{A}^d_{\text{out}}(v)$ leaving the node v which has the lowest travel time. In case, there is only one route, i.e., $|\mathcal{M}(t)| = 1$ at time $t \in [0, T]$; all flow is sent onto this route. The case considered corresponds to a situation in which drivers can physically see the next link that they will be travelling to and can therefore assess the travel time on that link, but not on future links.

Clearly, *shortest next link* is not Lipschitz-continuous w.r.t. $\boldsymbol{x}$ as it might jump from one link to another, and neither continuous. However, it still satisfies Definition 2.10. The only thing which has to be shown for this is the measurability of this routing, as the other properties follow by construction. There, the measurability follows by the continuity of the travel times $\tau_a[x_a]$ and by Lemma 4.23.

Routing 4.8 (weighted path distribution next link). *Given $v \in \mathcal{V}$ and $d \in \mathcal{D}$ we define* weighted path distribution next link *routing as*

$$\text{for } a \in \mathcal{A}^d_{\text{out}}(v): \quad \mathscr{R}^{c,d}_a[\boldsymbol{x}, \mathbf{ext}](t) := \frac{\exp(-\tau_a[x_a](t))}{\sum_{\tilde{a}\in\mathcal{A}^d_{\text{out}}(v)} \exp(-\tau_{\tilde{a}}[x_{\tilde{a}}](t))} \qquad t \in [0, T] \tag{4.3}$$

and $\mathcal{A}^d_{\text{out}}(v)$ as in Definition 2.2.

Remark 4.9 (weighted path distribution next link Routing 4.8). The routing in Routing 4.8 can be interpreted as a smoothed version of Routing 4.6. It assigns flow on the outgoing links weighted with the exponential function dependent on the travel time on the specific link. Clearly, it satisfies the Lipschitz-continuity in Definition 2.12 so that we indeed obtain existence and uniqueness of a solution on the entire network considered. However, one drawback might be that on every link $a \in \mathcal{A}^d_{\text{out}}(v)$ traffic is assigned, even if it might be a very small amount (when travel time is high). On the other hand, as we consider a macroscopic modelling of traffic flow, this might be reasonable.

Routing 4.10 (shortest path). *Given $v \in \mathcal{V}$ and $d \in \mathcal{D}$ we define* shortest path *routing for $t \in [0, T]$ as*

$$\text{for } \ \tilde{a} \in \left\{\boldsymbol{p}_1 : \boldsymbol{p} \in \arg\min_{\boldsymbol{p}\in\mathcal{P}^{v,d}} \boldsymbol{\tau}^{v,d}\left[\boldsymbol{\mathcal{X}}^{v,d}\right](t)\right\} =: \mathcal{M}(t), \quad \mathscr{R}^{c,d}_{\tilde{a}}[\boldsymbol{x}, \mathbf{ext}](t) := |\mathcal{M}(t)|^{-1} \tag{4.4}$$

with $\boldsymbol{\mathcal{X}}$ and $\boldsymbol{\tau}$ as in Definition 4.1 and Definition 4.2.

Remark 4.11 (shortest path Routing 4.10). In Routing 4.10 we consider the case in which the travel time on all available paths is known and therefore the shortest paths are known as well. This corresponds to a situation in which drivers use navigation apps (such as Google maps, Apple maps, Waze, etc.) to determine which path to use.

This routing can be interpreted as the global (w.r.t. the network) version of Routing 4.6. It is, as pointed out before, neither Lipschitz-continuous nor continuous (compare Remark 4.7), but one can obtain well-posedness of solutions if delayed. See also Remark 4.24 for why this routing is highly nontrivial.

Routing 4.12 (weighted path distribution). *Given $v \in \mathcal{V}$ and $d \in \mathcal{D}$ we define* weighted path distribution *routing as*

$$\text{for } a \in \mathcal{A}^d_{\text{out}}(v) \quad \mathscr{R}^{c,d}_a[\boldsymbol{x}, \mathbf{ext}](t) := \sum_{\substack{\boldsymbol{p}\in\mathcal{P}^{v,d}\\ \boldsymbol{p}_1=a}} \frac{\mathrm{e}^{-\left(\boldsymbol{\tau}^{v,d}[\boldsymbol{\mathcal{X}}^{v,d}](t)\right)_{\boldsymbol{p}}}}{\sum_{\tilde{\boldsymbol{p}}\in\mathcal{P}^{v,d}} \mathrm{e}^{-\left(\boldsymbol{\tau}^{v,d}[\boldsymbol{\mathcal{X}}^{v,d}](t)\right)_{\tilde{\boldsymbol{p}}}}}, \qquad t \in [0, T].$$

Remark 4.13 (weighted path distribution). Routing 4.12 uses as Routing 4.8 a logit function to distribute flow among possible paths. Flow is distributed in such a way that paths with lower travel time acquire more flow and paths with longer travel time acquire less flow. In this case, the utility of each path is equivalent to the exponential of (negative) travel time along that path.

This routing is again a global version (w.r.t. the network) of Routing 4.8 and also satisfies the Lipschitz-continuity in Definition 2.12 so that there exists—according to Theorem 3.4—a unique solution on the entire network.

Routing 4.14 (shortest mini path). *Given $(v, d) \in \mathcal{V} \times \mathcal{D}$ and $k \in \mathbb{N}_{\geq 1}$ we define* shortest mini path *routing for $t \in [0, T]$ as*

$$\text{for } \ a \in \left\{ \boldsymbol{p}_1 : \boldsymbol{p} \in \underset{\boldsymbol{p}\in\mathcal{P}^{v,d}}{\arg\min}\, \boldsymbol{\tau}^{v,d}_k \left[\boldsymbol{\mathcal{X}}^{v,d}\right](t) \right\} =: \mathcal{M}(t), \qquad \mathscr{R}^{c,d}_a[\boldsymbol{x}, \mathbf{ext}](t) := |\mathcal{M}(t)|^{-1},$$

where $\boldsymbol{\tau}^{v,d}_k$ is analogously to Definition 4.2 defined as

$$\begin{aligned} \boldsymbol{\tau}^{v,d}_k \left[\boldsymbol{\mathcal{X}}^{v,d}\right](t) &:= \left(\sum_{i=1}^{\min\{k,\mathrm{len}(\boldsymbol{p}_1)\}} \tau_{\boldsymbol{p}_{1,i}} \left[x_{\boldsymbol{p}_{1,i}}\right](t), \dots, \sum_{i=1}^{\min\{k,\mathrm{len}(\boldsymbol{p}_l)\}} \tau_{\boldsymbol{p}_{l,i}} \left[x_{\boldsymbol{p}_{l,i}}\right](t) \right) \quad t \in [0, T] \\ &= \left(\sum_{i=1}^{\min\{k,\mathrm{len}(\boldsymbol{p})\}} \tau_{\boldsymbol{p}_i} \left[x_{\boldsymbol{p}_i}\right](t) \right)_{\boldsymbol{p}\in\boldsymbol{P}^{c,d}_o} \qquad t \in [0, T]. \end{aligned}$$

Routing 4.15 (shortest end path and related routing). *In case the routing operator $\mathscr{R}^{c,d}_a$ for given $(a, c, d) \in \mathcal{A} \times \mathcal{C} \times \mathcal{D}$ does not depend on $\boldsymbol{x}_a$, $a \in \mathcal{A}_{\text{out}}(v)$, i.e.,*

$$\mathscr{R}^{c,d}_a[\boldsymbol{x}] \equiv \mathscr{R}^{c,d}_a\left[\tilde{\boldsymbol{x}}[\boldsymbol{x}]\right] \quad \forall \boldsymbol{x} \in C\left([0, T]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}_{\geq 0}\right)$$

with $\tilde{\boldsymbol{x}}_a[\boldsymbol{x}] :\equiv \{\begin{smallmatrix}\boldsymbol{x}_a & a\in\mathcal{A}\setminus\mathcal{A}_{\text{out}}(v)\\ \boldsymbol{0} & a\in\mathcal{A}_{\text{out}}(v)\end{smallmatrix}$ *no continuity or Lipschitz-continuity of* $\mathscr{R}$ *is required.* $\mathscr{R}$ *has only to satisfy Definition 2.10 and is not allowed to use information about the status of the network in future times to obtain a unique solution on the network. Existence and uniqueness are due to the fact that we have dynamics with a delay* > 0 *so that the independence of the routing operator from the status of the links leaving the corresponding nodes can be interpreted as a type of "delayed routing."*

One realization of such a routing operator would be considering the shortest path without paying attention to the travel time contribution of the leaving nodes, i.e., given $v \in \mathcal{V}$ *and* $d \in \mathcal{D}$ *we define* shortest end path *routing for* $t \in [0,T]$ *and* $k \in \mathbb{N}_{\geq 1}$ *as*

$$\textit{for}\ \ \tilde{a} \in \left\{\boldsymbol{p}_1 : \boldsymbol{p} \in \arg\min_{\boldsymbol{p}\in\mathcal{P}^{v,d}} \boldsymbol{\tau}^{v,d}_{\textit{end},k}\left[\boldsymbol{\mathcal{X}}^{v,d}\right](t)\right\} =: \mathcal{M}(t),\ \mathscr{R}^{c,d}_{\tilde{a}}[\boldsymbol{x},\mathbf{ext}](t) := |\mathcal{M}(t)|^{-1}$$

with $\boldsymbol{\tau}^{v,d}_{\textit{end},k}$ *similar to Definition 4.2 defined by*

$$\begin{aligned}\boldsymbol{\tau}^{v,d}_{\textit{end},k}\left[\boldsymbol{\mathcal{X}}^{v,d}\right](t) &:= \left(\sum_{i=k}^{\text{len}(\boldsymbol{p}_1)} \tau_{\boldsymbol{p}_{1,i}}\left[x_{\boldsymbol{p}_{1,i}}\right](t), \dots, \sum_{i=k}^{\text{len}(\boldsymbol{p}_l)} \tau_{\boldsymbol{p}_{l,i}}\left[x_{\boldsymbol{p}_{l,i}}\right](t)\right) \quad t \in [0,T]\\ &= \left(\sum_{i=k}^{\text{len}(\boldsymbol{p})} \tau_{\boldsymbol{p}_i}\left[x_{\boldsymbol{p}_i}\right](t)\right)_{\boldsymbol{p}\in\boldsymbol{P}^{c,d}_o} \quad t \in [0,T].\end{aligned}$$

Such a routing might be questionable from the point of view of applications as routing which is based on at most real time information should take particular care of the traffic situation in a close neighborhood as not so much can be said about the future. However, one argument for such a routing might be that all the nodes are sufficiently short so that the adjacent links to the considered nodes might be negligible.

Independent from the point of view of applications, this shows a nice mathematical property as the delay character (or hyperbolic character of having information only travelling with finite speed) automatically resolves any well-posedness problems of the ODE system.

Routing 4.16 (routing with delay). *All of the above introduced routing policies can incorporate delay by using travel time information from a previous time* $t - \tilde{t}$ *instead of current time* t*. Then, all the routings necessarily imply well-posedness and uniqueness of solutions as pointed out in Theorem 3.1, and no continuity or Lipschitz-continuity is required.*

Routing 4.17 (static user equilibrium). *Following [47] and the delay routing, we can determine the routing for a given and fixed time segment by using a stationary traffic assignment. For that, assume that we have for given* $K \in \mathbb{N}_{\geq 1}$ $\boldsymbol{t}_i \in [0,T], i \in \{1,\dots,K\}$, $\boldsymbol{t}_i < \boldsymbol{t}_{i+1}$*. Then, arriving at time* $t \in \{\boldsymbol{t}_i : i \in \{1,\dots,K\}\}$ *we know the status of the network* $\boldsymbol{x}(t)$ *and thus also the travel time* $\tau_a[x_a](t)$ *for every* $a \in \mathcal{A}$*. Then, we determine the routing between* $[\boldsymbol{t}_i, \boldsymbol{t}_{i+1})$ *as constant routing by the following optimization problem:*

Using the status of the network to determine the free-flow travel time on link $a \in \mathcal{A}$*, we can define stationary travel time functions* $\boldsymbol{tt}_a$ *as*

$$\boldsymbol{tt}_a(x) := \tau_a[x_a](t) + h_a x,\ \ a \in \mathcal{A}, x \in \mathbb{R}_{\geq 0}, \tag{4.5}$$

where h_a are the constants in Definition 2.18. As origin-destination pairs for the stationary model we use the flow located at the considered time at the node v, i.e., at $v \in \mathcal{V}$

$$\boldsymbol{d}_v^{c,d} := \boldsymbol{s}_v^{c,d}(t) + \sum_{a\in\mathcal{A}_{\text{in}}(v)} \boldsymbol{g}_a^{c,d}[\boldsymbol{u}_a, x_a, \dot{x}_a](t) \tag{4.6}$$

so that one can solve the stationary Wardrop's condition [62] by the following minimization problem:

$$\begin{aligned}
&\min_{\boldsymbol{y}} \sum_{a\in\mathcal{A}} \int_0^{\boldsymbol{y}_a} \boldsymbol{tt}_a(s)\,\mathrm{d}s, && \\
&\sum_{c\in\mathcal{C}} \boldsymbol{y}_a^c = \boldsymbol{y}_a, && a \in \mathcal{A},\\
&\textstyle\sum_{p\in\mathcal{P}^{v,d}} \boldsymbol{h}_{v,d,r}^c = \boldsymbol{d}_v^{c,d}, && (v,d) \in \mathcal{V}\times\mathcal{D},\\
&\boldsymbol{h} \geq \boldsymbol{0}, && \\
&\sum_{(v,d)\in\mathcal{V}\times\mathcal{D}} \sum_{r\in\mathcal{P}^{v,d}} \delta_{v,d,r,a}^c \boldsymbol{h}_{v,d,r}^c = \boldsymbol{y}_a, && a \in \mathcal{A}
\end{aligned}$$

with the tensor

$$\delta_{v,d,r,a}^c := \begin{cases} 1 & \text{route } r \in \mathcal{P}^{v,d} \text{ uses link } a, \\ 0 & \text{else} \end{cases} \quad a \in \mathcal{A},\ r \in \mathcal{P}^{v,d},\ (v,d) \in \mathcal{V}\times\mathcal{D}$$

and $\boldsymbol{h}_{v,d,r}^c$ the route flow between origin $v \in \mathcal{V}$ and $d \in \mathcal{D}$ for commodity $c \in \mathcal{C}$ and the route $r \in \mathcal{P}^{c,d}$. Having solved this convex optimization problem which admits a unique solution, one can obtain the routing by taking the ratios of the assigned flow, i.e., for $(c, v) \in \mathcal{C} \times \mathcal{V}$

$$\mathscr{R}_a^{c,d}(t) := \frac{\boldsymbol{y}_a^c}{\sum_{\tilde{a}\in\mathcal{A}_{\text{out}}^d(v)} \boldsymbol{y}_{\tilde{a}}^c},\ a \in \mathcal{A}_{\text{out}}^d(v).$$

Due to the proposed structure of solving these stationary problems for a constant assignment between $(\boldsymbol{t}_i, \boldsymbol{t}_{i+1})$, $i \in \{1, \ldots, K-1\}$ the framework fits into the introduced delay framework in Definition 2.12 so that we obtain by Theorem 3.1 a unique solution on the network.

One drawback of the proposed routing consists of the fact that travel time is assumed to increase affine linearly in time while in the usual stationary traffic assignment literature higher order polynomials or other significantly stronger increasing functions are considered. Clearly, we could also optimize over these functions, but to stick with affine linear increase of travel time in the framework of the time-dependent link-delay model, we chose the linear dependency. On the other hand, as the routing is redetermined after a small time (assuming that we have sufficiently many $\boldsymbol{t}_i$), this might be no problem after all.

A similar approach as in Routing 4.17 can be used to aim for the stationary social optimum at every time horizon.

Routing 4.18 (static social optimum). *Do the same as in Routing 4.17 but replace the objective function by*

$$\sum_{a\in\mathcal{A}} \int_0^{\boldsymbol{y}_a} \boldsymbol{tt}_a(s) + s\cdot \boldsymbol{tt}_a'(s)\,\mathrm{d}s = \sum_{a\in\mathcal{A}} \boldsymbol{y}_a \cdot \boldsymbol{tt}_a(\boldsymbol{y}_a).$$

Routing 4.19 (velocity- and energy-based routing). *In our analysis, we have not taken into account a dependency of the routing operator on velocity (i.e., $\dot{\boldsymbol{x}}$) or acceleration (i.e., $\ddot{\boldsymbol{x}}$ as long as it exists). This has its reason in the modeled dynamics: Having velocity showing up in the right-hand side of the ODDE system in* (2.15) *due to* (2.22) *due to the routing operator would complicate things, and the previously described approaches would not work anymore.*

For the second order derivative we would not even have existence as the entering data might not be sufficiently smooth.

However, by approximating velocity and acceleration by finite differences, we can model these impacts as well. Writing for $\delta_t \in (0,T)$

$$\begin{aligned}
\mathscr{R}_{ve}[\boldsymbol{x}](t) &:= \mathscr{R}\left[\tfrac{\boldsymbol{x}\circ \mathrm{P}_{[0,t]} - \boldsymbol{x}\circ \mathrm{P}_{[0,t-\delta_t]}}{\delta_t}\right](t), && t\in[0,T],\\
\mathscr{R}_{ac}[\boldsymbol{x}](t) &:= \mathscr{R}\left[\tfrac{\boldsymbol{x}\circ \mathrm{P}_{[0,t]} - 2\boldsymbol{x}\circ \mathrm{P}_{[0,t-\delta_t]} + \boldsymbol{x}\circ \mathrm{P}_{[0,t-2\delta_t]}}{(\delta_t)^2}\right](t), && t\in[0,T],
\end{aligned}$$

could give routing $\mathscr{R}_{ve}$ based on the average velocity and $\mathscr{R}_{as}$ based on average acceleration when $\mathscr{R}$ is any continuous or Lipschitz-continuous routing operator as in Definition 2.12.

Both variables, velocity and acceleration, represent important factors on the energy consumption. All the detailed theory, in particular Theorem 3.1, *Theorem* 3.4, *and Theorem* 3.8, *still holds true.*

Routing 4.20 (routing with memory). *Another instantiation of routing taking into account history in a more analytical framework consists of an averaging over time (nonlocal in time), i.e.,*

$$\mathscr{R}[\boldsymbol{x}](t) := \tilde{\mathscr{R}}\left[\tfrac{1}{t}\int_0^t \boldsymbol{x}(s)\gamma(s)\,\mathrm{d}s\right](t),\ \ t\in[0,T]$$

for a function $\tilde{\mathscr{R}}$ which satisfies (2.10) *and is Lipschitz-continuous or continuous. Then, Theorem* 3.8 *and Theorem* 3.4 *would still hold.*

Furthermore, one could think of memory-based/nonlocal approximation of the density within the routing operator, i.e., for $\varepsilon\in\mathbb{R}_{>0}$

$$\mathscr{R}[\boldsymbol{x}](t) := \hat{\mathscr{R}}\left[\tfrac{1}{\min\{\varepsilon,t\}}\int_{\max\{0,t-\varepsilon\}}^t \boldsymbol{x}(s)\gamma(s)\,\mathrm{d}s\right](t),\ \ t\in[0,T].$$

Routing 4.21 (routing with capacity constraints). *The DTA system presented in Definition* 2.18 *does not have any constraints w.r.t. the flow. For instance, capacity constraints on edges are not considered (as they could lead to a not well-posed system in the end). However, the routing operators allow us to also integrate capacity constraints in a "weak sense," meaning that they are followed as long as it is possible and otherwise a constant flow is assigned which might actually violate these constraints. To detail this, let $x_a^{\max}\in\mathbb{R}_{>0}, a\in\mathcal{A}$ be the capacity constraints on all the given edges in the network. Then, we define the routing at $v\in\mathcal{V}$ for $(c,d)\in\mathcal{C}\times\mathcal{D}$ as*

$$\mathscr{R}_a^{v,d}[\boldsymbol{x},\mathbf{ext}](t) := \frac{\max\{0,x_a^{\max}-x_a(t)\}+\frac{1}{|\mathcal{A}_{\text{out}}^d(v)|}}{1+\sum_{\tilde{a}\in\mathcal{A}_{\text{out}}^d(v)}\max\{0,x_{\tilde{a}}^{\max}-x_{\tilde{a}}(t)\}},\quad a\in\mathcal{A}_{\text{out}}^d(v),\ t\in[0,T].$$

Then, a proportion of traffic would be assigned equally to all the leaving arcs $a \in \mathcal{A}^d_{\text{out}}(v)$ *and another proportion dependent on the congestion state of the traffic, i.e., the measure* $x_a^{\max} - x_a(t)$*. If the latter expression is greater zero, flow would still be assigned, and the larger it is—the less traffic flow on the specific edge—the more flow would be assigned to the edge. Clearly, this operator is Lipschitz-continuous in the sense of Theorem 3.4 so that we obtain a unique solution on the entire network. One could again interpret it as a smoothed version of Routing 4.6.*

Routing 4.22 (routing explicitly depending on externalities). *Since the state and the capacity of several roads can change in course of time due to weather conditions, accidents, road blockades, or other events, it is reasonable to consider in every presented routing these time-dependent effects as a direct impact on the respective arc. Modeling these impacts, one could, for instance, realize them by a damping of the impact of the solution on the considered road. Thus, our routing operators* $\mathscr{R}_a^{c,d}$ *could be formulated by*

$$\mathscr{R}[\boldsymbol{x}, \mathbf{ext}] :\equiv \tilde{\mathscr{R}}[\operatorname{diag}(\mathbf{ext})\boldsymbol{x}],$$

where $\tilde{\mathscr{R}}$ *is an appropriate operator,* $\mathbf{ext} \in L^\infty((0,T); \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$*, and* $t \mapsto \operatorname{diag}(\mathbf{ext}(t))$ *is a diagonal matrix–valued function whose diagonal entries are the components of* $\mathbf{ext}$*.*

As the routing operator depends on $\mathbf{ext}$*, it is also possible to use* $\mathbf{ext}$ *for changing the network topology after a given amount of time, i.e., the previously described road accident, etc. might result in a shut-down of a specific road over a given time horizon. However, this is only possible if the feasibility of the network is still guaranteed, i.e., there still exists a route between every* $\mathcal{OD}$ *pair.*

To conclude this section, we provide the already mentioned lemma concerning the measurability of shortest path routing and an example which shows that the a priori link assignment defined within this routing has to be adapted w.r.t. the network parameters. Otherwise, we can obtain an ill-posed behavior of the flow. It is clear that the results can analogously be claimed also for the shortest next link routing.

Lemma 4.23 (measurability of $\arg\min$-mappings). *Let* $n \in \mathbb{N}_{\geq 1}$*,* $\mathcal{M} := \{1, \dots, n\}$*, and* $f_1, \dots, f_n \in C([0,T])$*. Then, the mapping* $[0,T] \ni t \mapsto |\arg\min_{i \in \mathcal{M}} f_i(t)|$ *is Lebesgue-measurable.*

Proof. Since we have a finite number of functions, we have $\arg\min_{i \in \mathcal{M}} f_i(t) \neq \emptyset$ for every $t \in [0,T]$. Now, let $k \in \mathcal{M}$ be arbitrary. Then, we obtain after defining $i^{\widetilde{\mathcal{M}}} := \min\{i : \ i \in \widetilde{\mathcal{M}}\}$ for $\widetilde{\mathcal{M}} \subset \mathcal{M}$

$$\begin{aligned}
&\left\{ t \in [0,T] : \ \left| \arg\min_{i \in \mathcal{M}} f_i(t) \right| = k \right\} \\
&= \bigcup_{\substack{\widetilde{\mathcal{M}} \subset \mathcal{M} \\ |\widetilde{\mathcal{M}}| = k}} \left(\bigcup_{i \in \widetilde{\mathcal{M}}} \{ t \in [0,T] : \ f_{i^{\widetilde{\mathcal{M}}}}(t) = f_i(t) \} \cap \bigcap_{j \in \mathcal{M} \setminus \widetilde{\mathcal{M}}} \{ t \in [0,T] : \ f_{i^{\widetilde{\mathcal{M}}}}(t) < f_j(t) \} \right).
\end{aligned}$$

All of these subsets of $[0,T]$ are Borel-measurable due to the continuity of the functions $f_1, \dots, f_n$. Thus, $t \mapsto |\arg\min_{i \in \mathcal{M}} f_i(t)|$ is Lebesgue-measurable. ∎

We conclude this section with a remark concerning the fact that a shortest path assignment in real time is unstable or even contradicting itself:

Remark 4.24 (instability/well-posedness of a shortest path routing operator). Consider again the simple network illustrated in Figure 4 with $T \in \mathbb{R}_{>0}$, $\mathcal{C} := \{1\}$, $\mathcal{O} := \{v_1, v_2, v_3\}$, $\mathcal{D} := \{v_4\}$, and define

$$\begin{aligned} h_{a_1} &:= 1, h_{a_2} := 3, h_{a_3} := 4, h_{a_4} := 2, & b_{a_1} &:= b_{a_2} := b_{a_3} := b_{a_4} := 1,\\ \boldsymbol{s}_{v_1}(t) &:= \boldsymbol{s}_{v_2}(t) := \boldsymbol{s}_{v_3}(t) := 1 & \forall t &\in [0, T]. \end{aligned}$$

Now, simplify the notation of the routing $\mathscr{R}_{a_1}^{v_1,v_4}$ in v_1 in the following way. Since we consider only one commodity and only one destination node, we omit the notation of the superscripts $1, v_4$. Next, since real routing can only happen at v_1 we also omit super index v_1. Furthermore, we assume that traffic is routed independent of **ext** and omit this dependency as well. We define a general shortest routing between v_1 and v_4 for $a \in \{a_1, a_2\}$ as $\mathscr{R}_a[\boldsymbol{x}] \in L^\infty((0, T); [0, 1])$ with $\mathscr{R}_{a_1}[\boldsymbol{x}](t) + \mathscr{R}_{a_2}[\boldsymbol{x}](t) = 1$ for every $t \in [0, T]$ and additionally

$$\mathscr{R}_a[\boldsymbol{x}](t) = \begin{cases} 0 & \text{if } a \notin \mathcal{M}(t),\\ 1 & \text{if } \mathcal{M}(t) = \{a\} \end{cases}$$

with

$$\mathcal{M}(t) := \Big\{\boldsymbol{p}_1 \in \{a_1, a_2\} : \boldsymbol{p} \in \arg\min_{\boldsymbol{p} \in \mathcal{P}^{v_1,v_4}} \boldsymbol{\tau}^{v_1,v_4}[\boldsymbol{\mathcal{X}}^{v_1,v_4}](t)\Big\}.$$

Plugging these data in Definition 2.18 we obtain, besides the initial conditions $\boldsymbol{x}_{a_1}(0) = \boldsymbol{x}_{a_2}(0) = 0$, the ODE system

$$\begin{aligned} \dot{\boldsymbol{x}}_{a_1}(t) &= \mathscr{R}_{a_1}[\boldsymbol{x}](t) - \boldsymbol{g}_{a_1}[\boldsymbol{u}_{a_1}, x_{a_1}, \dot{x}_{a_1}](t),\\ \dot{\boldsymbol{x}}_{a_2}(t) &= \big(1 - \mathscr{R}_{a_1}[\boldsymbol{x}](t)\big) - \boldsymbol{g}_{a_2}[\boldsymbol{u}_{a_2}, x_{a_2}, \dot{x}_{a_2}](t). \end{aligned}$$

Now, we show what structure the instantaneous shortest path routing imposes (at least for small times) such that the upper ODE system (together with the ODEs for $\boldsymbol{x}_{a_3}, \boldsymbol{x}_{a_4}$) holds a solution. And more, we illustrate that an even split of the inflow via the already mentioned routing in general will not produce a solution of the network model. For $t \in [0, T]$ we define

$$\tilde{r}(t) := \int_0^t \mathscr{R}_{a_1}[\boldsymbol{x}](s)\,\mathrm{d}s, \qquad \tilde{g}_a(t) := \int_0^t \boldsymbol{g}_a[\boldsymbol{u}_a, x_a, \dot{x}_a](s)\,\mathrm{d}s, \quad a \in \{a_1, a_2, a_3, a_4\},$$

where we omitted for readability reasons the functional dependencies incorporated in the respective integrands in $\tilde{r}$,$\tilde{g}$. We obtain them by rearranging of terms and adding equal terms in both elements of the set where arg min operates on

$$\begin{aligned} \mathcal{M}(t) &= \arg\min\big\{\tau_{a_1}[x_{a_1}](t) + \tau_{a_3}[x_{a_3}](t)\ ,\ \tau_{a_2}[x_{a_2}](t) + \tau_{a_4}[x_{a_4}](t)\big\}\\ &= \arg\min\big\{(h_{a_1} + h_{a_2})\tilde{r}(t)\ ,\ h_{a_1}\tilde{g}_{a_1}(t) - h_{a_3}(\tilde{g}_{a_1}(t) + t - \tilde{g}_{a_3}(t)) \qquad (4.7)\\ &\qquad + h_{a_2}(t - \tilde{g}_{a_2}(t)) + h_{a_4}(\tilde{g}_{a_2}(t) + t - \tilde{g}_{a_4}(t))\big\}. \qquad (4.8) \end{aligned}$$

For $t \in [0,1)$ (4.7)–(4.8) simplifies to

$$\arg\min\{(h_{a_1}+h_{a_2})\tilde{r}(t),(h_{a_2}+h_{a_4}-h_{a_3})t\} = \arg\min\{4\tilde{r}(t),t\}.$$

Now, since $4\tilde{r}$ and $(t \mapsto t)$ are absolutely continuous and they are both 0 in $t=0$, it can be seen by a contraposition argument that $4\tilde{r}(t)=t$ has to be fulfilled for every $t \in [0,1)$. Therefore, after differentiating both sides of the equation $4\tilde{r}(t)=t$, we obtain for $t \in [0,1)$

$$\mathscr{R}_{a_1}[\boldsymbol{x}](t) = \tfrac{h_{a_2}+h_{a_4}-h_{a_3}}{h_{a_1}+h_{a_2}} = \tfrac{1}{4}$$

which yields

$$\rho_{a_1}[x_{a_1}](t) = 1+\tfrac{5}{4}t,\ \rho_{a_2}[x_{a_2}](t) = 1+\tfrac{13}{4}t,\ \rho_{a_3}[x_{a_3}](t) = 1+5t,\ \rho_{a_4}[x_{a_4}](t) = 1+3t,$$

and for $t \in [1,\frac{9}{4})$ we obtain

$$\tilde{g}_{a_1}(t) = \int_0^{\rho_{a_1}[x_{a_1}]^{-1}(t)} \tfrac{1}{4}\,\mathrm{d}s = \tfrac{1}{5}(t-1), \qquad \tilde{g}_{a_2}(t) = \int_0^{\rho_{a_2}[x_{a_2}]^{-1}(t)} \tfrac{3}{4}\,\mathrm{d}s = \tfrac{13}{3}(t-1),$$
$$\tilde{g}_{a_3}(t) = \int_0^{\rho_{a_3}[x_{a_3}]^{-1}(t)} 1\,\mathrm{d}s = \tfrac{1}{5}(t-1), \qquad \tilde{g}_{a_4}(t) = \int_0^{\rho_{a_4}[x_{a_4}]^{-1}(t)} 1\,\mathrm{d}s = \tfrac{1}{3}(t-1).$$

Thus, (4.8) simplifies to

$$\begin{aligned}&\arg\min\left\{4\tilde{r}(t), t+\tfrac{1}{5}h_{a_1}(t-1)+\tfrac{13}{3}(h_{a_4}-h_{a_2})(t-1)-\tfrac{1}{3}h_{a_4}(t-1)\right\}\\ &= \arg\min\left\{4\tilde{r}(t), \tfrac{59}{195}t+\tfrac{136}{195}\right\}.\end{aligned}$$

Obviously, $4\tilde{r}(1)=1$, and analogously to the case $t \in [0,1)$, we obtain for $t \in [1,\frac{9}{4})$ that this equality must also hold for $t \in [1,\frac{9}{4})$ such that it has to hold $\mathscr{R}_{a_1}[\boldsymbol{x}](t) = \frac{59}{780}$ for $t \in [1,\frac{9}{4})$.

To sum it up, even if $\mathcal{M}(t) = \{a_1,a_2\}$ for every $t \in [0,\frac{9}{4})$, an even split of the inflow in the first node will not lead to a solution for shortest path routing. Even more, the necessary flow ratio which would need to be assigned is a function of the parameters of all involved links (and also sensitive w.r.t. additional inflow and commodity).

Thus, any kind of instantaneous shortest path assignment, if not properly "tuned," will result in an unstable behavior and contradict itself.

5. Conclusions.

The main contribution of this work is the mathematical framework to formalize nonanticipative routing operators in networks for which the routing is determined dynamically by information patterns capable of ingesting the state of each link and features constructed from it (such as its past, or its forecast). As demonstrated in the article, the corresponding mathematical framework necessary to provide existence and uniqueness proofs for the solutions to the problems are complex, even for a simplified link traffic flow model such as the link-delay model. While this model is a crude representation of vehicular traffic, it is important to note that the extension of such proofs for more complex traffic networks is not straightforward, in particular for spatiotemporal models such as PDEs commonly used in traffic flow theory. These difficulties would occur even more with multicommodity models inherently needed for DTA models. In follow-up work, we will extend the properties outlined in the current article to more complex models (in particular convection PDEs) and validate relevance on data obtained from microsimulation tools, calibrated from experimental data for the I210 corridor in California [32].

REFERENCES

[1] A. Ashwell, 9 *investigates traffic apps pushing drivers into neighborhood streets*, WSOC-TV (Channel 9), 2016.

[2] D. Bagby, *Reporter analysis: Waze directions send traffic through quiet streets*, Reporter Newspapers, 2016.

[3] X. J. Ban, J.-S. Pang, H. X. Liu, and R. Ma, *Modeling and solving continuous-time instantaneous dynamic user equilibria: A differential complementarity systems approach*, Transp. Res. B Methodol., 46 (2012), pp. 389–408.

[4] A. M. Bayen, P. Goatin, A. Keimer, and N. Laurent-Brouty, *Well-posedness of conservation laws on networks with finite buffers*, Commun. Math. Sci., to appear.

[5] M. Ben-Akiva, M. Bierlaire, J. Bottom, H. Koutsopoulos, and R. Mishalani, *Development of a route guidance generation system for real-time application*, in Proceedings of the 8th IFAC Symposium on Transportation Systems, 1997.

[6] M. Ben-Akiva, M. Bierlaire, H. Koutsopoulos, and R. Mishalani, *Dynamit: A simulation-based system for traffic prediction*, in Proceedings of the DACCORD Short Term Forecasting Workshop, 1998, pp. 1–12.

[7] D. E. Boyce, B. Ran, and L. J. Leblanc, *Solving an instantaneous dynamic user-optimal route choice model*, Transp. Sci., 29 (1995), pp. 128–142.

[8] R. Bradley, *Waze and the traffic panopticon*, The New Yorker, 2015.

[9] A. Bressan and K. T. Nguyen, *Conservation law models for traffic flow on a network of roads*, Netw. Heterog. Media, 10 (2015), pp. 255–293.

[10] A. Bressan and K. T. Nguyen, *Optima and equilibria for traffic flow on networks with backward propagating queues*, Netw. Heterog. Media, 10 (2015), pp. 717–748.

[11] G. Bretti, R. Natalini, and B. Piccoli, *Numerical approximations of a traffic flow model on networks*, Netw. Heterog. Media, 1 (2006), pp. 57–84.

[12] H. Brezis, *Functional Analysis, Sobolev Spaces and Partial Differential Equations*, Universitext, Springer, New York, 2011.

[13] T. Cabannes, M. Sangiovanni, A. Keimer, and A. M. Bayen, *Regrets in routing networks: Measuring the impact of routing apps in traffic*, ACM Trans. Spat. Algorithms Syst., 5 (2019), p. 9.

[14] T. Cabannes, F. Shyu, E. Porter, S. Yao, Y. Wang, M. A. S. Vincentelli, S. Hinardi, M. Zhao, and A. M. Bayen, *Measuring regret in routing: assessing the impact of increased app usage*, in Proceedings of the 21st International Conference on Intelligent Transportation Systems (ITSC), IEEE, 2018, pp. 2589–2594.

[15] H.-K. Chen, *Dynamic Travel Choice Models: A Variational Inequality Approach*, Springer Science & Business Media, New York, NY, 2012.

[16] H.-K. Chen and C.-F. Hsueh, *A model and an algorithm for the dynamic user-optimal route choice problem*, Transp. Res. B Methodol., 32 (1998), pp. 219–234.

[17] R. D. Driver, *Ordinary and Delay Differential Equations*, Springer Science & Business Media, New York, NY, 2012.

[18] T. L. Friesz, D. Bernstein, T. E. Smith, R. L. Tobin, and B.-W. Wie, *A variational inequality formulation of the dynamic network user equilibrium problem*, Oper. Res., 41 (1993), pp. 179–191.

[19] T. L. Friesz, K. Han, P. A. Neto, A. Meimand, and T. Yao, *Dynamic user equilibrium based on a hydrodynamic model*, Transp. Res. B Methodol., 47 (2013), pp. 102–126.

[20] T. L. Friesz, J. Luque, R. L. Tobin, and B.-W. Wie, *Dynamic network traffic assignment considered as a continuous time optimal control problem*, Oper. Res., 37 (1989), pp. 893–901.

[21] M. Garavello, K. Han, and B. Piccoli, *Models for Vehicular Traffic on Networks*, American Institute of Mathematical Sciences, Springfield, MO, 2016.

[22] M. Garavello and B. Piccoli, *Traffic Flow on Networks*, American Institute of Mathematical Sciences, Springfield, MO, 2006.

[23] M. O. Ghali and M. J. Smith, *A model for the dynamic system optimum traffic assignment problem*, Transp. Res. B Methodol., 29 (1995), pp. 155–170.

[24] M. Gugat, A. Keimer, G. Leugering, and Z. Wang, *Analysis of a system of nonlocal conservation laws for multi-commodity flow on networks*, Netw. Heterog. Media, 10 (2015), pp. 749–785.

[25] J. K. Hale, *Ordinary Differential Equations*, Robert E. Krieger Publishing Co., New York, NY, 1980.
[26] K. Han, T. L. Friesz, and T. Yao, *Existence of simultaneous route and departure choice dynamic user equilibrium*, Transp. Res. B Methodol., 53 (2013), pp. 17–30.
[27] K. Han, T. L. Friesz, and T. Yao, *A partial differential equation formulation of Vickrey's bottleneck model, part* I*: Methodology and theoretical analysis*, Transp. Res. B Methodol., 49 (2013), pp. 55–74.
[28] K. Han, T. L. Friesz, and T. Yao, *A partial differential equation formulation of Vickrey's bottleneck model, part* II*: Numerical analysis and computation*, Transp. Res. B Methodol., 49 (2013), pp. 75–93.
[29] S. Hendrix, *Traffic-weary homeowners and Waze are at war, again. Guess who's winning?*, The Washington Post, 2016.
[30] J. K. Ho, *A successive linear optimization approach to the dynamic traffic assignment problem*, Transp. Sci., 14 (1980), pp. 295–305.
[31] H. Holden and N. H. Risebro, *A mathematical model of traffic flow on a network of unidirectional roads*, SIAM J. Math. Anal., 26 (1995), pp. 999–1017.
[32] Berkeley ITS/PATH, University of California, https://connected-corridors.berkeley.edu, 2018.
[33] B. N. Janson, *Convergent algorithm for dynamic traffic assignment*, Transp. Res. Rec., 1328 (1991), pp. 69–80.
[34] B. N. Janson, *Dynamic traffic assignment for urban road networks*, Trans. Res. B Mathodol., 25 (1991), pp. 143–161.
[35] A. Keimer, N. Laurent-Brouty, F. Farokhi, H. Signargout, V. Cvetkovic, A. M. Bayen, and K. H. Johansson, *Information patterns in the modeling and design of mobility management services*, Proc. IEEE, 106 (2018), pp. 554–576.
[36] A. Keimer, L. Pflug, and M. Spinola, *Nonlocal scalar conservation laws on bounded domains and applications in traffic flow*, SIAM J. Math. Anal., 50 (2018), pp. 6271–6306.
[37] T. Knight, *Beset by traffic via navigation apps, Leonia restricts access*, Forward, 2018.
[38] J. Long, W. Y. Szeto, H.-J. Huang, and Z. Gao, *An intersection-movement-based stochastic dynamic user optimal route choice model for assessing network performance*, Transp. Res. B Methodol., 74 (2015), pp. 182–217.
[39] R. Ma, X. J. Ban, and J.-S. Pang, *Continuous-time dynamic system optimum for single-destination traffic networks with queue spillbacks*, Transp. Res. B Methodol., 68 (2014), pp. 98–122.
[40] H. S. Mahmassani and R. Herman, *Dynamic user equilibrium departure time and route choice on idealized traffic arterials*, Transp. Sci., 18 (1984), pp. 362–384.
[41] A. Marshall, *There are better ways to kill traffic than lying to Waze*, WIRED, 2016.
[42] M. Mccarty, *The road less traveled? Not since Waze came to Los Angeles*, NPR - All Things Considered, 2016.
[43] D. K. Merchant and G. L. Nemhauser, *A model and an algorithm for the dynamic traffic assignment problems*, Transp. Sci., 12 (1978), pp. 183–199.
[44] D. K. Merchant and G. L. Nemhauser, *Optimality conditions for a dynamic traffic assignment model*, Transp. Sci., 12 (1978), pp. 200–207.
[45] A. Nagurney and D. Zhang, *Projected Dynamical Systems and Variational Inequalities with Applications*, Springer Science & Business Media, New York, NY, 2012.
[46] X. Nie and H. M. Zhang, *Delay-function-based link models: their properties and computational issues*, Transp. Res. B Methodol., 39 (2005), pp. 729–751.
[47] M. Patriksson, *The Traffic Assignment Problem: Models and Methods*, Courier Dover Publications, Mineola, NY, 2015.
[48] R. Paulas, *For the good of society – and traffic! – delete your map app*, selectall, 2017.
[49] S. Peeta and H. S. Mahmassani, *System optimal and user equilibrium time-dependent traffic assignment in congested networks*, Ann. Oper. Res., 60 (1995), pp. 81–113.
[50] S. Peeta and T.-H. Yang, *Stability issues for dynamic traffic assignment*, Automatica, 39 (2003), pp. 21–34.
[51] S. Peeta and A. K. Ziliaskopoulos, *Foundations of dynamic traffic assignment: The past, the present and the future*, Netw. Spat. Econ., 1 (2001), pp. 233–265.
[52] B. Ran and D. E. Boyce, *A link-based variational inequality formulation of ideal dynamic user-optimal route choice problem*, Transp. Res. C Emerging Tech., 4 (1996), pp. 1–12.

[53] B. Ran and D. E. Boyce, *Modeling Dynamic Transportation Networks: An Intelligent Transportation System Oriented Approach*, Springer, Berlin, 1996.

[54] B. Ran and D. E. Boyce, *Dynamic Urban Transportation Network Models: Theory and Implications for Intelligent Vehicle-Highway Systems*, Springer Science & Business Media, New York, Ny, 2012.

[55] B. Ran, D. E. Boyce, and L. J. LeBlanc, *A new class of instantaneous dynamic user-optimal traffic assignment models*, Oper. Res., 41 (1993), pp. 192–202.

[56] B. Ran and T. Shimazaki, *A general model and algorithm for the dynamic traffic assignment problems*, in Transport Policy, Management & Technology Towards 2001: Selected Proceedings of the Fifth World Conference on Transport Research, 1989.

[57] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, Texts Appl. Math. 57, Springer-Verlag, New York, NY, 2011.

[58] M. J. Smith, *A new dynamic traffic model and the existence and calculation of dynamic user equilibria on congested capacity-constrained road networks*, Transp. Res. B Methodol., 27 (1993), pp. 49–63.

[59] Staff, *Ashcroft residents work to stop Waze 'craze' traffic*, WEHO ville, 2015.

[60] A. Walker, *Is it really possible to trick Waze to keep traffic off your street?*, Gizmodo, 2014.

[61] S. T. Waller and A. K. Ziliaskopoulos, *A chance-constrained based stochastic dynamic traffic assignment model: Analysis, formulation and solution algorithms*, Transp. Res. C Emerging Tech., 14 (2006), pp. 418–427.

[62] J. G. Wardrop, *Road paper. Some theoretical aspects of road traffic research*, Proc. Inst. Civil Eng., 1 (1952), pp. 325–362.

[63] Y. W. Xu, J. H. Wu, M. Florian, P. Marcotte, and D. L. Zhu, *Advances in the continuous dynamic network loading problem*, Transp. Sci., 33 (1999), pp. 341–353.

[64] E. Zeidler, *Nonlinear Functional Analysis,* I. *Fixed-Point Theorems*, Springer-Verlag, New York, 1986.

[65] E. Zeidler, *Applied Functional Analysis: Applications to Mathematical Physics*, Appl. Math. Sci., Springer, New York, NY, 1995.

[66] N. Zeveloff, *Israelis sue Waze navigation app for creating neighborhood traffic jam*, Forward, 2016.