Lax–Hopf Based Incorporation of Internal Boundary Conditions Into Hamilton-Jacobi Equation. Part II: Computational Methods

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Abstract—This article presents a new method for explicitly computing solutions to a Hamilton–Jacobi partial differential equation for which initial, boundary and internal conditions are prescribed as piecewise affine functions. Based on viability theory, a Lax–Hopf formula is used to construct analytical solutions for the individual contribution of each affine condition to the solution of the problem. The results are assembled into a Lax–Hopf algorithm which can be used to compute the solution to the partial differential equation at any arbitrary time at no other cost than evaluating a semi-analytical expression numerically. The method being semi-analytical, it performs at machine accuracy (compared to the discretization error inherent to finite difference schemes). The performance of the method is assessed with benchmark analytical examples. The running time of the algorithm is compared with the running time of a Godunov scheme.

Index Terms—Hamilton–Jacobi (HJ), initial conditions (ICs), Lax–Hopf formula, partial differential equation (PDE), piecewise affine (PWA), terminal conditions (TCs).

I. INTRODUCTION

A. Motivation and Background

T HE computation of numerical solutions to the Hamilton-Jacobi (HJ) partial differential equation (PDE) is a topic which has generated significant interest in the control and numerical analysis community. Most notably, solutions to HJ PDEs can be computed using level-set methods [40], [41], [43], [44], fast-marching methods [49], semi-lagrangian schemes [27], finite volume or finite element schemes [10], [13], [34], or viability schemes [14], [15], [48]. Some of the challenges which the solutions to these equations exhibit include kinks or discontinuities, which can lead to numerical difficulties for their computations. Numerous methods and numerical schemes have been proposed to solve these issues. By the nature of the problems in which these equations appear (control theory, differential games, fluid mechanics, vision, etc.), it is common for numerical frameworks developed to

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solve them to integrate boundary conditions (BCs) and *initial conditions* (ICs) or sometimes terminal conditions (TCs) for example in control theory [7], [35], [41].

However, the integration of initial or boundary conditions alone is not sufficient to solve new data reconstruction problems arising in the context of transportation engineering. This is in particular true for traffic monitoring systems, which are a specific example of a cyberphysical system, i.e., a system governed by both "physics" (flow of vehicles) and information, "cyber" flowing through it (in the present case about the state of traffic). In particular, the Global Positioning System (GPS) technology is progressively penetrating the smartphone fleet in use [33], [51], potentially enabling the ubiquitous mobile monitoring of transportation systems [25] in the near future. One key property of this monitoring paradigm is the sensing, which is Lagrangian (happening onboard the phone traveling in a car). In contrast, large scale distributed parameter systems such as the transportation network are traditionally monitored using Eulerian (fixed) sensors. Lagrangian sensors are attractive for the transportation infrastructure because their deployment does not rely on the usual costs of a public monitoring infrastructure such as the current loop detectors embedded in the pavement, which comes with maintenance costs.

The Lagrangian traffic measurements have to be integrated into a PDE through internal boundary conditions (IBCs), which are internal to the spatio-temporal domain of definition of these PDEs. Internal boundary conditions are less commonly used than initial or boundary conditions. This is mostly due to the fact that few classical problems in the aforementioned fields (control theory, differential games, fluid mechanics, vision, etc.) typically include moving data inside the physical domain. However, numerous systems nowadays typically include infrastructure which integrates Eulerian (fixed-control volume based) or Lagrangian (mobile-trajectory based) sensors. The fundamental challenge of integrating these different types of sensing data is the proper use of a constitutive model of the system, which should be able to integrate both types of data. The process of integrating Eulerian or Lagrangian sensing data into a flow model is called data assimilation or inverse modeling [36], [45], for which several approaches exist that include variational data assimilation [42], Newtonian relaxation [32], [36], [45] or Kalman filtering and its extensions [26], [37].

In a companion article [16], we described a model capable of mathematically handling initial, boundary and internal boundary conditions for the HJ PDE. The article develops a theoretical framework capable of handling both types of data (Eulerian and Lagrangian). However, the computation of numerical solutions for problems involving internal boundary conditions, unless instantiated for specific types of such conditions, still relies on traditional methods such as the ones mentioned earlier (level set methods, fast marching methods, or numerical viability methods), which, in the general case, do not leverage the specific form of the solution in the numerical computation (therefore leading to necessary approximation or numerical error). Because of the semi-analytical nature of the results outlined in the companion article [16], it is possible to compute solutions for the class of investigated HJ PDEs exactly (i.e., up to the machine accuracy of computing an analytical function numerically) for several classes of ICs, BCs and internal boundary conditions using a semi-analytically. A semi-analytical algorithm computes the solution analytically using a basic mathematical operation (minimization in the present case) on analytically computed parameters (i.e., parameters which are computed using a closed form expression).

Indeed, the present article extends the results of [16] to compute the analytical expression of the solution to the HJ PDE in the case in which all the conditions are given in piecewise affine (PWA) form. PWA conditions are important in engineering for several reasons. First, assuming PWA data for the initial and boundary conditions is common¹ in numerical analysis [38]. Second, the possibility of computing analytical solutions of PWA problems is of great interest in the control community, as it is a common way to model nonlinearities in systems governed by dynamical systems, see for example [9]. The specificity of the present article is that it finds an exact solution to the HJ PDE for PWA conditions, using the earlier contributions of [16], by instantiating the resulting optimization problems explicitly and computing their solutions exactly. Leveraging the tools developed in earlier work, the proposed method bypasses the need to construct a computational grid: by the analytical nature of the solution, it enables the computation of the solution at any arbitrary time directly. This is obviously a significant computational advantage over finite difference schemes. Previous alternate approaches included dynamic programming methods [23], [24], but the resulting solution could not be proved to be exact in general. Note that while the article makes the assumption that all ICs, BCs and internal boundary conditions are PWA, the model used for the HJ PDE can have an arbitrary concave (respectively convex) Hamiltonian, i.e., it does not need to be PWA. In particular, we give examples with quadratic Hamiltonians.

B. Mathematical Framework Used in This Article

This section summarizes the companion article [16] and can be omitted by readers familiar with [16]. In [16], we investigate the solution to the following *Moskowitz* HJ PDE:

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi \left(-\frac{\partial \mathbf{M}(t,x)}{\partial x} \right) = 0.$$
(1)

In the above equation, $\psi(\cdot)$ is a concave function defined on $[0, \omega]$ known as the *Hamiltonian* [3], [16], [50]. We assume that the Hamiltonian satisfies $\psi'(0) = \nu^{\flat}$ and $\psi'(\omega) = -\nu^{\sharp}$, where $\nu^{\flat} > 0$ and $\nu^{\sharp} > 0$, which implicitly assumes that $\psi(\cdot)$ is differentiable at 0 and ω . However, we do not assume that $\psi(\cdot)$

is differentiable² on $]0, \omega[$, and construct our analysis for this general set of concave $\psi(\cdot)$ functions.

Equation (1) can also be viewed as an integral formulation of the following first order hyperbolic conservation law [29], [38] describing the evolution of a density function ρ , for which we present new results later in the article

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} = 0.$$
 (2)

The formal link between the density function $\rho(\cdot, \cdot)$ and the Moskowitz function $\mathbf{M}(\cdot, \cdot)$ is given by

$$\mathbf{M}(t_2, x_2) - \mathbf{M}(t_1, x_1) = \int_{x_1}^{x_2} -\rho(t_1, x)dx + \int_{t_1}^{t_2} \psi(\rho(t, x_2))dt.$$
 (3)

In the context of transportation engineering, (2) is known as the *Lighthill-Whitham-Richards* (LWR) PDE [39], [46].

We define the set X as $X := [\xi, \chi] \subset \mathbb{R}$ where ξ represents the upstream boundary and χ represents the downstream boundary of the computational domain. Let $\mathcal{M}_0(\cdot, \cdot)$, $\gamma(\cdot, \cdot)$, $\beta(\cdot, \cdot)$ and $(\mu_p(\cdot, \cdot))_{p \in P}$ (where P is a finite set) be given lower semicontinuous functions from $\mathbb{R}_+ \times X$ to \mathbb{R} with the following domains of definition:

$$\begin{cases} \operatorname{Dom} (\mathcal{M}_0) := \{0\} \times X \\ \operatorname{Dom} (\gamma) := \mathbb{R}_+ \times \{\xi\} \\ \operatorname{Dom} (\beta) := \mathbb{R}_+ \times \{\chi\} \\ \operatorname{Dom} (\mu_p) \subset \mathbb{R}_+^* \times]\xi, \chi[. \end{cases}$$
(4)

In the above equation, \mathbb{R}_+ (respectively \mathbb{R}_+^*) denotes the set of positive (respectively strictly positive) real numbers.

Definition 1.1: (Mixed Initial-Boundary-Internal Boundary Conditions Problem): We are looking for the following solution **M** to the mixed initial-boundary-internal boundary conditions problem:

 \mathbf{M} is solution to the HJ PDE (1) in the

Barron – Jensen/Frankowska sense $\mathbf{M}(0, x) = \mathcal{M}_0(0, x) \quad \forall x \in X$ Initial condition $\mathbf{M}(t, \xi) = \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+$ Upstream boundary condition $\mathbf{M}(t, \chi) = \beta(t, \chi) \quad \forall t \in \mathbb{R}_+$ Downstream boundary condition $\mathbf{M}(t, x) = \mu_p(t, x) \quad \forall t \in \mathbb{R}_+$ Internal boundary conditions. (5)

The proper notion of weak solution used in the present article is the Barron-Jensen/Frankowska solution [6], [28]. The key feature of this weak solution (also used in [16], based on [3]) is the *lower semicontinuity* property. The link between this class of weak solutions and the viscosity solutions [5], [19], [20] was formally established by Frankowska [28].

One of the fundamental contributions of the present article as well as [16] is the use of control theoretic methods (in the present case viability theory [1] and set-valued analysis [4]) to

¹Obviously advanced finite difference and finite volume schemes can use more complex representation of the numerical solution on the discretization grid.

²Since $\psi(\cdot)$ is concave, it is differentiable almost everywhere [11], [47] on $]0, \omega[$. In the context of transportation engineering, the vast majority of articles make the assumption that ψ is piecewise affine (triangular). Therefore, this assumption (non-differentiability) is important to make for practical applications.

construct the proper solutions to the problem (5) of Definition 1.1. This solution is called the *viability episolution*, and is based on the concept of *capture basins*.

Definition 1.2: [1], [2] (Capture Basin): Given a dynamical system F and two sets \mathcal{K} (called the constraint set) and \mathcal{C} (called the target set) satisfying $\mathcal{C} \subset \mathcal{K}$, the capture basin $\operatorname{Capt}_F(\mathcal{K}, \mathcal{C})$ is the subset of states of \mathcal{K} from which there exists at least one evolution solution of F reaching the target \mathcal{C} in finite time while remaining in \mathcal{K} .

To construct the auxiliary dynamical system F used for the computation of the viability episolution, we need to define a convex transform of the Hamiltonian $\psi(\cdot)$ as follows:

Definition 1.3: (Convex Transform): For a concave function $\psi(\cdot)$ defined as previously, the convex transform φ^* is given by

$$\varphi^*(u) := \begin{cases} \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)] & \text{if } u \in [-\nu^\flat, \nu^\sharp] \\ +\infty & \text{otherwise.} \end{cases}$$
(6)

The function $\varphi^*(\cdot)$ is convex as the pointwise supremum of affine functions [11], [47], and is defined on the interval $\operatorname{Dom}(\varphi^*) := [-\nu^{\flat}, \nu^{\ddagger}]$. See also [12], [18] for additional information on the Legendre-Fenchel transform and algorithms for its numerical computation. Note that since $\psi(\cdot)$ is concave and satisfies $\psi'(0) = \nu^{\flat}$, the function $\varphi^*(\cdot)$ satisfies $\varphi^*(-\nu^{\flat}) := \sup_{p \in \operatorname{Dom}(\psi)} [-p\nu^{\flat} + \psi(p)] = 0$. Since $\psi(0) = 0$ and $0 \in [0, \omega]$, we have by definition (6) that $\varphi^*(\cdot) \ge 0$. Since $\varphi^*(\cdot)$ is convex, it is subdifferentiable [11] on $[-\nu^{\flat}, \nu^{\ddagger}]$, and its subderivative satisfies the Legendre–Fenchel inversion formula [3]

$$u \in -\partial_+\psi(\rho)$$
 if and only if $\rho \in \partial_-\varphi^*(u)$. (7)

Following [11], we use the following definition of the subderivative $\partial_{-}(\cdot)$ and the superderivative $\partial_{+}(\cdot)$:

$$v \in \partial_{-}\mathcal{F}(x_{0}) \text{ if and only if } \quad \forall x \in \text{Dom}(\mathcal{F}),$$

$$\mathcal{F}(x) \geq \mathcal{F}(x_{0}) + v(x - x_{0}) \qquad (8)$$

$$v \in \partial_{+}\mathcal{F}(x_{0}) \text{ if and only if } \quad \forall x \in \text{Dom}(\mathcal{F}),$$

$$\mathcal{F}(x) \leq \mathcal{F}(x_{0}) + v(x - x_{0}). \qquad (9)$$

Note that any convex (respectively concave) function $\mathcal{F}(\cdot)$ is subdifferentiable (respectively superdifferentiable) on its domain of definition [11].

One contribution of the articles [3], [16] was to propose a solution of (1) (i.e., problem (5)) using a new mathematical framework for this problem based on viability theory. For this, we define an *auxiliary dynamical system* F associated with the HJ PDE (1) as follows, referred to as the *characteristic system* [3], [16]:

Definition 1.4: (Auxiliary Dynamical System): Given a Hamiltonian $\psi(\cdot)$ with convex transform $\varphi^*(\cdot)$, we define an auxiliary dynamical system F associated with the HJ PDE (1)

$$\mathbf{F} := \begin{cases} \tau'(t) = -1\\ x'(t) = u(t)\\ y'(t) = -\varphi^*(u(t)) \end{cases} \quad \text{where } u(t) \in \mathrm{Dom}(\varphi^*) \ . \ (10)$$

Definition 1.5: (Constraint set Associated With a HJ PDE): For a HJ PDE (1) defined in the set $\mathbb{R}_+ \times X$, we define the constraint set $\mathcal{K} := \mathbb{R}_+ \times X \times \mathbb{R}$. We refer the reader to [3] for the construction of solutions associated with general epigraphical environment sets, and the interpretation of the resulting solutions. We recall the following definition:

Definition 1.6: (Target set Associated With a HJ PDE): For a HJ PDE (1) defined in $\mathbb{R}_+ \times X$, we define a *target function* as a lower semicontinuous function $\mathbf{c}(\cdot, \cdot)$ in a subset of $\mathbb{R}_+ \times X$. The target function \mathbf{c} defines an epigraphical target set as $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$. This set is the subset of triples $(t, x, y) \in \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \ge \mathbf{c}(t, x)$ (it is the *epigraph* of the function \mathbf{c}).

The above definitions enable us to construct the *viability episolutions* of the HJ PDE (1) using the concept of capture basins:

Definition 1.7: (Viability Episolution): Given a characteristic system F, a constraint set \mathcal{K} and a target set \mathcal{C} , respectively defined by Definitions 1.4, 1.5 and 1.6, the viability episolution **M** is defined by

$$\mathbf{M}(t,x) := \inf_{(t,x,y)\in \operatorname{Capt}_F(\mathcal{K},\mathcal{C})} y.$$
(11)

The viability episolution M_c associated with any given lower semicontinuous target function c is a solution to the Moskowitz HJ PDE (1) in the Barron-Jensen/Frankowska sense [3], [16]. By definition of the capture basin, the viability episolution M_c can be characterized [16] using the following generalized Lax-Hopf formula [3], [16], which will serve as a fundamental tool for this work to establish a semi-analytical solution to our problem

$$\mathbf{M}_{\mathbf{c}}(t,x) = \inf_{(u,T)\in \mathrm{Dom}(\varphi^*)\times\mathbb{R}_+} \left(\mathbf{c}(t-T,x+Tu) + T\varphi^*(u)\right).$$
(12)

C. Contributions of the Article

The first contribution of the article is the derivation of analytical expressions for the episolutions associated with affine initial, boundary, and internal boundary condition functions, for a general concave and continuous Hamiltonian. This result is new and provides analytical solutions for practical problems.

The second contribution of this article is the design of a semianalytical algorithm known as the Lax-Hopf algorithm. The Lax-Hopf algorithm can be used to numerically compute the solution to the Hamilton-Jacobi PDE (1) associated with any piecewise affine initial, boundary and internal boundary conditions problem by minimizing analytical functions. The fundamental advantage of this algorithm over dynamic programming methods [24] is the possibility to obtain exact results for any convex Hamiltonian and any grid type. We also show that this algorithm can be used to compute the density function $\rho(t, x) :=$ $-\partial \mathbf{M}(t,x)/\partial x$ associated with the Moskowitz function $\mathbf{M}(\cdot,\cdot)$ exactly wherever the latter is differentiable. The fundamental advantages of this algorithm with respect to finite difference schemes such as level-set methods [41], fast-marching methods [49] or Godunov schemes [30] are higher accuracy and lower computational cost.

The third contribution of the article is the derivation of explicit necessary and sufficient proper formulation conditions for piecewise affine initial, boundary and internal components.

The last contribution of the article is a numerical assessment of the performance of the method, for which the computational



Fig. 1. Illustration of a generic concave Hamiltonian ψ associated with the Hamilton–Jacobi PDE investigated in this article. The Hamiltonian ψ is only required to be concave and continuous, and is not necessarily differentiable everywhere nor piecewise affine.

cost is benchmarked against other methods such as finite difference schemes.

The rest of this article is organized as follows. Section II uses the Lax-Hopf formula to construct the components of the Hamilton-Jacobi solution for the initial condition (Section II-B), upstream boundary condition (Section II-C), downstream boundary condition (Section II-D) and internal boundary condition (Section II-E). The Lax-Hopf algorithm is constructed in Section III-D, after defining some prerequisites in Section III-A. A validation of the algorithm, as well as a computational cost comparison with a standard finite difference scheme (Godunov) are provided in Section III-E.

II. THE LAX-HOPF FORMULAS ASSOCIATED WITH AFFINE INITIAL, BOUNDARY AND INTERNAL BOUNDARY CONDITIONS

Following methods commonly used in the development of finite difference schemes, we derive the following specific results for affine initial, boundary and internal boundary conditions. Finite difference techniques nominally use discretizations of functions to be approximated, which occur on a computational grid. The present method makes similar assumptions, by representing the solution as piecewise affine. However, there are some fundamental differences between our proposed scheme and finite difference techniques:

- The proposed algorithm yields the exact value of the solution on discrete points. In addition, the locations of the components in the piecewise solution are computed analytically using Lax–Hopf formulas, and are not chosen *a priori*.
- 2) The method does not require any knowledge of the solution at intermediate time steps, i.e., there is no need for a grid.

We consider a general concave and continuous Hamiltonian $\psi(\cdot)$ satisfying $\psi(0) = \psi(\omega) = 0$. Note that the previous assumptions imply that the image $\text{Im}(\psi) := \{\psi(\rho), \rho \in [0, \omega]\}$ of $\psi(\cdot)$ is of the form $[0, \delta]$, where $\delta > 0$. A generic function $\psi(\cdot)$ satisfying these conditions is illustrated in Fig. 1.

A. Convexity Property of the Components Associated With a Convex Target Function

In this section, we consider a convex³ function $\mathbf{c}(t, x)$ defined on a compact domain $\mathcal{D} \subset \mathbb{R}_+ \times X$. Note that since **c** is convex and defined on a compact set, it is bounded below. Following [16], we define by $\mathbf{M}_{\mathbf{c}}(\cdot, \cdot)$ the component of the solution associated with $\mathbf{c}(\cdot, \cdot)$. For the present article, the function $\mathbf{M}_{\mathbf{c}}(\cdot, \cdot)$ is fully characterized by its Lax-Hopf representation, which can also serve as an alternate definition of the component:

Definition 2.1: (Component Function): For any target function $\mathbf{c}(\cdot, \cdot)$ as in Definition 1.6, the component $\mathbf{M}_{\mathbf{c}}(\cdot, \cdot)$ of $\mathbf{c}(\cdot, \cdot)$ is defined by

$$\forall (t,x) \in \mathbb{R}_+ \times X, \\ \mathbf{M}_{\mathbf{c}}(t,x) = \inf_{(u,T) \in \mathrm{Dom}(\varphi^*) \times \mathbb{R}_+} \left(\mathbf{c}(t-T,x+Tu) + T\varphi^*(u) \right).$$
(13)

Definition 2.2: (Variable Change for the Auxiliary Control): We define a new variable v as v = Tu, and define the cone $\mathcal{D} := \{ [-\nu^{\flat}t, \nu^{\sharp}t] \times \{t\} | t \in \mathbb{R}_+ \}.$

Note that Definition 2.2 implies that $(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+ \text{iff}(v, T) \in \mathcal{D}.$

Definition 2.3: (Auxiliary Objective Function): We define the function $f(\cdot, \cdot, \cdot, \cdot)$ as

$$\forall (t, x, v, T) \in \mathbb{R}_+ \times X \times \mathcal{D}, \\ f(t, x, v, T) := \mathbf{c}(t - T, x + v) + T\varphi^*\left(\frac{v}{T}\right).$$

Since φ^* is convex, its associated perspective function $(v,T) \rightarrow T\varphi^*(v/T)$ is convex [11] for T > 0. Since the function $(t, x, v, T) \rightarrow (t - T, x + v)$ is affine and $\mathbf{c}(\cdot, \cdot)$ is convex, the function $(t, x, v, T) \rightarrow \mathbf{c}(t - T, x + v)$ is convex [11], [47]. Hence the function $f(\cdot, \cdot, \cdot, \cdot)$ is convex as the sum of two convex functions. The function $f(\cdot, \cdot, \cdot, \cdot)$ is also bounded below since the function \mathbf{c} is bounded below and the function φ^* is positive [16]. By definition of f, we can rewrite (13) as

$$\mathbf{M}_{\mathbf{c}}(t,x) = \inf_{(v,T)\in\mathcal{D}} f(t,\ x,\ v,\ T).$$
(14)

Equation (14) implies

.

$$\mathcal{E}pi(\mathbf{M}_{\mathbf{c}}) = \{(t, x, y) | \exists (v, T) \in \mathcal{D}s.t.(t, x, v, T, y) \in \mathcal{E}pi(f) \}.$$
(15)

Proposition 2.4: (Convexity Property): The component $M_c(\cdot, \cdot)$ associated with the convex target function $c(\cdot, \cdot)$ is convex.

Proof: Since the function $f(\cdot, \cdot, \cdot, \cdot)$ is convex, its epigraph $\mathcal{E}pi(f)$ is also convex. Since the set $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$ is nonempty, the epigraph of $\mathbf{M}_{\mathbf{c}}$ is nonempty by the inclusion $\mathcal{C} \subset \operatorname{Capt}_F(\mathcal{K}, \mathcal{C}) := \mathcal{E}pi(\mathbf{M}_{\mathbf{c}})$ (see [1] for a proof of this property).

Hence, (15) implies that the epigraph of M_c is convex, since it is the projection of a convex set on a subspace [11], [47].

B. Analytical Lax-Hopf Formula Associated With an Affine Initial Condition

This section now analytically computes the value of a component associated with an affine initial condition. The method chosen follows the procedure outlined below:

- 1) Write the Lax-Hopf formula associated with the affine initial condition.
- 2) Write the minimization problem associated with this instantiation of the Lax-Hopf formula.
- 3) Analytically find a minimizer of the optimization program.

³In this article, unless stated otherwise, the convexity property of a function of several variables refers to the global convexity property (i.e., convexity with respect to all the variables).

Definition 2.5: (Affine Initial Condition): We consider the affine initial condition $\mathcal{M}_{0,i}(0,x)$, where *i* is an integer

$$\mathcal{M}_{0,i}(0,x) = \begin{cases} a_i x + b_i & \text{if } x \in [\bar{\alpha}_i, \ \bar{\alpha}_{i+1}] \\ +\infty & \text{otherwise.} \end{cases}$$
(16)

The derivation assumes that t > 0. This assumption is necessary since the following minimization involves a division by t. Note that the case t = 0 is of no interest since the proper formulation property implies that $\forall x \in [\bar{\alpha}_i, \bar{\alpha}_{i+1}], \mathbf{M}_{\mathcal{M}_{0i}}(0, x) = a_i x + b_i$.

Proposition 2.6: (Lax-Hopf Formula for an Affine Initial Condition): The Lax–Hopf formula associated with the initial condition (16) can be expressed as

$$\mathbf{M}_{\mathcal{M}_{0},i}(t,x) = \inf_{u \in \mathrm{Dom}(\varphi^{*}) \cap [(\bar{\alpha}_{i}-x)/t,(\bar{\alpha}_{i+1}-x)/t]} (a_{i}(x+tu) + b_{i}(x) + t\varphi^{*}(u)), \ \forall (t,x) \in \mathbb{R}^{+}_{+} \times X.$$
(17)

Proof: The Lax-Hopf formula associated with the initial condition component reads [16]

$$\mathbf{M}_{\mathcal{M}_{0i}}(t,x) = \inf_{u \in \text{Dom}(\varphi^*)} (a_i(x+tu) + b_i + t\varphi^*(u))$$

such that $(x+tu) \in [\bar{\alpha}_i, \bar{\alpha}_{i+1}].$

This formula is valid for all $(t,x) \in \mathbb{R}_+ \times X$. Since t > 0, the condition $(x + tu) \in [\bar{\alpha}_i, \bar{\alpha}_{i+1}]$ is equivalent to $u \in [(\bar{\alpha}_i - x)/t, (\bar{\alpha}_{i+1} - x)/t]$, which in turn implies (17).

This component has a domain of definition, which can be explicitly characterized.

Proposition 2.7: (Domain of Definition of an Affine Initial Condition Component): The domain of definition of $\mathbf{M}_{\mathcal{M}_{0}}(\cdot, \cdot)$ is given by the following formula:

$$Dom(\mathbf{M}_{\mathcal{M}_{0},i}) = \left\{ (t,x) \in \mathbb{R}_{+}^{*} \times X \\ \text{such that } \bar{\alpha}_{i} - \nu^{\sharp} t \leq x \leq \bar{\alpha}_{i+1} + \nu^{\flat} t \right\}.$$
(18)

Proof: The Lax-Hopf formula (17) implies

$$Dom(\mathbf{M}_{\mathcal{M}_0,i}) := \left\{ (t,x) \in \mathbb{R}^*_+ \times X \\ \text{such that } \exists u \in Dom(\varphi^*) \cap \left[\frac{\bar{\alpha}_i - x}{t}, \frac{\bar{\alpha}_{i+1} - x}{t} \right] \right\}.$$

Equation (18) is obtained using the above formula, and noting that $Dom(\varphi^*) = [-\nu^{\flat}, \nu^{\sharp}].$

The analytical computation of the solution can be done by minimizing an auxiliary function, which we now define.

Definition 2.8: (Auxiliary Objective Function): For all $(a_i, b_i, t, x) \in \mathbb{R}^2 \times \text{Dom}(\mathbf{M}_{\mathcal{M}_0, i})$, we define an objective function $\zeta_{a_i, b_i, t, x}(\cdot)$ by the following formula:

$$\forall u \in \text{Dom}(\varphi^*), \ \zeta_{a_i, b_i, t, x}(u) \coloneqq a_i(x + tu) + b_i + t\varphi^*(u). \ (19)$$

Given this definition, (17) becomes

$$\forall (t,x) \in \mathbb{R}^*_+ \times X, \mathbf{M}_{\mathcal{M}_0,i}(t,x)$$

=
$$\inf_{u \in \text{Dom}(\varphi^*) \cap [(\bar{\alpha}_i - x)/t, (\bar{\alpha}_{i+1} - x)/t]} \zeta_{a_i,b_i,t,x}(u).(20)$$

The function $\zeta_{a_i,b_i,t,x}(\cdot)$ is convex as the sum of two convex functions, and thus subdifferentiable on $\text{Dom}(\varphi^*)$ in the sense of (8). The subderivative of $\zeta_{a_i,b_i,t,x}(\cdot)$ is given by

$$\forall u \in \text{Dom}(\varphi^*),$$

$$\partial_{-}\zeta_{a_i,b_i,t,x}(u) = \{w | \exists v \in \partial_{-}\varphi^*(u), w = a_i t + vt\}$$

$$:= t \cdot (\{a_i\} + \partial_{-}\varphi^*(u))$$
(21)

with a slight abuse of notation for the summation of the two sets in the second equality. This last expression can now be used to analytically compute the minimizer.

Proposition 2.9: (Explicit Minimization of $\zeta_{a_i,b_i,t,x}(\cdot)$): We now assume that a_i in the target function \mathcal{M}_{0_i} given by (16) satisfies the condition $-a_i \in \text{Dom}(\psi) := [0, \omega]$. Since $\psi(\cdot)$ is concave, it is also superdifferentiable on its domain of definition, and thus $\forall \rho \in [0, \omega], \partial_+\psi(\rho) \neq \emptyset$.

Let $u_0(a_i)$ be an element of $-\partial_+\psi(-a_i) \neq \emptyset$. Note that the Legendre-Fenchel inversion formula (7) implies that $u_0(a_i) \in$ $\text{Dom}(\varphi^*)$ and $-a_i \in \partial_-\varphi^*(u_0(a_i))$. Using this definition of $u_0(a_i)$, the function $\zeta_{a_i,b_i,t,x}(\cdot)$ has the following minimizer over $\text{Dom}(\varphi^*) \cap [(\bar{\alpha}_i - x)/t, (\bar{\alpha}_{i+1} - x)/t]$:

$$\begin{cases} u = u_0(a_i) & \text{if } u_0(a_i) \in \left\lfloor \frac{\bar{\alpha}_i - x}{t}, \frac{\bar{\alpha}_{i+1} - x}{t} \right\rfloor \\ u = \frac{\bar{\alpha}_i - x}{t} & \text{if } u_0(a_i) \leq \frac{\bar{\alpha}_i - x}{t} \\ u = \frac{\bar{\alpha}_{i+1} - x}{t} & \text{if } u_0(a_i) \geq \frac{\bar{\alpha}_{i+1} - x}{t}. \end{cases}$$
(22)

Proof: The function $\zeta_{a_i,b_i,t,x}(u)$ is minimal for a given $u \in \text{Dom}(\varphi^*)$ if and only if $0 \in \partial_-\zeta_{a_i,b_i,t,x}(u)$ by [11]. By (21), this happens if and only if for this $u, -a_i \in \partial_-\varphi^*(u)$. Using the Legendre-Fenchel inversion formula [3], we can rewrite $u := u_0(a_i) \in -\partial_+\psi(-a_i)$ as $-a_i \in \partial_-\varphi^*(u_0(a_i))$, and thus $u_0(a_i)$ minimizes $\zeta_{a_i,b_i,t,x}(\cdot)$ over $\text{Dom}(\varphi^*)$. Hence, since $\zeta_{a_i,b_i,t,x}(\cdot)$ is convex, $\zeta_{a_i,b_i,t,x}(u)$ is decreasing for $u \leq u_0(a_i)$ and increasing for $u \geq u_0(a_i)$, which implies (22).

Proposition 2.10: (Computation of $\mathbf{M}_{\mathcal{M}_0,i}(\cdot,\cdot)$): Let $u_0(a_i)$ be defined as in Proposition 2.9. For all $(t,x) \in \text{Dom}(\mathbf{M}_{\mathcal{M}_0,i})$, the expression $\mathbf{M}_{\mathcal{M}_0,i}(t,x)$ can be computed using the following formula:

$$\mathbf{M}_{\mathcal{M}_{0},i}(t,x) = \begin{cases} (i) \quad t\psi(-a_{i}) + a_{i}x + b_{i} \\ if u_{0}(a_{i}) \in \left[\frac{\bar{\alpha}_{i}-x}{t}, \frac{\bar{\alpha}_{i+1}-x}{t}\right] \\ (ii) \quad a_{i}\bar{\alpha}_{i} + b_{i} + t\varphi^{*}\left(\frac{\bar{\alpha}_{i}-x}{t}\right) \\ if u_{0}(a_{i}) \leq \frac{\bar{\alpha}_{i}-x}{t} \\ (iii) \quad a_{i}\bar{\alpha}_{i+1} + b_{i} + t\varphi^{*}\left(\frac{\bar{\alpha}_{i+1}-x}{t}\right) \\ if u_{0}(a_{i}) \geq \frac{\bar{\alpha}_{i+1}-x}{t}. \end{cases}$$

$$(23)$$

Proof: The cases (*ii*) and (*iii*) of (23) are trivially obtained by combining (17) and (22). Since the function $-\psi(\cdot)$ is convex, it is identical [11] to its Fenchel biconjugate

$$\forall \rho \in [0, \omega], \psi(\rho) = \inf_{u \in \text{Dom}(\varphi^*)} \left(-\rho u + \varphi^*(u) \right).$$

The function $g : u \rightarrow a_i u + \varphi^*(u)$ is convex, and thus subdifferentiable on $\text{Dom}(\varphi^*)$. By definition of $u_0(a_i), 0 \in \partial_-g(u_0(a_i))$. This last property implies that $u_0(a_i)$ minimizes $g(\cdot)$ over $\text{Dom}(\varphi^*)$, and thus that $\psi(-a_i) = a_i u_0(a_i) + \varphi^*(u_0(a_i))$. Hence, the case (i) of



Fig. 2. Left: Illustration of the construction of a $u_0(a_i)$ from the knowledge of a_i . The transform $\varphi^*(u_0(a_i))$ corresponds to the value intercepted on the vertical axis by the tangent line of slope $-u_0(a_i)$ to the graph of ψ in $-a_i$. Right: The (t, x) domain of the episolution corresponding to the affine initial condition (16) can be separated in three different areas. The domain highlighted in light gray corresponds to the case (i) in (23). The domain highlighted in medium gray corresponds to the case (iii), and the remaining domain in dark gray corresponds to the case (ii). The domain of the initial condition is represented by a dashed line.

(23) is obtained by combining (17), (22), and the property $\psi(-a_i) = a_i u_0(a_i) + \varphi^*(u_0(a_i))$.

Fig. 2 illustrates the different domains of (23) for the episolution associated with an affine initial condition defined by (16).

C. Analytical Lax-Hopf Formula Associated With an Affine Upstream Boundary Condition

Definition 2.11: (Affine Upstream Boundary Condition): We consider the affine upstream boundary condition $\gamma_j(t,\xi)$, where j is an integer

$$\gamma_j(t,\xi) = \begin{cases} c_j t + d_j & \text{if } t \in [\bar{\gamma}_j, \bar{\gamma}_{j+1}] \\ +\infty & \text{otherwise.} \end{cases}$$
(24)

In the following derivation, we consider that $x > \xi$. We also assume that the target function (24) satisfies the condition $c_j \in$ $\text{Im}(\psi) = [0, \delta]$. Note that when the previous condition is satisfied, the boundary condition component associated to γ_j is properly formulated [16], and thus $\forall t \in [\bar{\beta}_j, \bar{\beta}_{j+1}], \mathbf{M}_{\gamma_j}(t, \xi) =$ $\gamma_j(t, \xi)$.

Definition 2.12: (Density Associated With c_j): Recalling that Im $(\psi) := [0, \delta]$, we define ρ_c as

$$\rho_c = \inf_{\rho \in [0,\omega] \text{ such that } \psi(\rho) = \delta} \rho$$

Since $c_j \in \text{Im}(\psi) = [0, \delta]$, there exists $\rho_j \in [0, \rho_c]$ such that $\psi(\rho_j) = c_j$. Note that since $\psi(\cdot)$ is concave and $\delta > 0$, $\psi(\cdot)$ is increasing on $[0, \rho_c]$, and thus $\partial_+\psi(\rho_j) \cap \mathbb{R}_+ \neq \emptyset$.

- Let $u_0(\rho_j)$ be an element of $-\partial_+\psi(\rho_j) \cap \mathbb{R}_- \neq \emptyset$.
- Let $T_0(\rho_i, x)$ be defined as

$$T_0(\rho_j, x) \coloneqq \begin{cases} \frac{\xi - x}{u_0(\rho_j)} & \text{if } u_0(\rho_j) \neq 0\\ +\infty & \text{if } u_0(\rho_j) = 0. \end{cases}$$
(25)

Proposition 2.13: (Computation of $\mathbf{M}_{\gamma_j}(\cdot, \cdot)$): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\gamma_j})$, the expression $\mathbf{M}_{\gamma_j}(t,x)$ can be computed using the following formula:

$$\mathbf{M}_{\gamma_{j}}(t,x) = \begin{cases} (i) \quad t\psi(\rho_{j}) + \rho_{j}(\xi - x) + d_{j} \\ if \, T_{0}(\rho_{j}, x) \in [t - \bar{\gamma}_{j+1}, t - \bar{\gamma}_{j}] \\ (ii) \quad \psi(\rho_{j})\bar{\gamma}_{j} + d_{j} + (t - \bar{\gamma}_{j})\varphi^{*}\left(\frac{\xi - x}{t - \bar{\gamma}_{j}}\right) \\ if \, t - \bar{\gamma}_{j} \leq T_{0}(\rho_{j}, x) \\ (iii) \quad \psi(\rho_{j})\bar{\gamma}_{j+1} + d_{j} + (t - \bar{\gamma}_{j+1})\varphi^{*}\left(\frac{\xi - x}{t - \bar{\gamma}_{j+1}}\right) \\ if \, T_{0}(\rho_{j}, x) \leq t - \bar{\gamma}_{j+1}. \end{cases}$$
(26)

Proof: Equation (26) can be obtained from (35), observing that the affine downstream boundary condition (24) can be viewed as an affine internal boundary condition of the form (32), where

$$\begin{cases} \delta_{l} = \bar{\gamma}_{j} \\ \bar{\delta}_{l+1} = \bar{\gamma}_{j+1} \\ x_{l} = \xi \\ v_{l} = 0 \\ g_{l} = c_{j} \\ h_{l} = c_{j} \bar{\gamma}_{j} + d_{j} \end{cases}$$

$$(27)$$

D. Analytical Lax-Hopf Formula Associated With an Affine Downstream Boundary Condition

Definition 2.14: (Affine Downstream Boundary Condition): We consider the affine downstream boundary condition $\beta_k(t, \chi)$, where k is an integer

$$\beta_k(t,x) = \begin{cases} e_k t + f_k & \text{if } t \in [\bar{\beta}_k, \bar{\beta}_{k+1}] \\ +\infty & \text{otherwise.} \end{cases}$$
(28)

In the following derivation, we consider that $x < \chi$. We also assume that the target function (28) satisfies the condition $e_k \in \text{Im}(\psi) = [0, \delta]$. Note that when the previous condition is satisfied, the boundary condition component associated to β_k is properly formulated [16], and thus $\forall t \in [\bar{\gamma}_k, \bar{\gamma}_{k+1}]$, $\mathbf{M}_{\beta_k}(t, \chi) = \beta_k(t, \chi)$.

Definition 2.15: (Density Associated With e_k): Recalling that $Im(\psi) = [0, \delta]$, we define ρ_c as

$$\rho_c = \sup_{\rho \in [0,\omega] \text{ such that } \psi(\rho) = \delta} \rho.$$

Since $e_k \in \text{Im}(\psi) = [0, \delta]$, there exists $\rho_k \in [\rho_c, \omega]$ such that $\psi(\rho_k) = e_k$. Note that since $\psi(\cdot)$ is concave and $\delta > 0$, $\psi(\cdot)$ is decreasing on $[\rho_c, \omega]$, and thus $\partial_+\psi(\rho_k) \cap \mathbb{R}_- \neq \emptyset$.

- Let $u_0(\rho_k)$ be an element of $-\partial_+\psi(\rho_k) \cap \mathbb{R}_+$.
- Let $T_0(\rho_k, x)$ be defined as

$$T_0(\rho_k, x) := \begin{cases} \frac{\xi - x}{u_0(\rho_k)} & \text{if } u_0(\rho_k) \neq 0\\ +\infty & \text{if } u_0(\rho_k) = 0. \end{cases}$$
(29)

We have by the Legendre–Fenchel inversion formula that $\rho_k \in \partial_-\varphi^*(u_0(\rho_k))$, which implies that $u_0(\rho_k) \in \text{Dom}(\varphi^*)$.

Remark: Note that the definition of ρ_c differs from the previous section for functions $\psi(\cdot)$ which are not strictly concave. This is sometimes referred as "lower critical density" (Section II-C) and "upper critical density" (Section II-D), but we have kept the same notation since the two corresponding densities are only intermediate variables in our derivations. \Box

Proposition 2.16: (Computation of $\mathbf{M}_{\beta_k}(\cdot, \cdot)$): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\beta_k})$, the expression $\mathbf{M}_{\beta_k}(t,x)$ can be computed using the following formula:

$$\mathbf{M}_{\beta_{k}}(t,x) = \begin{cases} (i) & t\psi(\rho_{k}) + \rho_{k}(\chi - x) + f_{k} \\ if T_{0}(\rho_{k}, x) \in [t - \bar{\beta}_{k+1}, t - \bar{\beta}_{k}] \\ (ii) & \psi(\rho_{k})\bar{\beta}_{k} + f_{k} + (t - \bar{\beta}_{k})\varphi^{*}\left(\frac{\chi - x}{t - \bar{\beta}_{k}}\right) \\ if t - \bar{\beta}_{k} \leq T_{0}(\rho_{k}, x) \\ (iii) & \psi(\rho_{k})\bar{\beta}_{k+1} + f_{k} + (t - \bar{\beta}_{k+1})\varphi^{*}\left(\frac{\chi - x}{t - \bar{\beta}_{k+1}}\right) \\ if T_{0}(\rho_{k}, x) \leq t - \bar{\beta}_{k+1}. \end{cases}$$
(30)

Proof: Equation (30) can be obtained from (35), observing that the affine downstream boundary condition (28) can be viewed as an affine internal boundary condition of the form (32), where

$$\begin{cases} \bar{\delta}_{l} = \bar{\beta}_{k} \\ \bar{\delta}_{l+1} = \bar{\beta}_{k+1} \\ x_{l} = \chi \\ v_{l} = 0 \\ g_{l} = e_{k} \\ h_{l} = e_{k} \bar{\beta}_{k} + f_{k} \end{cases}$$
(31)

E. Analytical Lax-Hopf Formula Associated With an Affine Internal Boundary Condition

The previous section explained how to compute the value of a component analytically from the initial and boundary conditions. We now treat the problem of internal boundary conditions using a similar approach, which is one of the contributions of this article. As will appear in this section, the algebra involved in doing this mathematical construction is more involved than the previous case.

Definition 2.17: (Affine Internal Boundary Condition): We consider the following affine internal boundary condition $\mu_l(\cdot, \cdot)$, where *l* is an integer:

$$\mu_l(t,x) = \begin{cases} g_l(t-\bar{\delta}_l) + h_l & \text{if } x = x_l + v_l(t-\bar{\delta}_l) \\ & \text{and } t \in [\bar{\delta}_l, \bar{\delta}_{l+1}] \\ +\infty & \text{otherwise.} \end{cases}$$
(32)

We assume that the constants g_l and v_l in (32) satisfy $0 \le g_l \le \varphi^*(-v_l)$ and $0 \le v_l < \nu^{\flat}$. The constants g_l and v_l represent the rate of label change and the speed of the internal boundary condition respectively. The internal boundary condition is located at x_l and has the value h_l at the initial time $\overline{\delta}_l$.

Proposition 2.18: (Lax-Hopf Formula for Affine Internal Boundary Condition): The Lax-Hopf formula (33) associated with the internal boundary condition (32) can be expressed as

$$\mathbf{M}_{\mu_{l}}(t,x) = \inf_{\substack{T \in \mathbb{R}_{+} \cap [t-\overline{\delta}_{l+1}, t-\overline{\delta}_{l}]}} \left(g_{l}(t-T-\overline{\delta}_{l}) + h_{l} + T\varphi^{*}\left(\frac{x_{l}+v_{l}(t-\overline{\delta}_{l}-T)-x}{T}\right), \quad \forall (t,x) \in \mathbb{R}_{+} \times X.$$
(33)

Proof: This formula is the instantiation of the Lax-Hopf formula proved in [16] for a constant velocity v_l and a constant label variation rate g_l .

The two following definitions enable us to express the internal boundary condition component $\mathbf{M}_{\mathbf{M}_{\mu_l}}$ analytically (see Figs. 3 and 4).

Definition 2.19: (Densities Associated With v_l and g_l):

- We define the function f_{vl}(·) as f_{vl}: ρ→ψ(ρ) ρv_l. The function f_{vl} is concave as the sum of concave functions, and attains its maximum value φ^{*}(-v_l) (by definition of the function φ^{*}(·)) for a given ρ := ρ_l.
- Note that since $v_l \in [0, \nu^{\flat}[$, the function f_{v_l} satisfies $f_{v_l}(0) = 0$ and $f_{v_l}(\omega) \leq 0$. By assumption, we also have $g_l \leq \varphi^*(-v_l)$, and since $f_{v_l}(\cdot)$ is concave and continuous, there exist two solutions $\rho_1(v_l, g_l) \in [0, \rho_l]$ and $\rho_2(v_l, g_l) \in [\rho_l, \omega]$ such that $f_{v_l}(\rho_p(v_l, g_l)) = g_l$ for $p \in \{1, 2\}$ (see Fig. 5).



Fig. 3. Left: Illustration of the construction of a $u_0(\rho_j)$ from a known c_j . The transform $\varphi^*(u_0(\rho_j))$ corresponds to the value intercepted on the vertical axis by the tangent line of slope $-u_0(\rho_j)$ to the graph of ψ in ρ_j . Right: The (t, x) domain of the episolution corresponding to the affine upstream boundary condition (24) can be separated in three different areas. The domain highlighted in light gray corresponds to the case (i) in (26). The domain highlighted in dark gray corresponds to the case (ii), and the remaining domain in medium gray condition is represented by a dashed line.



Fig. 4. Left: Illustration of the construction of a $u_0(\rho_k)$ from a known e_k . The transform $\varphi^*(u_0(\rho_k))$ corresponds to the value intercepted on the vertical axis by the tangent line of slope $-u_0(\rho_k)$ to the graph of ψ in ρ_k . Right: The (t, x) domain of the episolution corresponding to the affine downstream boundary condition (28) can be separated in three different areas. The domain highlighted in light gray corresponds to the case (i) in (30). The domain highlighted in dark gray corresponds to the case (ii), and the remaining domain in medium gray corresponds to the case (iii). The domain of the downstream boundary condition is represented by a dashed line.



Fig. 5. Left: Illustration of the construction of a $u_1(v_l, g_l)$ and $u_2(v_l, g_l)$ from known v_l and g_l . Right: The (t, x) domain of the episolution corresponding to the affine internal boundary condition (32) can be separated in four different areas. The domains highlighted in dark gray correspond to the case (iii) in (35). The domains highlighted in light gray correspond to the case (i) and (i). The remaining domain in medium gray corresponds to the case (iv). The domain of the internal boundary condition is represented by a dashed line.

• For $p \in \{1, 2\}$, we also define $u_p(v_l, g_l)$ as elements of $-\partial_+\psi(\rho_p(v_l, g_l))$.

Definition 2.20: (Capture Times Associated With $u_p(v_l, g_l)$, for $p \in \{1, 2\}$):

• We define $T_p(t, x, v_l, g_l)$ for $p \in \{1, 2\}$ as

$$T_p(t, x, v_l, g_l) \coloneqq \begin{cases} \frac{x_l + v_l(t - \overline{\delta}_l) - x}{u_p(v_l, g_l) + v_l} & \text{if } u_p(v_l, g_l) \neq -v_l \\ +\infty & \text{if } u_p(v_l, g_l) = -v_l. \end{cases}$$
(34)

A justification for Definitions 2.19 and 2.20 is given in Appendix.

Proposition 2.21: (Computation of $\mathbf{M}_{\mu_l}(\cdot, \cdot)$): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\mu_l})$, the expression $\mathbf{M}_{\mu_l}(t,x)$ can be computed using the following formula:

$$\begin{split} \mathbf{M}_{\mu_{l}}(t,x) &= \\ \begin{cases} (i) & \psi(\rho_{1}(v_{l},g_{l}))(t-\bar{\delta}_{l}) + (x_{l}-x)\rho_{1}(v_{l},g_{l}) + h_{l} \\ & \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \leq x \\ & \text{and } T_{1}(t,x,v_{l},g_{l}) \in \left[t-\bar{\delta}_{l+1},t-\bar{\delta}_{l}\right] \\ (ii) & \psi(\rho_{2}(v_{l},g_{l}))(t-\bar{\delta}_{l}) + (x_{l}-x)\rho_{2}(v_{l},g_{l}) + h_{l} \\ & \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \geq x \\ & \text{and } T_{2}(t,x,v_{l},g_{l}) \in \left[t-\bar{\delta}_{l+1},t-\bar{\delta}_{l}\right] \\ (iii) & h_{l} + (t-\bar{\delta}_{l})\varphi^{*}\left(\frac{x_{l}-x}{t-\bar{\delta}_{l}}\right) \\ & \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \leq x \text{ and } T_{1}(t,x,v_{l},g_{l}) \geq t-\bar{\delta}_{l} \\ (iv) & g_{l}(\bar{\delta}_{l+1}-\bar{\delta}_{l}) + h_{l} + (t-\bar{\delta}_{l+1})\varphi^{*}\left(\frac{x_{l}+v_{l}(\bar{\delta}_{l+1}-\bar{\delta}_{l})-x}{t-\bar{\delta}_{l+1}}\right) \\ & \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \leq x \text{ and } T_{1}(t,x,v_{l},g_{l}) \leq t-\bar{\delta}_{l+1} \\ (iv) & g_{l}(\bar{\delta}_{l+1}-\bar{\delta}_{l}) + h_{l} + (t-\bar{\delta}_{l+1})\varphi^{*}\left(\frac{x_{l}+v_{l}(\bar{\delta}_{l+1}-\bar{\delta}_{l})-x}{t-\bar{\delta}_{l+1}}\right) \\ & \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \leq x \text{ and } T_{2}(t,x,v_{l},g_{l}) \leq t-\bar{\delta}_{l+1} \\ (x_{l} + v_{l}(t-\bar{\delta}_{l}) \geq x \text{ and } T_{2}(t,x,v_{l},g_{l}) \leq t-\bar{\delta}_{l+1}. \end{split}$$

$$(35)$$

Proof: The proof of (35) is available in Appendix. ■ Fig. 5 illustrates the domains of (35), for the episolution to an affine internal boundary condition defined by (32).

F. Closed Form Expression of the Derivatives of the Functions $\mathbf{M}_{\mathcal{M}_0,i}(\cdot,\cdot), \gamma_j(\cdot,\cdot), \beta_k(\cdot,\cdot)$ and $\mathbf{M}_{\mu_l}(\cdot,\cdot)$

The episolutions $\mathbf{M}_{\mathcal{M}_0,i}(\cdot,\cdot)$, $\mathbf{M}_{\gamma_j}(\cdot,\cdot)$, $\mathbf{M}_{\beta_k}(\cdot,\cdot)$ and $\mathbf{M}_{\mu_l}(\cdot,\cdot)$ are convex since they are associated with convex target functions defined on a compact subset of $\mathbb{R}_+ \times X$. Hence, these functions are differentiable almost everywhere on their domains of definition. The spatial derivatives of the above functions can be computed (whenever φ^* is differentiable and using $\varphi^{*'}$ as the notation for the derivative of φ^*) explicitly as

$$\frac{\partial \mathbf{M}_{\mathcal{M}_{0},i}(t,x)}{\partial x} = \begin{cases}
a_{i} & \text{if } u_{0}(a_{i}) \in]\frac{\bar{\alpha}_{i}-x}{t}, \frac{\bar{\alpha}_{i+1}-x}{t}[\\
-\varphi^{*'}(\frac{\bar{\alpha}_{i}-x}{t}) & \text{if } u_{0}(a_{i}) < \frac{\bar{\alpha}_{i}-x}{t} \\
-\varphi^{*'}(\frac{\bar{\alpha}_{i+1}-x}{t}) & \text{if } u_{0}(a_{i}) > \frac{\bar{\alpha}_{i+1}-x}{t}.
\end{cases}$$
(36)

In the previous formula, $u_0(a_i)$ is an element of $-\partial_+\psi(-a_i)$

$$\frac{\partial \mathbf{M}_{\gamma_j}(t,x)}{\partial x} = \begin{cases}
-\rho_j & \text{if } T_0(\rho_j,x) \in [t - \bar{\gamma}_{j+1}, t - \bar{\gamma}_j] \\
-\varphi^{*'} \left(\frac{\xi - x}{t - \bar{\gamma}_j}\right) & \text{if } t - \bar{\gamma}_j < T_0(\rho_j,x) \\
-\varphi^{*'} \left(\frac{\xi - x}{t - \bar{\gamma}_{j+1}}\right) & \text{if } T_0(\rho_j,x) < t - \bar{\gamma}_{j+1}.
\end{cases}$$
(37)

In the previous formula, ρ_j and T_0 are computed by Definition 2.12

$$\frac{\partial \mathbf{M}_{\mathbf{M}_{\beta_{k}}}(t,x)}{\partial x} = \begin{cases}
-\rho_{k} & \text{if } T_{0}(\rho_{k},x) \in]t - \overline{\beta}_{k+1}, t - \overline{\beta}_{k}[\\
-\varphi^{*'}\left(\frac{\chi-x}{t-\overline{\beta}_{k}}\right) & \text{if } t - \overline{\beta}_{k} < T_{0}(\rho_{k},x) \\
-\varphi^{*'}\left(\frac{\chi-x}{t-\overline{\beta}_{k+1}}\right) & \text{if } T_{0}(\rho_{k},x) < t - \overline{\beta}_{k+1}.
\end{cases}$$
(38)

In the previous formula, ρ_k and T_0 are computed by Definition 2.15

$$\frac{\partial \mathbf{M}_{\mathbf{M}_{\mu_{l}}}(t,x)}{\partial x} = \begin{cases}
-\rho_{1}(v_{l},g_{l}) \\
\text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) < x \text{ and } T_{1}(t,x,v_{l},g_{l}) \in]t - \bar{\delta}_{l+1}, t - \bar{\delta}_{l}[\\
-\rho_{2}(v_{l},g_{l}) \\
\text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) > x \text{ and } T_{2}(t,x,v_{l},g_{l}) \in]t - \bar{\delta}_{l+1}, t - \bar{\delta}_{l}[\\
-\varphi^{*'}\left(\frac{x_{l}-x}{t - \bar{\delta}_{l}}\right) \\
\text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) < x \text{ and } T_{1}(t,x,v_{l},g_{l}) > t - \bar{\delta}_{l} \text{ or } \\
\text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) > x \text{ and } T_{2}(t,x,v_{l},g_{l}) > t - \bar{\delta}_{l} \\
-\varphi^{*'}\left(\frac{x_{l}+v_{l}(\bar{\delta}_{l+1}-\bar{\delta}_{l})-x}{t - \bar{\delta}_{l+1}}\right) \\
\text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) < x \text{ and } T_{1}(t,x,v_{l},g_{l}) < t - \bar{\delta}_{l+1} \\
\text{or if } x_{l} + v_{l}(t - \bar{\delta}_{l}) > x \text{ and } T_{2}(t,x,v_{l},g_{l}) < t - \bar{\delta}_{l+1}.
\end{cases}$$
(39)

In the previous formula, ρ_1 , ρ_2 , T_1 and T_2 are computed by Definition A.4..

These formulas are essential: they enable instantaneous computations of solutions to the (2), which can be made extensive use of for data assimilation [17]. They also provide an instantaneous way of solving (2) from $t = t_0$ to any arbitrary time $t = t_1$ without marching the whole grid interval $[t_0, t_1]$ in time.

III. THE LAX-HOPF ALGORITHM

In the previous section, we derive closed form episolutions to affine initial, boundary and internal boundary conditions when the functions $\psi(\cdot)$ and $\varphi^*(\cdot)$ have a closed form expression (see chapter 3 in [29] for examples of commonly used functions $\psi(\cdot)$ and [3] for closed form expressions of their transforms $\varphi^*(\cdot)$). Hence, we can compute numerical solutions to a given mixed initial-boundary conditions problem in which the initial and boundary conditions are piecewise affine⁴ as a minimization of analytical functions. This process can be formalized as a *Lax-Hopf algorithm*, which is a semi-analytical method. The accuracy of the algorithm is the accuracy of a numerical computation of a closed form function in numerical software (i.e very close to machine accuracy with no discretization error inherent to finite difference schemes).

Definition of the Initial, Boundary and Internal Boundary Conditions Components

We consider piecewise affine initial, boundary and internal boundary conditions, and define the finite sets I, J, K and $(L_p)_{p\in P}$ as $I := \{1, \ldots, i_{\max}\}, J := \{1, \ldots, j_{\max}\}, K :=$ $\{1, \ldots, k_{\max}\}$ and $L_p := \{1, \ldots, l_{p_{\max}}\}$ for $p \in P$. While the previous section can handle infinite horizon problems, only finite horizon problems can be implemented numerically. Therefore in the rest of the article, we assume that $i_{\max} < +\infty$, $j_{\max} < +\infty, k_{\max} < +\infty$ and $l_{p_{\max}} < +\infty$ for all $p \in P$. Assuming a finite time horizon is equivalent to assuming that the target C does not have a domain of definition which extends beyond that finite horizon. In the initial-boundary-internal

⁴Note that we can also compute the exact solution to a mixed initial-boundary conditions problem in which the initial and boundary conditions are piecewise polynomial using the same method, provided that we can compute closed form episolutions to polynomial initial, boundary and internal boundary conditions.

boundary conditions context, it boils down to assigning $+\infty$ to the functions γ , β and μ for times higher than the horizon.

Definition 3.1: (Piecewise Affine Initial, Boundary, and Internal Boundary Conditions): Let $(\bar{\alpha}_i)_{i \in \mathbb{N}}, (\bar{\gamma}_j)_{j \in \mathbb{N}}, (\bar{\beta}_k)_{k \in \mathbb{N}}$ and $(\bar{\delta}_{p_l})_{l \in \mathbb{N}}$ be strictly increasing sequences satisfying

$$\begin{cases} \bar{\alpha}_i \in X & \forall i \in \{1, \dots, i_{\max + 1}\} \\ \bar{\gamma}_j \in \mathbb{R}_+ & \forall j \in \{1, \dots, j_{\max + 1}\} \\ \bar{\beta}_k \in \mathbb{R}_+ & \forall k \in \{1, \dots, k_{\max + 1}\} \\ \bar{\delta}_{pl} \in \mathbb{R}_+ & \forall l \in \{1, \dots, l_{p_{\max + 1}}\}, \quad \forall p \in P. \end{cases}$$

$$(40)$$

We consider the following continuous piecewise affine initial, boundary, and internal boundary conditions:

$$\mathcal{M}_{0}(t,x) = \begin{cases} a_{i}x + b_{i} & \text{if } t = 0, \\ & \text{and } \exists i \in I \text{ such that } x \in [\bar{\alpha}_{i}, \bar{\alpha}_{i+1}] \\ +\infty & \text{otherwise} \end{cases}$$

$$\gamma(t,x) = \begin{cases} c_{j}t + d_{j} & \text{if } x = \xi \\ & \text{and } \exists j \in J \text{ such that } t \in [\bar{\gamma}_{j}, \bar{\gamma}_{j+1}] \\ +\infty & \text{otherwise} \end{cases}$$

$$\beta(t,x) = \begin{cases} e_{k}t + f_{k} & \text{if } x = \chi \\ & \text{and } \exists k \in K \text{ such that } t \in [\bar{\beta}_{k}, \bar{\beta}_{k+1}] \\ +\infty & \text{otherwise} \end{cases}$$

$$\mu_{p}(t,x) = \begin{cases} g_{pl}(t - \bar{\delta}_{pl}) + h_{pl} & \text{if } \exists l \in L_{p} \text{ such that} \\ & x = v_{pl}(t - \bar{\delta}_{pl}) + x_{pl} \\ & \text{and } t \in [\bar{\delta}_{pl}, \delta_{pl+1}] \\ +\infty & \text{otherwise.} \end{cases}$$

$$(41)$$

Note that the internal boundary condition μ_p is now also indexed by p, which denotes the moving boundary associated with μ_p . The function μ_p is piecewise affine for any p, and its corresponding affine blocks are denoted by μ_{pl} , and defined by

$$\mu_{p_l}(t,x) = \begin{cases} g_{p_l}(t-\bar{\delta_{p_l}}) + h_l & \text{if } x = x_{p_l} + v_{p_l}(t-\bar{\delta_{p_l}}) \\ & \text{and } t \in [\bar{\delta_{p_l}}, \bar{\delta_{p_{l+1}}}] \\ +\infty & \text{otherwise.} \end{cases}$$
(42)

This definition enables us to consider multiple internal boundary condition constraints, indexed by p. For each p, the second index l represents an affine block of the internal boundary condition.

Equation (41) is simply an algebraically compact way of integrating the components respectively defined by (16), (24), (28) and (32)

$$\mathcal{M}_{0}(\cdot, \cdot) = \inf_{i \in I} \left(\mathcal{M}_{0_{i}}(\cdot, \cdot) \right), \quad \gamma(\cdot, \cdot) = \inf_{j \in J} \left(\gamma_{j}(\cdot, \cdot) \right),$$

$$\beta(\cdot, \cdot) = \inf_{k \in K} \left(\beta_{k}(\cdot, \cdot) \right) \quad \text{and} \quad \mu_{p}(\cdot, \cdot) = \inf_{l \in L_{p}} \left(\mu_{p_{l}}(\cdot, \cdot) \right). (43)$$

For consistency with the definition of our time-space domain $\mathbb{R}_+ \times X$, we assume that the following conditions are satisfied:

$$\begin{cases} \bar{\alpha}_{1} = \xi \quad \text{(upstream boundary)} \\ \bar{\alpha}_{i_{\max}+1} = \chi \quad \text{(downstream boundary)} \\ \bar{\gamma}_{1} = \bar{\beta}_{1} = 0 \quad \text{(initial time)} \\ \bar{\gamma}_{j_{\max}+1} = \bar{\beta}_{k_{\max}+1} \quad \text{(finite horizon)} \\ \xi \leq x_{p_{l}}, 0 \leq \bar{\delta}_{p_{l}} \leq \bar{\beta}_{k_{\max}+1} \\ \text{and } x_{p_{l}} + v_{p_{l}}(\delta_{p_{l}+1} - \bar{\delta}_{p_{l}}) \leq \chi \forall l \in L_{p}, \forall p \in P \\ \text{(inclusion of the trajectory p in the domain).} \end{cases}$$
(44)

The above initial, upstream, downstream and internal boundary conditions (41) entirely define the mixed initial-boundary-internal boundary conditions problem (5). This problem has a solution if and only if the proper formulation and compatibility conditions detailed in the companion article [16] are satisfied. The compatibility conditions for the present problem are given by Theorem 6.4 of [16], and do not become simpler because of the specific form of the initial, boundary and internal boundary conditions. The proper formulation conditions, expressed by Example 4.7, and proposition 5.7 for triangular Hamiltonians, can however be written explicitly in the present case due to the specific form of the boundary and internal boundary conditions. We do not consider the initial condition component, since this component is unconditionally properly formulated [16].

A. Proper Formulation of the Boundary and Internal Boundary Condition Components

This section solves a very important problem for a target: is the capture problem associated with a target well posed? For general targets, the proper formulation conditions cannot be expressed in an explicit form [16]. However, for the specific case of piecewise boundary and internal boundary conditions, these conditions can be expressed explicitly.

Proposition 3.2: (Proper Formulation of the Initial and Boundary Condition Components): The boundary condition components $\mathbf{M}_{\gamma}(\cdot, \cdot)$ and $\mathbf{M}_{\beta}(\cdot, \cdot)$ associated with the boundary condition functions $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ defined by (41) are properly formulated if and only if the following properties are satisfied:

$$\begin{cases} (i) & c_j \leq \delta & \forall j \in J \text{ proper formulation of } \gamma_j \\ (ii) & e_k \leq \delta & \forall k \in K \text{ proper formulation of } \beta_k. \end{cases}$$
(45)

Proof: Note that (41) implies that the functions $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are continuous, and defined on $[0, \overline{\gamma}_{j_{\max}+1}] \times \{\xi\}$ and $[0, \overline{\beta}_{k_{\max}+1}] \times \{\chi\}$ respectively. The proper formulation conditions [16] for the upstream and downstream boundary condition components read

$$\forall t \in \mathbb{R}_+, \ \forall T \in [0, t], \ \gamma \left(t - T, \xi\right) + T\delta \ge \gamma \left(t, \xi\right) \\ \forall t \in \mathbb{R}_+, \ \forall T \in [0, t], \ \beta \left(t - T, \chi\right) + T\delta \ge \beta \left(t, \chi\right).$$
(46)

The conditions (46) correspond to growth conditions on the functions $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$. Since the functions $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are continuous, these growth conditions are satisfied if and only if the conditions (*i*) and (*ii*) of (45) are satisfied.

Proposition 3.2 states that the piecewise affine functions $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ are properly formulated if and only if the affine functions $\gamma_j(\cdot, \cdot)$ and $\beta_k(\cdot, \cdot)$ are properly formulated for all $(j,k) \in J \times K$. A similar property exists for the internal boundary condition component for triangular Hamiltonians.

Proposition 3.3: (Proper Formulation of the Internal Boundary Condition Component (Triangular Hamiltonians)): We assume that the internal boundary condition function μ_p defined by (41) and (42) satisfies the following property:

$$g_{p_l} \le \varphi^*(-v_{p_l}) \quad \forall l \in L_p \text{ proper formulation of } \mu_{p_l}(\cdot, \cdot).$$
(47)

When the condition (47) is satisfied, the internal component μ_p is properly formulated [16].

Proof: The proper formulation condition [16] for the internal boundary condition component p read

$$\forall t \in \left[\bar{\delta_{p_1}}, \ \bar{\delta_{p_{l_{p_{\max}}+1}}} \right],$$

$$\inf_{T \in [0, t - \bar{\delta_{p_1}}]} \left(T \varphi^* \left(\frac{\bar{x}_p(t-T) - \bar{x}_p(t)}{T} \right) - \int_{t-T}^t R_p(\tau) d\tau \right) \ge 0.$$

$$(48)$$

In order to show that $\mu_p(\cdot, \cdot)$ is properly formulated when condition (47) holds, we need to prove that condition (48) is satisfied. In formula (48), the *trajectory* $\bar{x}_p(\cdot)$ and *trajectory label* $R_p(\cdot)$ functions associated with the point p are defined by $\begin{cases} \bar{x}_p(t) = x_{p_l} + v_{p_l}(t - \bar{\delta}_{p_l}) & \text{if } \exists l \in L_p \text{ s.t. } t \in [\bar{\delta}_{p_l}, \bar{\delta}_{p_{l+1}}] \\ R_p(t) = g_{p_l} & \text{if } \exists l \in L_p \text{ s.t. } t \in [\bar{\delta}_{p_l}, \bar{\delta}_{p_{l+1}}]. \end{cases}$

Let us consider $t \in [\overline{\delta_{p_1}}, \overline{\delta_{p_{l_{p_{\max}}}}}]$ and $T \in]0, t - \overline{\delta_{p_1}}]$. We define $a \in \mathbb{N}_*$ and $b \in \mathbb{N}_*$ such that $\overline{\delta_{p_a}} \leq t - T \leq \overline{\delta_{p_{a+1}}}$ and $\overline{\delta_{p_b}} \leq t \leq \overline{\delta_{p_{b+1}}}$. For all $l \in [a, b]$, we also define t_l as

$$t_{l} := \begin{cases} \bar{\delta}_{pl+1} - \bar{\delta}_{pl} & \text{if } l > a \text{ and } l < b\\ \bar{\delta}_{pa+1} - (t - T) & \text{if } l = a\\ t - \bar{\delta}_{pb} & \text{if } l = b. \end{cases}$$
(50)

By construction, we have $T = \sum_{l=a}^{b} t_l$. Using the above definitions, the quantities $(\bar{x}_p(t-T) - \bar{x}_p(t))/T$ and $\int_{t-T}^t R_p(\tau) d\tau$ can be expressed as

$$\frac{\bar{x}_p(t-T) - \bar{x}_p(t)}{T} = -\frac{\sum_{l=a}^{b} t_l v_l}{\sum_{l=a}^{b} t_l} \quad \text{and} \quad \int_{t-T}^t R_p(\tau) d\tau = \sum_{l=a}^{b} t_l g_{p_l}.$$
(51)

Equation (51) enables us to write condition (48) as

$$\inf_{T \in [0,t-\delta_{p_1}]} \left(T\varphi^* \left(\frac{\bar{x}_p(t-T) - \bar{x}_p(t)}{T} \right) - \sum_{l=a}^b t_l g_{p_l} \right) \ge 0.$$
(52)

We now prove that condition (47) implies (52). Using the concavity of φ^* for a triangular Hamiltonian ψ , (51) and the fact that $T = \sum_{l=a}^{b} t_l$, we can write Jensen's inequality as

$$T\varphi^*\left(\frac{\bar{x}_p(t-T)-\bar{x}_p(t)}{T}\right) \ge \sum_{l=a}^b t_l \varphi^*(-v_{p_l}).$$
(53)

Since by (47) we have $\forall l \in L_p, \varphi^*(-v_{p_l}) \ge g_{p_l}$, (53) implies that

$$T\varphi^*\left(\frac{\bar{x}_p(t-T) - \bar{x}_p(t)}{T}\right) \ge \sum_{l=a}^b t_l g_{p_l}.$$
 (54)

The above condition implies (52), by taking the infimum over $T \in [0, t - \overline{\delta_{p_1}}]$, for any given $t \in [\overline{\delta_{p_1}}, \overline{\delta_{p_{l_{p_{max}}}}}]$.

Proposition 3.3 states that the piecewise affine function $\mu_p(\cdot, \cdot)$ defined by (41) is properly formulated when the Hamiltonian $\psi(\cdot)$ is triangular, and the affine functions $\mu_{p_l}(\cdot, \cdot)$ are properly formulated (47) for all $l \in L_p$.

When the Hamiltonian $\psi(\cdot)$ is not triangular, the proper formulation of $\mu_p(\cdot, \cdot)$ defined by (41) cannot be checked using (47). However, the proper formulation of $\mu_p(\cdot, \cdot)$ can still be verified in practice. Specifically, the proper formulation of $\mu_p(\cdot, \cdot)$ reads

$$\mathbf{M}_{\mu_{p_m}}(t, v_{p_l}(t - \bar{\delta_{p_l}}) + x_{p_l}) \ge g_{p_l}(t - \bar{\delta_{p_l}}) + h_{p_l} \\
\forall t \in [\bar{\delta_{p_l}}, \bar{\delta_{p_{l+1}}}], \quad \forall l \in L_p, \forall m \in L_p.$$
(55)

Equation (55) is similar to equation (xii) of (56). Hence, we can check (55) by solving a finite number of convex programs of the form (57).

B. Compatibility Conditions

This section defines the compatibility conditions under which the initial, boundary and internal boundary conditions are compatible with each other.

Proposition 3.4: (compatibility Conditions for Piecewise Affine Initial, Boundary and Internal Boundary Conditions): We define the functions $\mathbf{M}_{\mathcal{M}_0,i}(\cdot, \cdot)$, $\mathbf{M}_{\gamma_j}(\cdot, \cdot)$, $\mathbf{M}_{\beta_k}(\cdot, \cdot)$ and $\mathbf{M}_{\mu_l}(\cdot, \cdot)$ by (23), (26), (30) and (35). Given these definitions, the compatibility conditions between the initial, boundary and internal boundary conditions read [16]

(i)
$$a_1\bar{\alpha}_1 + b_1 = c_1\bar{\gamma}_1 + d_1$$

(ii) $a_i \quad \bar{\alpha}_i \quad + 1 + b_i \quad = e_1\gamma_1 + f_1$

(*iii*)
$$\mathbf{M}_{\mathsf{M}_0,i}(t,\xi) \ge c_j t + d_j$$

$$\begin{array}{l} \forall t \in [\bar{\gamma}_j, \bar{\gamma}_{j+1}], \ \forall j \in J, \ \forall i \in I \\ (iv) \qquad \mathbf{M}_{\mathcal{M}_0, \underline{i}}(t, \underline{\chi}) \ge e_k t + f_k \end{array}$$

$$\forall t \in [\bar{\beta}_k, \bar{\beta}_{k+1}], \forall k \in K, \forall i \in I$$

$$\begin{array}{ccc} \forall Y_{\gamma_j}(t,\chi) \geq c_k t + f_k \\ \forall t \in [\overline{\beta}_k, \overline{\beta}_{k+1}], \ \forall k \in K, \ \forall j \in J \\ (w) \qquad \mathbf{M}_{2}, (t,\xi) \geq a, t + d, \end{array}$$

$$\begin{array}{ll} (vii) & \mathbf{M}_{\gamma_j}(t, v_{p_l}[t - \delta_{p_l}) + x_{p_l}) \ge g_{p_l}(t - \delta_{p_l}) + h_{p_l} \\ & \forall t \in [\bar{\delta}_{p_l}, \bar{\delta}_{p_{l+1}}], \forall l \in L_p, \forall p \in P, \forall j \in J \end{array}$$

(viii)
$$\begin{aligned} \mathbf{M}_{\beta_k}(t, v_{p_l}(t - \delta_{p_l}) + x_l) &\geq g_{p_l}(t - \delta_{p_l}) + h_{p_l} \\ \forall t \in [\delta_{p_l}, \delta_{p_{l+1}}], \forall l \in L_p, \forall p \in P, \forall k \in K \end{aligned}$$

$$\begin{array}{ll} (ix) & \mathbf{M}_{\mu_{p_l}}(t,\xi) \ge c_j t + d_j \\ & \forall t \in [\bar{\gamma}_j, \bar{\gamma}_{j+1}], \, \forall p \in P, \, \forall l \in L_p \end{array}$$

$$\begin{array}{ll} (x) & \mathbf{M}_{\mu_{p_l}}(t,\xi) \geq e_k t + f_k \\ \forall t \in [\bar{\beta}_k, \bar{\beta}_{k+1}], \, \forall \underline{p} \in P, \, \forall l \in L_p \end{array}$$

$$\begin{array}{ll} (xi) & \mathbf{M}_{\mathcal{M}_0,i}(t, v_{p_l}(t - \bar{\delta_{p_l}}) + x_{p_l}) \ge g_{p_l}(t - \bar{\delta_{p_l}}) + h_{p_l} \\ & \forall t \in [\bar{\delta_{p_l}}, \bar{\delta_{p_{l+1}}}], \, \forall i \in I, \, \forall l \in L_p, \, \forall p \in P \end{array}$$

$$\begin{array}{ll} (xii) & \mathbf{M}_{\mu_{q_m}}(t, v_{p_l}(t - \bar{\delta_{p_l}}) + x_{p_l}) \ge g_{p_l}(t - \bar{\delta_{p_l}}) + h_{p_l} \\ & \forall t \in [\bar{\delta}_{p_l}, \bar{\delta_{p_{l+1}}}], \ \forall l \in L_p, \ \forall p \in P, \ \forall m \in L_q, \\ & \forall q \in P \setminus \{p\}. \end{array}$$

(56)

The above proposition is the specific instantiation of Theorem 6.4 of [16] for piecewise affine initial, boundary and internal boundary condition functions.

The conditions (i) and (ii) can be easily checked analytically. The other conditions involve inequalities of the form $\mathbf{M}_{\mathbf{c}}(t, x(t)) \ge at + b, \forall t \in [c, d]$, where $\mathbf{M}_{\mathbf{c}}(\cdot, \cdot)$ is convex by Proposition 2.4, and $x(\cdot)$ is affine (possibly constant). Since $\mathbf{M}_{\mathbf{c}}(\cdot, \cdot)$ is convex and $x(\cdot)$ is affine, $\mathbf{M}_{\mathbf{c}}(t, x(t))$ is a convex function of t. Hence an inequality of the above form can be verified by checking that the solution of the following convex program is positive:

$$\begin{array}{ll} \min & \mathbf{M}_{\mathbf{c}}(t, x(t)) - at - b \\ \text{s.t.} & t \in [c, d]. \end{array}$$
 (57)

This program is not in a standard convex form. However, it can easily be solved numerically (for instance using gradient descent methods) since it involves the minimization of a convex function over a convex set. Note that the number of problems of the form (57) that need to be solved grows polynomially with the number of initial, boundary, and internal boundary condition affine blocks.

C. Construction of the Lax-Hopf Algorithm

Finally, this section outlines the algorithm which results from the previous formulas and can be used for the computation of analytical solutions to the HJ PDE (1), as well as the computation of analytical solutions to the first order conservation law (2).

Proposition 3.5: (Numerical Computation of the Moskowitz Function $\mathbf{M}(\cdot, \cdot)$): When the compatibility conditions (56) are satisfied, the inf-morphism property [16] states that the viability episolution \mathbf{M} to the mixed initial-boundary-internal boundary conditions problem (5) is given by the following formula:

$$\forall (t,x) \in \mathbb{R}_+ \times X, \quad \mathbf{M}(t,x) = \min\left[\min_{i \in I} \mathbf{M}_{\mathcal{M}_0,i}(t,x), \min_{j \in J} \mathbf{M}_{\gamma_j}(t,x), \min_{k \in K} \mathbf{M}_{\beta_k}(t,x), \min_{p \in P} \min_{l \in L_p} \mathbf{M}_{\mu_{p_l}}(t,x)\right].$$
(58)

Proof: This proposition is the instantiation of the results of [16] for piecewise affine initial, boundary and internal conditions.

In addition to computing the solution to (1), (36)–(39) enable us to compute the solution to (2), using the following results.

Proposition 3.6: (Numerical Computation of the Spatial Derivative of $\mathbf{M}(\cdot, \cdot)$): Let us consider a Moskowitz function $\mathbf{M}(\cdot, \cdot)$ computed using (58), and $(t, x) \in \mathbb{R}_+ \times X$ such that $\mathbf{M}(\cdot, \cdot)$ is differentiable at (t, x). Since the Moskowitz function $\mathbf{M}(\cdot, \cdot)$ is the minimum of the convex functions $\mathbf{M}_{\mathcal{M}_0,i}(\cdot, \cdot)$, $\mathbf{M}_{\gamma_j}(\cdot, \cdot)$, $\mathbf{M}_{\beta_k}(\cdot, \cdot)$ and $\mathbf{M}_{\mu_l}(\cdot, \cdot)$ for $(i, j, k, l) \in I \times J \times K \times L$, there exists a component \mathbf{M}_a which is equal to the Moskowitz function at (t, x), i.e., $\mathbf{M}(t, x) = \mathbf{M}_a(t, x)$. We assume that $\mathbf{M}_a(\cdot, \cdot)$ is differentiable at (t, x). Given these assumptions, we have the following property:

$$\frac{\partial \mathbf{M}(t,x)}{\partial x} = \frac{\partial \mathbf{M}_a(t,x)}{\partial x}.$$
(59)

Proof: Let us define the function $w(\cdot, \cdot)$ as $w(\cdot, \cdot) := \mathbf{M}_a(\cdot, \cdot) - \mathbf{M}(\cdot, \cdot)$. Since $\mathbf{M}(\cdot, \cdot)$ and $\mathbf{M}_a(\cdot, \cdot)$ are both differentiable at (t, x), $w(\cdot, \cdot)$ is also differentiable at (t, x). By definition of $\mathbf{M}(\cdot, \cdot)$, the function $w(\cdot, \cdot)$ is positive, and satisfies w(t, x) = 0. Hence, (t, x) minimizes $w(\cdot, \cdot)$, which yields $\partial g(t, x)/\partial x = 0$. This last equality implies (59).

Since $\mathbf{M}(\cdot, \cdot)$ is the minimum of a finite number of convex functions, it is differentiable almost everywhere [11], and its associated density function $\rho(\cdot, \cdot)$ is thus defined almost everywhere on $\mathbb{R}_+ \times X$. Fig. 6 shows a comparison between the results obtained with the Lax-Hopf algorithm and the Godunov scheme for a benchmark example adapted from [8]. As can be



Fig. 6. Illustration of the computation of the density function. Top: Moskowitz function computed by the Lax-Hopf algorithm, using a triangular Hamiltonian and piecewise affine initial and boundary conditions. The value of the Moskowitz function is represented by a level of gray. Center: associated density function computed using the Lax-Hopf algorithm. Bottom: computation of the density function corresponding to the same problem using the Godunov scheme [21], [30], [38]. As can be seen, the Lax-Hopf method perfectly captures shocks, which are otherwise smoothed out by the Godunov scheme.

seen in this figure, the Lax-Hopf algorithm does not induce diffusion errors inherent to finite difference schemes such as the Godunov scheme. The Godunov scheme is a first order accurate numerical scheme used to compute the density $\rho(t, x)$ solution to the conservation (2), see for example [27], [38], [50].

D. Examples of Numerical Computations Using the Lax-Hopf Algorithm

The striking difference in terms of computational cost between the Lax-Hopf algorithm and any finite difference scheme, such as the Godunov scheme, is that one does not need intermediate computations for times $M \in \{1, \ldots, n_t\}$ to compute the solution at time step n_t . In other words, no iteration is needed to compute the value of the solution at any given time. Note



Fig. 7. Comparison between the Lax–Hopf algorithm and the analytical solution of problem (60). The solutions at times t = 0, t = 5, t = 10 and t = 15 are represented in the upper left, upper right, lower left and lower right subfigures respectively. In each of these subfigures, the analytical solution is represented by a dashed line, and the solution yielded by the Lax-Hopf algorithm is represented using dots. The difference between the two solutions is of the order of machine error, and thus not visible on these figures.

however that unlike finite difference schemes or dynamic programming, the computational cost of the Lax-Hopf algorithm is related to the number of piecewise affine elements in the initial, boundary and internal boundary conditions.

1) Validation of the Lax-Hopf Algorithm (Density Function): We compare the Lax-Hopf algorithm and the Godunov scheme [27], [30], [50] (and its specific instantiation as the Daganzo cell transmission model [21], [22]), which is widely used by the transportation research community.

In this implementation, we consider a (non piecewise affine) Greenshields Hamiltonian [31], defined by $\psi(\rho) = \nu/\rho^* \rho(\rho^* - \rho)$, where $\nu = 1$ and $\rho^* = 4$ (dummy values). We consider the following initial and upstream boundary condition functions:

$$\begin{cases} a := (-2, -4, -1) \\ b := (0, 20, -40) \\ \bar{\alpha} := (0, 10, 20, 30) \end{cases} \begin{cases} c := (2) \\ d := (0) \\ \bar{\gamma} := (0, 20). \end{cases}$$
(60)

These initial and upstream boundary conditions were used previously in [50]. It is easy to check using (45) and (56) that the boundary condition component is properly formulated, and that the compatibility conditions are satisfied in this case.

We compute the Moskowitz and density functions solution to the initial and upstream boundary conditions problem (60) using the Lax-Hopf algorithm (58) and (59), and compare the results with the analytical formula derived in [50]. The results are illustrated in Fig. 7.

As can be seen in Fig. 7, the numerical solution of the LWR PDE using the Lax-Hopf algorithm is identical to the analytical solution computed by the method of characteristics in [8]. In addition to its high accuracy, the Lax–Hopf algorithm is not limited by the *Courant Friedrichs Lewy* (CFL) time step size condition inherent to many finite difference schemes, and can thus compute the solution at a given time faster than finite difference schemes, such as the Godunov scheme.

The Godunov scheme is only stable when the CFL condition $\nu\Delta t \leq \Delta x$ is satisfied, where Δt and Δx represent



Fig. 8. Computational time comparison between the Lax–Hopf algorithm and the Godunov scheme (60). This figure represents the time required to compute the solution of problem (60) at time t = 15, using both the Godunov scheme (dots) and the Lax–Hopf algorithm (dashed line).

the discretized time and space steps. We consider the mixed initial-boundary-internal boundary conditions problem (60) as previously, and compute the solution at time t = 15 using the Godunov scheme and the Lax-Hopf algorithm, for different space resolutions Δx . The computational times are shown in Fig. 8. For fairness of the comparison, all algorithms presented here have been implemented in the same programming language (Matlab), and run on the same platform (Thinkpad T61 running Windows XP).

Fig. 8 shows that the Lax–Hopf algorithm is significantly faster than the Godunov scheme when high accuracy is required. Indeed, the Lax–Hopf algorithm can compute the solution at time t = 15 using only the knowledge of the initial and boundary conditions. In contrast, the Godunov scheme has to compute the solution for each time step Δt , which is upper-constrained by the CFL condition, and thus cannot be arbitrary large.

2) Integration of Internal Boundary Conditions: In this implementation, we consider a triangular Hamiltonian [16] with parameters $\nu^{\flat} = 1$, $\gamma = 1$, $\omega = 6$, $\nu^{\ddagger} = 1/5$ and $\delta = \varphi^{*}(0) = 1$. We also consider initial, upstream and downstream boundary condition functions defined by (41), and associated with the following set of parameters $a_i, b_i, \bar{\alpha}_i, c_j, d_j, \bar{\gamma}_j, e_k, f_k, \bar{\beta}_k$:

$$\begin{cases} a := \left(-1, -\frac{7}{2}, -\frac{1}{10}, -\frac{7}{5}\right) \\ b := \left(0, -\frac{25}{2}, -\frac{43}{2}, \frac{9}{2}\right) \\ \bar{\alpha} := \left(0, 5, 10, 20, 25\right) \\ c := \left(1, \frac{1}{2}, \frac{4}{5}, \frac{7}{10}\right) \\ d := \left(0, \frac{3}{2}, -\frac{9}{5}, -\frac{3}{10}\right) \\ \bar{\gamma} := \left(0, 3, 11, 15, 20\right) \\ e := \left(0, \frac{2}{5}, 0, \frac{4}{5}\right) \\ f := \left(-\frac{61}{2}, -\frac{289}{10}, -\frac{221}{10}, -\frac{365}{10}\right) \\ \bar{\beta} := \left(0, 4, 17, 18, 20\right). \end{cases}$$
(61)

Since $\forall j \in J, c_j \leq \varphi^*(0)$ and $\forall k \in K, e_k \leq \varphi^*(0)$, the upstream and downstream boundary components $\gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ associated with the specific numerical example (61) are properly formulated. Additionally, it can be shown using (57) that the functions $\mathcal{M}_0(\cdot, \cdot), \gamma(\cdot, \cdot)$ and $\beta(\cdot, \cdot)$ defined by (41) and associated with the numerical example (61) satisfy conditions (56). We first compute the solution to (1) associated with (61) numerically using the Lax-Hopf algorithm. The results are shown in Fig. 9.

Fig. 9. Example of integration of an internal boundary condition into the solution of the HJ PDE (1). Left: Computation of the solution to the mixed initial-boundary conditions problem with parameters listed in (61). Right: Computation of the solution to the mixed initial-boundary-internal boundary conditions problem (61) and (62). The initial, boundary and internal boundary conditions are represented by solid lines.

We now incorporate a single internal boundary condition, defined by the following parameters:

$$\begin{cases} v_1 \coloneqq \left(\frac{2}{5}, 0, \frac{1}{2}, \frac{1}{2}\right) \\ g_1 \coloneqq \left(\frac{1}{5}, 1, \frac{1}{4}, 0\right) \\ h_1 \coloneqq \left(-18, -19, -20, -21, -21\right) \\ \bar{\delta}_1 \coloneqq \left(3, 8, 9, 14, 15\right). \end{cases}$$
(62)

The explicit expression of $\varphi^*(\cdot)$ can be found in [16]

$$\forall x \in [\nu^{\flat}, \nu^{\sharp}], \quad \varphi^*(x) = \nu^{\flat}(x - v).$$

Using this expression, it is easy to check using (47) that the condition (62) is properly formulated. Additionally, it can be shown using (57) that the functions $\mathcal{M}_0(\cdot, \cdot)$, $\gamma(\cdot, \cdot)$, $\beta(\cdot, \cdot)$ and $\mu_1(\cdot, \cdot)$ defined by (41) and associated with the numerical example (61) and (62) satisfy conditions (56). As can be seen in Fig. 9, the incorporation of the internal boundary condition modifies the value of the solution around it, and enables us to add new information to the solution.

IV. CONCLUSION

This article presents a Lax-Hopf based method to compute solutions to a Hamilton-Jacobi partial differential equation for which not only initial and boundary conditions are prescribed but also internal boundary conditions. Using previous results obtained for general functions, the article computes an analytical form of the solution of the PDE in the case of affine prescribed conditions. By nature of the algorithm, the accuracy of the method surpasses any finite difference scheme, since it consists in the numerical evaluation of a function. The computational cost is of comparable magnitude in the general case, and almost zero if information is only requested from the algorithm for specific times (versus for a time range), in contrast to finite difference schemes which require to grid the whole spatio-temporal space according to constraints driven by stability issues. The performance of the method is assessed in practice using benchmark analytical examples. Extensions of this method could include more general classes of functions, beyond piecewise affine conditions.

Subsequent work has included the integration of this numerical scheme into the development of the *Mobile Millennium* system, a traffic information system launched on November 10, 2008 from the Berkeley campus. *Mobile Millennium* gathers positioning data from cellular phones, integrates it into traffic flow models, and broadcasts it back to the phones in real-time [52]. In particular, one key aspect currently under investigation is the possibility of explicitly using the piecewise affine solutions obtained in this article to solve variational data assimilation problems. The explicit nature of the obtained solution makes it possible to integrate the nonlinearity of the model in the analytical evaluation of the solution, while leaving the unknown variable of the inverse modeling problem appear linearly, a desirable feature, which has already been used in practice for travel time estimation [17]. Extension of these ideas are very promising for estimation problems using Lagrangian data.

APPENDIX ANALYTICAL LAX-HOPF FORMULA ASSOCIATED WITH AN AFFINE INTERNAL BOUNDARY CONDITION

We recall the definition of the affine internal boundary condition

$$\mu_l(t,x) = \begin{cases} g_l(t-\bar{\delta}_l) + h_l & \text{if } x = x_l + v_l(t-\bar{\delta}_l) \\ & \text{and } t \in [\bar{\delta}_l, \bar{\delta}_{l+1}] \\ +\infty & \text{otherwise.} \end{cases}$$
(63)

For the computation of the corresponding component $\mathbf{M}_{\mu_l}(t,x)$, we assume that (t,x) satisfy $x \neq x_l + v_l(t - \overline{\delta}_l)$. In addition, we assume that the constants g_l and v_l satisfy $0 \leq g_l \leq \varphi^*(-v_l)$. The internal boundary condition component has a domain of definition, which can be computed analytically as follows.

Proposition A.1: (Domain of Definition of an Affine Internal Boundary Condition Component): The domain of definition of $\mathbf{M}_{\mu_l}(\cdot, \cdot)$ is given by the following formula:

$$Dom(\mathbf{M}_{\mu_l}) = \{(t, x) \in \mathbb{R}_+ \times X \text{ such that } t \ge \bar{\delta}_l \\ and \ x_l - \nu^{\sharp}(t - \bar{\delta}_l) \le x \le x_l + \nu^{\flat}(t - \bar{\delta}_l)\}$$
(64)

Proof: The Lax-Hopf formula (33) implies

$$Dom(\mathbf{M}_{\mu_l}) := \{(t, x) \in \mathbb{R}_+ \times X \text{ s.t.} \\ \exists T \in \mathbb{R}^*_+ \cap [t - \bar{\delta}_{l+1}, t - \bar{\delta}_l] \text{ and} \\ \frac{x_l + v_l(t - \bar{\delta}_l - T) - x}{T} \in Dom(\varphi^*) \}$$

Since T > 0, the condition $(x_l + v_l(t - \overline{\delta}_l - T) - x)/T \in$ $[-\nu^{\flat}, \nu^{\sharp}] \quad \text{is equivalent} \\ (x_l + v_l(t - \overline{\delta}_l) - x) / (\nu^{\sharp} + v_l)$ $Dom(\varphi^*)$ to T \geq and T $(x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l)$. Hence, \geq $Dom(\varphi^*)$ if and only if the set (t,x) \in $\mathbb{R}^*_+ \cap [t - \overline{\delta}_{l+1}, t - \overline{\delta}_l] \cap [(x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp})]$ $+v_l), +\infty[\cap[(x_l+v_l(t-\bar{\delta}_l)-x)/(-\nu^{\flat}+v_l),+\infty]$ is not empty, which implies

$$\max\left(0, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{-\nu^{\flat} + v_l}, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{\nu^{\sharp} + v_l}\right) \le t - \overline{\delta}_l.$$

This last inequality implies (64).

The method followed next also makes use of an auxiliary objective function, which is later used to explicitly find the minimizer.

Definition A.2: (Auxiliary Objective Function): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\mu_l})$, we define the function $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$ as

$$\forall T \in \mathbb{R}^*_+, \kappa_{\overline{\delta}_l, g_l, h_l, x_l, v_l, t, x}(T) := \left(g_l(t - T - \overline{\delta}_l) + h_l + T\varphi^*\left(\frac{x_l + v_l(t - \overline{\delta}_l - T) - x}{T}\right)\right).$$
(65)

Given this definition, (33) becomes

$$\mathbf{M}_{\mu_{l}}(t,x) = \kappa_{\overline{\delta}_{l},g_{l},h_{l},x_{l},v_{l},t,x}(T)$$

$$\inf_{T \in \left[\max\left(0,(x_{l}+v_{l}(t-\overline{\delta}_{l})-x)/(-\nu^{\flat}+v_{l}),(x_{l}+v_{l}(t-\overline{\delta}_{l})-x)/(\nu^{\sharp}+v_{l}),t-\overline{\delta}_{l+1}\right),t-\overline{\delta}_{l}\right]} (66)$$

Since $\varphi^*(\cdot)$ is convex, the function $h : u \to \varphi^*(u - v_l)$ is convex, and its associated perspective function $T \to Th(x_l + v_l(t - \overline{\delta}_l) - x/T)$ is also convex for T > 0 by [11]. Hence the function $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$ is convex as the sum of two convex functions. The subderivative of $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$ is given by

$$\begin{aligned} \forall T \in \left[\max\left(0, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{\neg \nu^{\flat} + v_l}, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{\nu^{\sharp} + v_l}, t - \overline{\delta}_{l+1}\right), \\ t - \overline{\delta}_l \right], \\ \partial_{-\kappa_{\overline{\delta}_l}, g_l, h_l, x_l, v_l, t, x}(T) &= \left\{ w | \exists v \in \partial_{-\varphi}^* (\frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} - v_l), \\ w &= -g_l + \varphi^* \left(\frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} - v_l \right) - \frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} v \right\} \\ &= -g_l + \varphi^* \left(\frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} - v_l \right) \\ &- \frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} \partial_{-\varphi}^* \left(\frac{x_l + v_l(t - \overline{\delta}_l) - x}{T} - v_l \right) \end{aligned}$$
(67)

with a slight abuse of notation for the second equality as previously. Because of the higher complexity of this case, we need to define intermediate quantities used in the explicit minimization.

- Definition A.3: (Densities Associated With v_l and g_l):
- We define the function f_{vl}(·) as f_{vl}: ρ→ψ(ρ) ρvl. The function f_{vl} is concave as the sum of concave functions, and attains its maximum value φ*(-vl) (by definition of the function φ*(·)) for a given ρ := ρl.
- Note that since $v_l \in [0, \nu^{\flat}[$, the function f_{v_l} satisfies $f_{v_l}(0) = 0$ and $f_{v_l}(\omega) \leq 0$. By assumption, we also have $g_l \leq \varphi^*(-v_l)$, and since $f_{v_l}(\cdot)$ is concave and continuous, there exist two solutions $\rho_1(v_l, g_l) \in [0, \rho_l]$ and $\rho_2(v_l, g_l) \in [\rho_l, \omega]$ such that $f_{v_l}(\rho_p(v_l, g_l)) = g_l$ for $p \in \{1, 2\}$ (see Fig. 5).
- For $p \in \{1, 2\}$, we also define $u_p(v_l, g_l)$ as elements of $-\partial_+\psi(\rho_p(v_l, g_l))$. Note that since f_{v_l} is concave, it is increasing on $[0, \rho_l]$ and decreasing on $[\rho_l, \omega]$, which implies that $u_1(v_l, g_l) \leq -v_l$ and $u_2(v_l, g_l) \geq -v_l$. Note also that the Legendre-Fenchel inversion formula implies that $u_p(v_l, g_l) \in \text{Dom}(\varphi^*)$ for $p \in \{1, 2\}$.

Definition A.4: (Capture Times Associated With $u_p(v_l, g_l)$, for $p \in \{1, 2\}$):

• We define
$$T_p(t, x, v_l, g_l)$$
 for $p \in \{1, 2\}$ as

$$T_p(t, x, v_l, g_l) \coloneqq \begin{cases} \frac{x_l + v_l(t - \overline{\delta}_l) - x}{u_p(v_l, g_l) + v_l} & \text{if } u_p(v_l, g_l) \neq -v_l \\ +\infty & \text{if } u_p(v_l, g_l) = -v_l. \end{cases}$$
(68)

- The definition of $T_p(\cdot, \cdot, \cdot, \cdot)$ implies that $T_1(t, x, v_l, g_l) \ge 0$ if and only if $x_l + v_l(t \overline{\delta}_l) x \le 0$, and that $T_2(t, x, v_l, g_l) \ge 0$ if and only if $x_l + v_l(t \overline{\delta}_l) x \ge 0$.
- Note also that since $u_p(v_l, g_l) \in [-\nu^{\flat}, \nu^{\sharp}]$, we have $T_1(t, x, v_l, g_l) \geq (x_l + v_l(t \overline{\delta}_l) x)/(-\nu^{\flat} + v_l)$ when $x_l + v_l(t - \overline{\delta}_l) - x \leq 0$ and $T_2(t, x, v_l, g_l) \geq (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp} + v_l)$ when $x_l + v_l(t - \overline{\delta}_l) - x \geq 0$.

The previous definitions can now be used to compute the explicit minimizer.

Proposition A.5: (Explicit Minimization of $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\mu_l})$, the function $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$ has the following minimizer over $[\max(0, (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l), (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp} + v_l), t - \overline{\delta}_{l+1}), t - \overline{\delta}_l]$:

$$\begin{cases} (i) \quad T_{1}(t, x, v_{l}, g_{l}) & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \leq 0 \\ & \text{and } T_{1}(t, x, v_{l}, g_{l}) \in [t - \bar{\delta}_{l+1}, t - \bar{\delta}_{l}] \\ (ii) \quad t - \bar{\delta}_{l} & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \leq 0 \\ & \text{and } T_{1}(t, x, v_{l}, g_{l}) \geq t - \bar{\delta}_{l} \\ (iii) \quad t - \bar{\delta}_{l+1} & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \leq 0 \\ & \text{and } T_{1}(t, x, v_{l}, g_{l}) \leq t - \bar{\delta}_{l+1} \\ (iv) \quad T_{2}(t, x, v_{l}, g_{l}) & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \geq 0 \\ & \text{and } T_{2}(t, x, v_{l}, g_{l}) \in [t - \bar{\delta}_{l+1}, t - \bar{\delta}_{l}] \\ (v) \quad t - \bar{\delta}_{l} & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \geq 0 \\ & \text{and } T_{2}(t, x, v_{l}, g_{l}) \geq t - \bar{\delta}_{l} \\ (vi) \quad t - \bar{\delta}_{l+1} & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \geq 0 \\ & \text{and } T_{2}(t, x, v_{l}, g_{l}) \leq t - \bar{\delta}_{l} \\ (vi) \quad t - \bar{\delta}_{l+1} & \text{if } x_{l} + v_{l}(t - \bar{\delta}_{l}) - x \geq 0 \\ & \text{and } T_{2}(t, x, v_{l}, g_{l}) \leq t - \bar{\delta}_{l+1}. \end{cases}$$

Proof: The function $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(T)$ is minimal for a given T > 0 if and only if $0 \in \partial_{-}\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(T)$. Since $u_p(v_l,g_l) \in -\partial_{+}\psi(\rho_p(v_l,g_l))$ for $p \in \{1,2\}$, we have by the Legendre-Fenchel inversion formula that $\rho_p(v_l,g_l) \in \partial_{-}\varphi^*(u_p(v_l,g_l))$. This last formula imply that $0 \in \partial_{-}(\varphi^*(\cdot) - \cdot \rho_p(v_l,g_l))(u_p(v_l,g_l))$, and thus that

$$\psi(\rho_p(v_l, g_l)) = \inf_{u \in \text{Dom}(\varphi^*)} [\varphi^*(u) - \rho_p(v_l, g_l)u]$$
$$= \varphi^*(u_p(v_l, g_l)) - \rho_p(v_l, g_l)u_p(v_l, g_l)$$

Since we consider only positive capture times T, we have to consider two situations:

• If $x_l + v_l(t - \bar{\delta}_l) - x \leq 0$, we have that $T_2(t, x, v_l, g_l) \leq 0$ and $T_1(t, x, v_l, g_l) \geq 0$. The relations $\psi(\rho_1(v_l, g_l)) - \rho_1(v_l, g_l)v_l = g_l, \ \rho_1(v_l, g_l) \in \partial_-\varphi^*(u_1(v_l, g_l))$ and $\psi(\rho_1(v_l, g_l)) = \varphi^*(u_1(v_l, g_l)) - \rho_1(v_l, g_l)u_1(v_l, g_l)$ imply that $-g_l + \varphi^*(u_1(v_l, g_l)) - (u_1(v_l, g_l) + v_l)\rho_1(v_l, g_l) = 0$. Hence, using our definition of $T_1(t, x, v_l, g_l) := (x_l + v_l(t - \overline{\delta}_l) - x)/(u_1(v_l, g_l) + v_l)$, we have that

$$0 = -g_l + \varphi^* \left(\frac{x_l + v_l(t - \bar{\delta}_l) - x}{T_1(t, x, v_l, g_l)} - v_l \right) - \frac{x_l + v_l(t - \bar{\delta}_l) - x}{T_1(t, x, v_l, g_l)} \rho_1(v_l, g_l).$$

Using (67), we have

$$0 \in \partial_{-}\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(T_1(t,x,v_l,g_l)),$$

and thus $T_1(t, x, v_l, g_l)$ minimizes $\kappa_{\bar{\delta}_l, g_l, h_l, x_l, v_l, t, x}(T)$ for positive times T.

The cases (i), (ii) and (iii) in (69) are obtained using the convexity of $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$. Note that in our situation, Definition 2.20 implies that $T_1(t,x,v_l,g_l) \ge (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l)$.

Since $x_l + v_l(t - \overline{\delta}_l) - x \leq 0$, we also have that $(x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp} + v_l) \leq 0$. Hence, we have that

$$T_1(t, x, v_l, g_l)$$

$$\geq \max\left(0, \frac{x_l + v_l(t - \bar{\delta}_l) - x}{-\nu^\flat + v_l}, \frac{x_l + v_l(t - \delta_l) - x}{\nu^\sharp + v_l}\right).$$

Hence, the condition $T_1(t, x, v_l, g_l) \geq \max(0, (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^\flat + v_l), (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^\sharp + v_l), t - \overline{\delta}_{l+1})$ is satisfied if and only if $T_1(t, x, v_l, g_l) \geq t - \overline{\delta}_{l+1}$. Note also that if the previous condition is not satisfied, then $\max(0, (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^\flat + v_l), (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^\sharp + v_l), t - \overline{\delta}_{l+1}) = t - \overline{\delta}_{l+1}$.

• If $x_l + v_l(t - \overline{\delta}_l) - x \ge 0$, we have that $T_1(t, x, v_l, g_l) \le 0$ and $T_2(t, x, v_l, g_l) \ge 0$. The relations $\psi(\rho_1(v_l, g_l)) - \rho_1(v_l, g_l)v_l = g_l, \ \rho_2(v_l, g_l) \in \partial_-\varphi^*(u_2(v_l, g_l))$ and $\psi(\rho_2(v_l, g_l)) = \varphi^*(u_2(v_l, g_l)) - \rho_2(v_l, g_l)u_2(v_l, g_l)$ imply that $-g_l + \varphi^*(u_1(v_l, g_l)) - (u_1(v_l, g_l) + v_l)\rho_1(v_l, g_l) = 0$. Hence, using our definition of $T_2(t, x, v_l, g_l) := (x_l + v_l(t - \overline{\delta}_l) - x)/(u_2(v_l, g_l) + v_l)$, we have that

$$0 = -g_l + \varphi^* \left(\frac{x_l + v_l(t - \bar{\delta}_l) - x}{T_2(t, x, v_l, g_l)} - v_l \right) - \frac{x_l + v_l(t - \bar{\delta}_l) - x}{T_2(t, x, v_l, g_l)} \rho_2(v_l, g_l).$$

Using (67), we have

$$0 \in \partial_{-}\kappa_{\bar{\delta}_l,q_l,h_l,x_l,v_l,t,x}(T_2(t,x,v_l,g_l)),$$

and thus $T_2(t, x, v_l, g_l)$ minimizes $\kappa_{\overline{\delta}_l, g_l, h_l, x_l, v_l, t, x}(T)$ for positive times T.

The cases (iv), (v) and (vi) in (69) are obtained using the convexity of $\kappa_{\overline{\delta}_l,g_l,h_l,x_l,v_l,t,x}(\cdot)$. Note that in our situation, Definition 2.20 implies that $T_2(t,x,v_l,g_l) \ge (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp} + v_l)$.

Since $x_l + v_l(t - \overline{\delta}_l) - x \ge 0$, we also have that $(x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l) \le 0$. Hence, we have that

$$\geq \max\left(0, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{-\nu^{\flat} + v_l}, \frac{x_l + v_l(t - \overline{\delta}_l) - x}{\nu^{\sharp} + v_l}\right).$$

Hence, the condition $T_2(t, x, v_l, g_l) \ge \max(0, (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l), (x_l + v_l(t - \overline{\delta}_l) - x)/(\nu^{\sharp} + v_l), t - \overline{\delta}_{l+1})$ is satisfied if and only if $T_2(t, x, v_l, g_l) \ge t - \overline{\delta}_{l+1}$. Note also that if the previous condition is not satisfied, then $\max(0, (x_l + v_l(t - \overline{\delta}_l) - x)/(-\nu^{\flat} + v_l), t - \overline{\delta}_{l+1}).$

Once the minimizer is computed, it can be used to find the explicit expression of the value function.

Proposition A.6: (Computation of $\mathbf{M}_{\mu_l}(\cdot, \cdot)$): For all $(t,x) \in \text{Dom}(\mathbf{M}_{\mu_l})$, the expression $\mathbf{M}_{\mu_l}(t,x)$ can be computed using the following formula:

$$\mathbf{M}_{\mu_{l}}(t,x) = \begin{cases}
(i) & \psi(\rho_{1}(v_{l},g_{l}))(t-\bar{\delta}_{l}) + (x_{l}-x)\rho_{1}(v_{l},g_{l}) + h_{l} \\
& \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \leq x \\
& \text{and } T_{1}(t,x,v_{l},g_{l}) \in [t-\bar{\delta}_{l+1},t-\bar{\delta}_{l}] \\
(ii) & \psi(\rho_{2}(v_{l},g_{l}))(t-\bar{\delta}_{l}) + (x_{l}-x)\rho_{2}(v_{l},g_{l}) + h_{l} \\
& \text{if } x_{l} + v_{l}(t-\bar{\delta}_{l}) \geq x \\
& \text{and } T_{2}(t,x,v_{l},g_{l}) \in [t-\bar{\delta}_{l+1},t-\bar{\delta}_{l}]
\end{cases}$$
(70)

$$\begin{array}{ll} (iii) \quad h_l + (t - \overline{\delta}_l)\varphi^* \left(\frac{x_l - x}{t - \delta_l}\right) \\ & \text{if } x_l + v_l(t - \overline{\delta}_l) \leq x \text{ and } T_1(t, x, v_l, g_l) \geq t - \overline{\delta}_l \\ & \text{or if } x_l + v_l(t - \overline{\delta}_l) \geq x \text{ and } T_2(t, x, v_l, g_l) \geq t - \overline{\delta}_l \\ (iv) \quad g_l(\overline{\delta}_{l+1} - \overline{\delta}_l) + h_l + (t - \overline{\delta}_{l+1})\varphi^* \left(\frac{x_l + v_l(\overline{\delta}_{l+1} - \overline{\delta}_l) - x}{t - \overline{\delta}_{l+1}}\right) \\ & \text{if } x_l + v_l(t - \overline{\delta}_l) \leq x \text{ and } T_1(t, x, v_l, g_l) \leq t - \overline{\delta}_{l+1} \\ & \text{or if } x_l + v_l(t - \overline{\delta}_l) \geq x \text{ and } T_2(t, x, v_l, g_l) \leq t - \overline{\delta}_{l+1}. \end{array}$$

Proof: The cases
$$(iii)$$
 and (iv) in (71) are trivially obtained

by combining (66) and (69). The cases (*ii*) and (*iii*) in (71) are trivially obtained by combining (66) and (69).

The cases (i) and (ii) in (70) are also obtained by combining (66) and (69). By combining the formula $\varphi^*((x_l + v_l(t - \overline{\delta}_l - T_p(t, x, v_l, g_l)) - x)/T_p(t, x, v_l, g_l)) = \psi(\rho_p(v_l, g_l)) + (x_l + v_l(t - \overline{\delta}_l - T_p(t, x, v_l, g_l)) - x)/T_p(t, x, v_l, g_l)\rho_p(v_l, g_l)$ and the definition of $\rho_p(v_l, g_l)$, we have $-g_l + \varphi^*((x_l + v_l(t - \overline{\delta}_l - T_p(t, x, v_l, g_l)) - x)/T_p(t, x, v_l, g_l)) = (\rho_p(v_l, g_l)x_l + v_l(t - \overline{\delta}_l) - x)/T_p(t, x, v_l, g_l).$ Using again the definition of $\rho_p(v_l, g_l)$, we have

 $g_l - \rho_p(v_l, g_l)v_l = \psi(\rho_p(v_l, g_l))$, which after some algebra leads to the cases (i) and (ii) in (70).

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