

# Lax–Hopf Based Incorporation of Internal Boundary Conditions Into Hamilton–Jacobi Equation.

## Part I: Theory

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**Abstract**—This article proposes a new approach for computing a semi-explicit form of the solution to a class of *Hamilton–Jacobi (HJ) partial differential equations (PDEs)*, using control techniques based on viability theory. We characterize the epigraph of the value function solving the HJ PDE as a capture basin of a target through an auxiliary dynamical system, called “characteristic system”. The properties of capture basins enable us to define *components* as building blocks of the solution to the HJ PDE in the Barron/Jensen-Frankowska sense. These components can encode initial conditions, boundary conditions, and internal “boundary” conditions, which are the topic of this article. A generalized Lax–Hopf formula is derived, and enables us to formulate the necessary and sufficient conditions for a mixed initial and boundary conditions problem with multiple internal boundary conditions to be well posed. We illustrate the capabilities of the method with a data assimilation problem for reconstruction of highway traffic flow using Lagrangian measurements generated from *Next Generation Simulation (NGSIM)* traffic data.

**Index Terms**—Hamilton–Jacobi (HJ), next generation simulation (NGSIM), partial differential equations (PDEs).

### I. INTRODUCTION

#### A. Background and Motivation

A common mathematical tool for modeling *distributed parameter systems* is *partial differential equations (PDEs)*. They provide an efficient way of representing physical phenomena in a mathematically compact manner, which integrates the distributed features of the systems of interest. Among PDEs, a specific class stands out, *conservation laws* [40], which model phenomena in which a balance equation governs the physics (for example mass balance, momentum balance, etc.). Hyperbolic scalar conservation laws appear naturally in hydrodynamics, gas dynamics, or biological systems, etc. In one dimension (for example to model irrigation channels, gas in pipes, or blood in small blood vessels), hyperbolic scalar conservation laws have

a direct counterpart in *Hamilton–Jacobi (HJ)* theory [30], which we use in this article.

Conservation laws use an *Eulerian* framework, i.e., a (static) *control volume* based representation of the physics, in which conservation is expressed. This is in contrast with *Lagrangian* approaches, which are *trajectory*-based, i.e., can be used to characterize the evolution of quantities (such as particles) along trajectories. When solving PDEs, it is common to use *boundary conditions (BCs)*, which are inherently Eulerian, and *initial conditions (ICs)*. BCs and ICs can be integrated with the PDE in a Cauchy problem [30], which if well posed leads to existence and uniqueness of a solution. For phenomena which are evolving in time, it is not common to prescribe data other than at the boundary of the domain (BCs). Beyond the mathematical reasons for this fact (to do with the Cauchy problem), there are also historical or technological reasons: usually, for conservation laws, data is only prescribed (and eventually controlled) at the boundary of the domain of interest. Moreover, for sensing data, while it might be possible to place sensors other than at the boundary of the domain, the use of *mobile* sensors which travel inside the domain along trajectories is still relatively new, as is the field of *Lagrangian sensing* for distributed parameter systems. The fusion of Lagrangian and Eulerian data into Eulerian distributed parameter model is precisely the focus of the current article.

The fundamental problem in prescribing *internal* data inside the domain for a conservation law (or equivalently for its Hamilton–Jacobi counterpart) is the potentially introduced discontinuities due to this additional information, which might result from incompatibilities of the data with BCs or ICs. While such discontinuities are standard in the definition of solutions to hyperbolic scalar conservation laws (see the *entropy solution* defined by Oleinik [48] for conservation laws in unbounded domains, and later in [9], [39] for bounded domains), prescribing additional data which might introduce other discontinuities is to our best knowledge posed as such an open problem.

The present article solves the problem of prescribing *internal conditions* for the Hamilton–Jacobi counterpart of scalar hyperbolic conservation laws. The *viscosity solution* to the HJ PDE goes back to the seminal work of Crandall, Evans and Lions [24], [25], which finds its applications in numerous fields, including control theory [7], [8]. The theory of viscosity solutions has shaped engineering by the application of distributions to solve this problem, on which a significant theoretical framework was built since the 1990’s, particularly in control theory. For the present work, and because of the nature of the internal

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conditions we would like to prescribe for the PDE, we use *viability theory* [2], which is a set valued approach [5], [6] to the same problems. The benefit of using this theory (which has some links with nonsmooth analysis [20]) is the possibility of posing the problem as a *capture problem* [3]: control theoretic properties give an elegant way of integrating this data in the Cauchy problem. The equivalence between different approaches to the Hamilton-Jacobi equations proposed by viscosity theory, viability theory, and nonsmooth analysis has been investigated in the literature, see for example [11], [16], [31], [42]. So even if our work directly refers viability theory, by the article [31] it is de facto mathematically related to viscosity techniques.

This article is a follow up article to earlier work [4] in which we use viability techniques for solving Dirichlet problems with inequality constraints (obstacles) for a class of HJ PDEs enclosing this particular equation. In the work [4], the hypograph of the solution of this class of HJ PDEs was defined as the capture basin of a target associated with initial and boundary conditions under an auxiliary control system, viable in an environment associated with some inequality constraints. From the tangential condition characterizing capture basins, the authors of [4] proved that this solution is the unique upper semicontinuous solution to the HJ PDE in the Barron/Jensen-Frankowska sense. We show how this framework allows us to translate properties of capture basins into corresponding properties of the solutions to this problem. For instance, this approach provides a representation formula of the solution, which boils down to the Lax-Hopf formula in the absence of constraints.

In the context of traffic flow theory, this work is of particular interest. Indeed, macroscopic highway traffic flow models go back to the pioneering work of Lighthill, Whitham and Richards [41], [49], in which highway traffic flow is modeled with a nonlinear first order hyperbolic PDE with concave flux function, called the *Lighthill-Whitham-Richards* (LWR) PDE. This model is the seminal model for numerous highway traffic flow studies available in the literature today [1], [12], [26]–[29], [37], [45]–[47]. It models the evolution of the density of vehicles on a highway using a flux function obtained from empirical measurements [29].

An alternate macroscopic formulation of highway traffic flow was later introduced by Newell [45]–[47]. It consists in numbering vehicles upon entry on the highway and following the isolines of the functions representing vehicles numbers at all considered times and locations. The properties of the resulting HJ PDE have been recently studied in [28], [29]. The corresponding solutions are sometimes referred to as the Moskowitz surface [28], [29], [43]. The solution of this HJ PDE has no shocks, but is not necessarily differentiable. It is only upper semicontinuous. Actually, the non differentiability of the cumulated vehicle number function is closely related to the presence of the shocks of the solution to the LWR PDE (see for instance [17]–[19]).

The problem of fusion of Eulerian and Lagrangian data has a particular relevance for traffic. Indeed, the convergence of communication and multi-media platforms (for example the Nokia N95, the iPhone, or the G1 Google phone) have enabled a crucial new component for large scale infrastructure systems monitoring: mobility tracking. While most of the sensing in large

scale infrastructure systems is static, the use of GPS in phones such as the Nokia N95 as probe sensors on highways opens the door for a new class of highway sensing applications: Lagrangian sensing, or mobile sensing [34], [35], [54]. The fundamental difficulty posed by this type of sensing is that it provides mobile information about a distributed parameter system (flow on a highway) internal to the domain of interest i.e., onboard of moving vehicles, in addition to the data traditionally available at the boundary of the domain (for example traditional loop detector data [36], [44]).

### B. Problem Statement

The results presented in this article hold for *any* Hamilton-Jacobi PDE with concave (respectively convex) Hamiltonian. The modification of all results shown in the article for convex Hamiltonians is straightforward, and mostly consists in inverting signs and changing epigraphs into hypographs [4], [9], [39]. We illustrate the results in the context of highway traffic flow to emphasize the applicability of the method to practical problems.

Specifically, we solve the following mathematical problem:

Given a Hamilton-Jacobi PDE with concave Hamiltonian, given an initial condition, given upstream and downstream boundary conditions, and given additional data internal to the domain, how to define the proper (lower-semicontinuous) Barron Jensen/Frankowska solution to the problem, which satisfies all conditions above? Additionally, when are the above conditions compatible?

Control theory (and in particular viability theory) provides the appropriate tools to solve this problem, and to construct a semi-analytical solution to the problem, using a Lax-Hopf formula developed specifically for this problem.

The rest of this article is organized as follows. The remainder of this section presents two HJ PDEs, the first one which serves as a basis for this work, from the literature, and the second one which is investigated in the present work. Section II presents the control framework of viability theory used to solve the Moskowitz HJ PDE. Section III introduces a generalized Lax-Hopf formula used to compute the solutions explicitly. Section IV presents a useful *inf-morphism* property, and introduces *components* as building blocks of the solution. Section V introduces the concept of *internal* components, which encode the effects of internal conditions on the solution. Section VI presents the construction of the solutions to mixed initial-boundary-internal conditions problems. The same section also details the necessary and sufficient conditions for these problems to be well posed in the Barron-Jensen/Frankowska sense. Section VII illustrates the previous results on a practical traffic flow data assimilation problem, using a high resolution dataset.

1) *Inhomogeneous Hamilton-Jacobi Equation:* We consider a function  $\mathbf{N}(\cdot, \cdot)$  satisfying a HJ PDE with a concave Hamiltonian, following the framework set forth in [4]. The HJ PDE is inhomogeneous, with a source term

$$\frac{\partial \mathbf{N}(t, x)}{\partial t} + \psi \left( \frac{\partial \mathbf{N}(t, x)}{\partial x} \right) = \psi(v(t)). \quad (1)$$

The concave function  $\psi$  is called the *Hamiltonian*. The term  $\psi(v(\cdot))$  on the right hand side of (1) can be interpreted as a

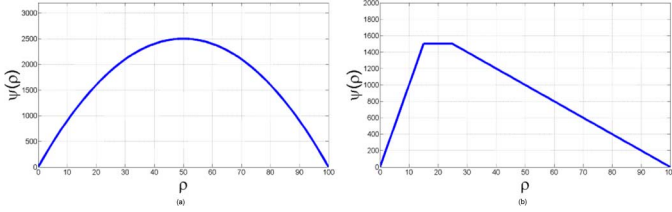


Fig. 1. Illustration of the Greenshields and trapezoidal Hamiltonians. Numerical values are represented in the context of transportation, *i.e.*, the variable  $\rho$  is homogeneous to the vehicle density (in percent of the maximal density). The Hamiltonian  $\psi(\rho)$  is represented in vehicles per hour. Left: representation of a Greenshields Hamiltonian. Right: representation of a trapezoidal Hamiltonian.

source term, where  $v(\cdot)$  is a given boundary condition to be made explicit later.

In the context of transportation engineering,  $\mathbf{N}(t, x)$  represents the *cumulated number of vehicles* (CVN), which is the total number of vehicles located between the upstream boundary of the highway  $\xi$  and the location  $x$ , at time  $t$ . In the same context, the Hamiltonian  $\psi(\cdot)$  is referred to as *flux function* or *fundamental diagram* [26], [27], usually assumed to be concave. The function  $v(\cdot)$  corresponds to the density of vehicles prescribed at the upstream boundary of the highway.

*Example 1.1: Trapezoidal Hamiltonian* [26], [27], [53]: One of the simplest Hamiltonians used in the literature is the trapezoidal model

$$\psi(\rho) = \begin{cases} \nu^b \rho & \text{if } \rho \leq \gamma^b \\ \delta & \text{if } \rho \in [\gamma^b, \gamma^\sharp] \\ \nu^\sharp(\omega - \rho) & \text{if } \rho \geq \gamma^\sharp \end{cases}$$

where  $\nu^b$ ,  $\nu^\sharp$ ,  $\omega$ ,  $\delta$ ,  $\gamma^b$  and  $\gamma^\sharp$  are constants and satisfy the following relations:  $\delta \leq (\omega \nu^b \nu^\sharp / \nu^b + \nu^\sharp)$  (called capacity in the transportation engineering literature),  $\gamma^b := (\delta / \nu^b)$  (called lower critical density in the transportation engineering literature), and  $\gamma^\sharp := (\nu^\sharp \omega - \delta / \nu^\sharp)$  (called upper critical density in the transportation engineering literature). When  $\gamma^b = \gamma^\sharp$ , the Hamiltonian is triangular, as used in the Daganzo cell transmission model [26], [27], [53].

*Example 1.2: Greenshields Hamiltonian* [1], [33]: Another possible model for the Hamiltonian  $\psi(\cdot)$  is called the *Greenshields Hamiltonian* and is given by

$$\forall \rho \in [0, \omega], \quad \psi(\rho) := \frac{\nu}{\omega} \rho (\omega - \rho) \quad (2)$$

where  $\omega$  and  $\nu$  are model parameters, respectively referred to as jam density and free flow velocity in the transportation literature.

The Greenshields and trapezoidal Hamiltonians are illustrated in Fig. 1. In this article, we assume *once for all* that the Hamiltonians are concave functions, and vanish for  $\rho = 0$  and  $\rho = \omega$  (jam density in the context of transportation engineering). We also assume that the vehicles evolve in the set  $X$ , which is an interval of  $\mathbb{R}$  defined by  $X = [\xi, \chi]$ , and that the time  $t$  is ranging in  $\mathbb{R}_+$ .

The results of existence, uniqueness, and the proper characterization of the solution to (1) are available in [4].

2) *Solution to the Homogeneous HJ PDE: The Moskowitz Function:* The present article focuses on a modification of (1), which also appears in the literature [28], [45]–[47].

*Definition 1.3: Moskowitz Function:* The Moskowitz function  $\mathbf{M}(t, x)$  is defined from the function  $\mathbf{N}(t, x)$  by the following variable change:

$$\mathbf{M}(t, x) = -\mathbf{N}(t, x) + \int_0^t \psi(v(u)) du. \quad (3)$$

By applying the variable change  $\mathbf{N}(t, x) \rightarrow -\mathbf{M}(t, x) + \int_0^t \psi(v(u)) du$ , we obtain the following HJ PDE [29] for the Moskowitz function:

$$\frac{\partial \mathbf{M}(t, x)}{\partial t} - \psi \left( -\frac{\partial \mathbf{M}(t, x)}{\partial x} \right) = 0. \quad (4)$$

In the context of transportation, it was first introduced in [43], and appeared later in the famous Newell trilogy [45]–[47], which does not mention it in reference to the field of HJ equations however. The link with HJ equations was made a few years later by Daganzo [28], [29]. In the context of traffic flow modeling, the Moskowitz function is a possible macroscopic description of traffic flow, in which the traffic is described by a surface representing the so-called *cumulative number of vehicles* [45] on the highway. Conversely, microscopic descriptions of the traffic flow consist in following individual vehicles trajectories. Microscopic descriptions of traffic flow can be linked with the Moskowitz formulation using the concept of *label functions*.

*Definition 1.4: Label Functions:* Let  $x_A(\cdot) : [\bar{t}_{\min}, \bar{t}_{\max}] \rightarrow X$  be a given continuous function (denoted in the remainder of the article as *trajectory function*) representing the trajectory of a point  $A$ . Let  $\mathbf{M}(\cdot, \cdot)$  be a solution of the HJ PDE (4). The label function  $l_A(\cdot)$  associated with the trajectory function  $x_A(\cdot)$  is defined by

$$\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}], \quad l_A(t) := \mathbf{M}(t, x_A(t)). \quad (5)$$

In the context of transportation engineering, the label function associated with a trajectory encodes the behavior of a vehicle  $A$  with respect to the rest of the traffic flow. We provide further physical interpretations of the label function in Section V.

*Proposition 1.5: Single Valued Selections:* Let the Moskowitz function  $\mathbf{M}$  be given on  $[\bar{t}_{\min}, \bar{t}_{\max}] \times X$ . Let  $I(\cdot)$  be the image of  $\mathbf{M}$  for a fixed  $t$ :  $I(t) = \mathbf{M}(t, X)$ . Let us consider a function  $l_A : [\bar{t}_{\min}, \bar{t}_{\max}] \rightarrow \mathbb{R}$  satisfying  $\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}]$ ,  $l_A(t) \in I(t)$ . There exists a function  $x_A(\cdot) : [\bar{t}_{\min}, \bar{t}_{\max}] \rightarrow X$  satisfying  $\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}]$ ,  $\mathbf{M}(t, x_A(t)) = l_A(t)$ . If  $\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}]$ , the function  $x \rightarrow \mathbf{M}(t, x)$  is strictly monotonic, the selection  $x_A(\cdot)$  such that  $\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}]$ ,  $\mathbf{M}(t, x_A(t)) = l_A(t)$  is unique.

*Proof:* Let  $t \in [\bar{t}_{\min}, \bar{t}_{\max}]$ . Since  $l_A(t) \in I(t)$ , there exists  $x_A(t)$  in  $X$  satisfying  $\mathbf{M}(t, x_A(t)) = l_A(t)$  for all  $t$  in  $[\bar{t}_{\min}, \bar{t}_{\max}]$ . If furthermore for all  $t$  in  $[\bar{t}_{\min}, \bar{t}_{\max}]$ , the function  $\mathbf{M}(t, \cdot)$  is strictly monotonic,  $\mathbf{M}(t, \cdot)$  is a bijection from  $X$  to  $I(t)$ , and there exists a unique  $x_A(t)$  satisfying  $\mathbf{M}(t, x_A(t)) = l_A(t)$ . ■

Note that for general functions  $\mathbf{M}(\cdot, \cdot)$  and  $l_A(\cdot)$ , there is no guarantee on the continuity of the selection  $x_A(\cdot)$  on  $[\bar{t}_{\min}, \bar{t}_{\max}]$ . The existence and uniqueness of the solution  $\mathbf{M}(\cdot, \cdot)$  to (4) is presented later in this article.

## II. VIABILITY EPISOLUTIONS

While the solution to the HJ PDE with initial and boundary conditions is well known and well studied in the literature [15], [17]–[20], [24] (see in particular the reference book [8]), the

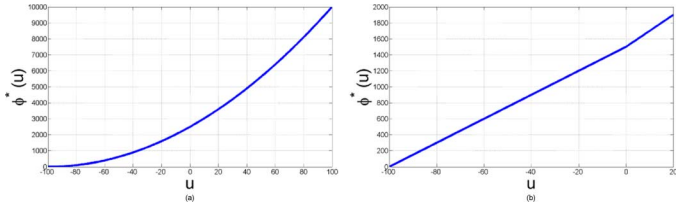


Fig. 2. Illustration of the convex transforms associated with the Greenshields and trapezoidal Hamiltonians. Left: representation of the function  $\varphi^*$  associated with a Greenshields Hamiltonian. Right: representation of the function  $\varphi^*$  associated with a trapezoidal Hamiltonian.

mathematical properties of the solution to (1) or (4) requires specific treatments when trying to introduce internal boundary conditions, as shown later. In the present context, we introduce a specific control framework based on viability theory [2], [3], which enables us to add this type of conditions to a traditional Cauchy problem [30]. We first recall a definition from viability theory [2], [3], which we later use in the article.

**Definition 2.1:** [2], [3] *Capture Basin:* Given a dynamical system  $F$  and two sets  $\mathcal{K}$  (called the constraint set) and  $\mathcal{C}$  (called the target set) satisfying  $\mathcal{C} \subset \mathcal{K}$ , the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$  is the subset of states of  $\mathcal{K}$  from which there exists at least one evolution solution of  $F$  reaching the target  $\mathcal{C}$  in finite time while remaining in  $\mathcal{K}$ .

Definition 2.1 will be used throughout the article. Note that there are several ways to compute the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$  numerically, in particular using the capture basin algorithm [15], [51]. We also need to recall the following definition from convex analysis:

**Definition 2.2:** *Convex Transform:* Given a concave function  $\psi$  with domain  $\text{Dom}(\psi)$ , we define a transform of  $\psi$  (denoted  $\varphi^*$ ), as follows:

$$\varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)]. \quad (6)$$

Note that (6) in Definition 2.2 differs from the traditional definition of the Legendre-Fenchel transform by a sign change. The convex transforms  $\varphi^*$  associated with Greenshields and trapezoidal Hamiltonians can be computed analytically, and are represented in Fig. 2. More details regarding the function  $\varphi^*$  are available in [4], [14], [23]. In the following, we also define  $\text{Dom}(\varphi^*)$  as the domain of the function  $\varphi^*$  (i.e.,  $\text{Dom}(\varphi^*) = \{u \in \mathbb{R} \text{ such that } \varphi^*(u) < +\infty\}$ ). The link of the capture basin with the HJ PDE of interest can be made using a dynamical system, which we now introduce:

**Definition 2.3:** *Auxiliary Dynamical System:* Given a Hamiltonian  $\psi(\cdot)$  with convex transform  $\varphi^*(\cdot)$ , we define an auxiliary dynamical system  $F$  associated with the HJ PDE (4):

$$F := \begin{cases} \tau'(t) = -1 \\ x'(t) = u(t) \\ y'(t) = -\varphi^*(u(t)) \end{cases} \quad \text{where } u(t) \in \text{Dom}(\varphi^*). \quad (7)$$

This dynamical system is both Marchaud and Lipschitz [4]. The function  $u(\cdot)$  is called auxiliary control of the dynamical system  $F$ .

To be rigorous, we have to mention *once and for all* that the controls  $u(\cdot)$  are measurable integrable functions with values in  $\text{Dom}(\varphi^*)$ , and thus, ranging  $L^1(0, +\infty; \text{Dom}(\varphi^*))$ , and that the above system of differential equations is valid for almost

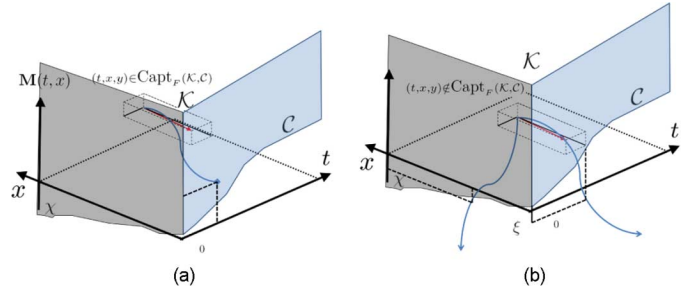


Fig. 3. Illustration of the capture basin associated with an epigraphical target. Left: element  $(t, x, y)$  of the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$ : there exists an evolution starting from  $(t, x, y)$  and reaching  $\mathcal{C}$  in finite time while remaining in  $\mathcal{K} := \mathbb{R}_+ \times X \times \mathbb{R}$ . Right: element  $(t, x, y)$  not belonging to the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$ : all evolutions starting from  $(t, x, y)$  exit the set  $\mathcal{K}$  before reaching  $\mathcal{C}$  (only two evolutions are represented for clarity).

all  $t \geq 0$ . We now introduce specific expressions for  $\mathcal{K}$  and  $\mathcal{C}$ , which intervene in the definition of the proper capture basin used to define the solution to the HJ PDE.

**Definition 2.4:** *Constraint Set Associated With a HJ PDE:* For a HJ PDE (4) defined in the set  $\mathbb{R}_+ \times X$ , we define the constraint set  $\mathcal{K}$  as  $\mathcal{K} := \mathbb{R}_+ \times X \times \mathbb{R}$ .

We refer the reader to [4] for the construction of solutions associated with general epigraphical environment sets, and the interpretation of the resulting solutions. We recall the following definition:

**Definition 2.5:** *Target Set Associated With a HJ PDE:* For a HJ PDE (4) defined in  $\mathbb{R}_+ \times X$ , we define a *target function* as a lower semicontinuous function  $\mathbf{c}(\cdot, \cdot)$  in a subset of  $\mathbb{R}_+ \times X$ . The target function  $\mathbf{c}$  defines an epigraphical target set as  $\mathcal{C} := \text{Epi}(\mathbf{c})$ . This set is the subset of triples  $(t, x, y) \in \mathbb{R}_+ \times X \times \mathbb{R}$  such that  $y \geq \mathbf{c}(t, x)$  (it is the *epigraph* of the function  $\mathbf{c}$ ).

Note that the target set  $\mathcal{C} = \text{Epi}(\mathbf{c})$  associated with a target function  $\mathbf{c}$  is closed, since it is the epigraph of a lower semicontinuous function. The definition 2.1 of the capture basin can now be applied to the specific target  $\mathcal{C}$  given by Definition 2.5 in the constraint set  $\mathcal{K}$  given by Definition 2.4 with the dynamics (7), as illustrated in Fig. 3.

**Definition 2.6:** *Viability Episolution:* Given a characteristic system  $F$ , a constraint set  $\mathcal{K}$  and a target set  $\mathcal{C}$ , respectively defined by Definitions 2.3, 2.4 and 2.5, the *viability episolution*  $\mathbf{M}$  is defined by

$$\mathbf{M}(t, x) := \inf_{(t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{C})} y. \quad (8)$$

Note that by definition, the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$  of a target  $\mathcal{C}$  viable in the environment  $\mathcal{K}$  is the subset of initial states  $(t, x, y)$  for which there exists a measurable control  $u(\cdot)$  such that its associated evolution

$$s \mapsto \left( t - s, x + \int_0^s u(\tau) d\tau, y - \int_0^s \varphi^*(u(\tau)) d\tau \right) \quad (9)$$

is viable in  $\mathcal{K}$  until it reaches the target  $\mathcal{C}$ . We illustrate the notion of viability episolution associated with a given target function in Fig. 4. It is called “episolution” of the HJ PDE (4) because it is defined by its epigraph, i.e., by (8) which states that the graph of  $\mathbf{M}(\cdot, \cdot)$  is the lower envelope of the capture basin  $\text{Capt}_F(\mathcal{K}, \mathcal{C})$ . The viability episolution  $\mathbf{M}$  defined by (8) is shown later in theorem 4.8 to be a Barron-Jensen/Frankowska solution to (4). Furthermore  $\mathbf{M}$  is differentiable, it is a classical solution to (4).

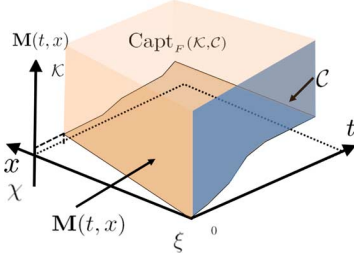


Fig. 4. Illustration of a viability episolution. We represent in the same figure a target  $\mathcal{C}$  and its associated viability episolution  $\mathbf{M}$ . The episolution is the lower boundary of the capture basin  $\text{Capt}_P(\mathcal{K}, \mathcal{C})$ , shaded in this figure.

### III. GENERALIZED LAX-HOPF FORMULA

The definitions in the previous section explain the construction of a function  $\mathbf{M}(\cdot, \cdot)$  from a target function  $\mathbf{c}(\cdot, \cdot)$ . In this section, we use this concept of viability episolution to construct a semi-explicit Lax-Hopf formula associated with this solution. This Lax-Hopf formula will be used later in the article (Section VI) to explicitly instantiate the compatibility conditions which the initial, boundary and internal conditions should satisfy for well posedness of the problem. Theorem 3.1 below is necessary for the rest of the article since the Lax-Hopf formula presented in [4] is explicitly written for initial and boundary conditions, and not for internal boundary conditions. Also, the HJ PDE (1) solved in [4] is not homogeneous, and its source term (translating the upstream boundary condition) appears explicitly in the Lax-Hopf formula which prevents a straight application of the results presented in [4]. The HJ PDE (4) introduced in this article enables us to remove this explicit dependency.

**Theorem 3.1: Generalized Lax Hopf Formula:** The viability episolution  $\mathbf{M}_{\mathbf{c}}$  associated with a target  $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$ , for a given lower semicontinuous function  $\mathbf{c}$ , and defined by (8) can be expressed as:

$$\mathbf{M}_{\mathbf{c}}(t, x) = \inf_{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} (\mathbf{c}(t - T, x + Tu) + T\varphi^*(u)). \quad (10)$$

*Proof:* We fix  $(t, x) \in \mathbb{R}_+ \times X$ , and define  $R$  as the set of elements  $(u(\cdot), T, y)$  belonging to  $L^1(0, \infty; \text{Dom}(\varphi^*)) \times \mathbb{R}_+ \times \mathbb{R}$  and satisfying viability property (11)

$$\forall s \in [0, T] \left( t - s, x + \int_0^s u(\tau) d\tau, y - \int_0^s \varphi^*(u(\tau)) d\tau \right) \in \mathcal{K}. \quad (11)$$

Equations (8) and (9) thus imply the following formula:

$$\mathbf{M}_{\mathbf{c}}(t, x) = \inf_{(u(\cdot), T, y) \in R \text{ such that } (t-T, x + \int_0^T u(\tau) d\tau, y - \int_0^T \varphi^*(u(\tau)) d\tau) \in \mathcal{E}pi(\mathbf{c})} y. \quad (12)$$

Since the graph of the target function  $\mathbf{c}$  (denoted  $\text{Graph}(\mathbf{c})$ ) is the lower boundary of  $\mathcal{E}pi(\mathbf{c})$ , we have

$$\begin{aligned} & \left( (t-T, x + \int_0^T u(\tau) d\tau, y - \int_0^T \varphi^*(u(\tau)) d\tau) \in \mathcal{E}pi(\mathbf{c}) \right. \\ & \text{and } \left. (t-T, x + \int_0^T u(\tau) d\tau, z - \int_0^T \varphi^*(u(\tau)) d\tau) \in \text{Graph}(\mathbf{c}) \right) \\ & \Rightarrow z \leq y. \end{aligned} \quad (13)$$

Since  $\text{Graph}(\mathbf{c}) \subset \mathcal{E}pi(\mathbf{c})$ , (13) and (12) imply

$$\mathbf{M}_{\mathbf{c}}(t, x) = \inf_{(u(\cdot), T, y) \in R \text{ such that } (t-T, x + \int_0^T u(\tau) d\tau, y - \int_0^T \varphi^*(u(\tau)) d\tau) \in \text{Graph}(\mathbf{c})} y. \quad (14)$$

Since  $\mathbf{c}$  is infinite outside of its domain of definition, and given the definition of  $\text{Graph}(\mathbf{c})$ , (14) can be expressed as follows:

$$\mathbf{M}_{\mathbf{c}}(t, x) = \inf_{(u(\cdot), T, y) \in R} \left[ \mathbf{c} \left( t - T, x + \int_0^T u(\tau) d\tau \right) + \int_0^T \varphi^*(u(\tau)) d\tau \right]. \quad (15)$$

The indexing by  $y$  in the inf of the formula above is dummy, since it does not intervene in the infimum. We have left it as is, in order to avoid defining another set similar to  $R$  just for this formula only. We now define the following constant control  $\hat{u}$  on the time interval  $[0, T]$  as follows:

$$\hat{u} := \frac{1}{T} \int_0^T u(\tau) d\tau. \quad (16)$$

The control  $\hat{u}$  is the average value of the control function  $u(\cdot)$  on the time interval  $[0, T]$ . In the following, we slightly abuse the notation by calling  $\hat{u}(\cdot)$  the constant function  $t \rightarrow \hat{u}$ . Note that by convexity of  $\mathcal{K}$ ,  $(\hat{u}(\cdot), T, y) \in R$  if  $(u(\cdot), T, y) \in R$ .

We define  $y(u(\cdot), T)$  and  $y(\hat{u}(\cdot), T)$ , respectively, as the values of the term minimized in (15) obtained for the control functions  $u(\cdot)$  and  $\hat{u}(\cdot)$ , and for the capture time  $T$

$$\begin{cases} y(u(\cdot), T) = \mathbf{c}(t - T, x + \int_0^T u(\tau) d\tau) + \int_0^T \varphi^*(u(\tau)) d\tau \\ y(\hat{u}(\cdot), T) = \mathbf{c}(t - T, x + T\hat{u}) + T\varphi^*(\hat{u}). \end{cases} \quad (17)$$

Since  $\varphi^*$  is convex and lower semicontinuous, Jensen inequality implies

$$\varphi^* \left( \frac{1}{T} \int_0^T u(\tau) d\tau \right) \leq \frac{1}{T} \int_0^T \varphi^*(u(\tau)) d\tau \quad (18)$$

and thus, since  $T\hat{u} = \int_0^T u(\tau) d\tau$

$$y(\hat{u}(\cdot), T) \leq y(u(\cdot), T). \quad (19)$$

Equation (19) thus implies that one can replace the search of the infimum over the class of measurable functions  $u(\cdot)$  by the search of the infimum over the set of constant functions  $\hat{u}(\cdot)$ .

Hence, we can write (15) as

$$\mathbf{M}_{\mathbf{c}}(t, x) = \inf_{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} (\mathbf{c}(t - T, x + Tu) + T\varphi^*(u)) \quad (20)$$

which for the remainder of the article enables us to restrict ourselves to the set of constant controls. ■

Note that in practice, given a constant control function  $u$ , the parameters  $T$  used for the minimization in (10) can be restricted to the elements of the set  $S_c(t, x, u)$  defined by formula (21)

$$S_c(t, x, u) := \{s \in \mathbb{R}_+ \text{ such that } (t - s, x + su) \in \text{Dom}(\mathbf{c})\}. \quad (21)$$

Indeed, when  $T \notin S_c(t, x, u)$ ,  $\mathbf{c}(t - T, x + Tu)$  is infinite. We could also alternatively define for any  $(t, x)$  the set  $R_c(t, x)$  as  $R_c(t, x) := \{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+ \text{ s.t. } (t - T, x + Tu) \in \text{Dom}(\mathbf{c})\}$ . Note that the parameters  $(u, T)$  used for the minimization in (10) can also be restricted to the elements of  $R_c$  (when  $(u, T) \notin R_c(t, x)$ ,  $\mathbf{c}(t - T, x + Tu)$  is infinite). When  $\forall u \in \text{Dom}(\varphi^*)$ ,  $S_c(t, x, u) = \emptyset$ , (20) involves a minimization on an empty set, and  $\mathbf{M}_c(t, x)$  is infinite. Since  $\mathbf{c}(t - T, x + Tu) = +\infty$  when  $(t - T, x + Tu) \notin \text{Dom}(\mathbf{c})$ , we can write (20) as

$$\mathbf{M}_c(t, x) = \inf_{\{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+ \text{ such that } T \in S_c(t, x, u)\}} (\mathbf{c}(t - T, x + Tu) + T\varphi^*(u)) \quad (22)$$

or alternatively as

$$\mathbf{M}_c(t, x) = \inf_{\{(u, T) \in R_c(t, x)\}} (\mathbf{c}(t - T, x + Tu) + T\varphi^*(u)). \quad (23)$$

While (23) is algebraically simpler than (22), we only consider expression (22) in the remainder of the article for compatibility with [4].

#### IV. INF-MORPHISM PROPERTY AND ITS CONSEQUENCES

The major contribution of this article is the definition of the components of the solution to the HJ PDE (4), which encodes initial, boundary and internal conditions (Sections IV-B and V). For this, we need to introduce an inf-morphism property, which results from the sup-morphism property derived in [4].

##### A. Inf-Morphism Property

It is well known [2], [3] that for a given environment  $\mathcal{K}$ , the capture basin of a finite union of targets is the union of the capture basins of these targets:

$$\text{Capt}_F \left( \mathcal{K}, \bigcup_{i \in I} \mathcal{C}_i \right) = \bigcup_{i \in I} \text{Capt}_F(\mathcal{K}, \mathcal{C}_i) \quad (24)$$

where  $I$  is a finite set. This property can be translated in epigraphical form:

*Proposition 4.1: Inf-Morphism Property:* Let  $\mathbf{c}_i$  ( $i$  belongs to a finite set  $I$ ) be a family of functions whose epigraphs are the targets  $\mathcal{C}_i$ . Since the epigraph of the minimum of the functions  $\mathbf{c}_i$  is the union of the epigraphs of the functions  $\mathbf{c}_i$ , the target  $\mathcal{C} := \bigcup_{i \in I} \mathcal{C}_i$  is the epigraph of the function  $\mathbf{c} := \inf_{i \in I} \mathbf{c}_i$ . Hence, (24) implies the following property [4]:

$$\forall t \geq 0, \quad x \in X, \quad \mathbf{M}_c(t, x) = \inf_{i \in I} \mathbf{M}_{\mathbf{c}_i}(t, x). \quad (25)$$

In [4], the authors prove a sup-morphism property, used to construct the solution of the corresponding HJ PDE (1). Equation (25) can be obtained similarly, noting that hypographs have to be changed to epigraphs, and supremums to infimums. The inf-morphism property enables us to compute the episolution

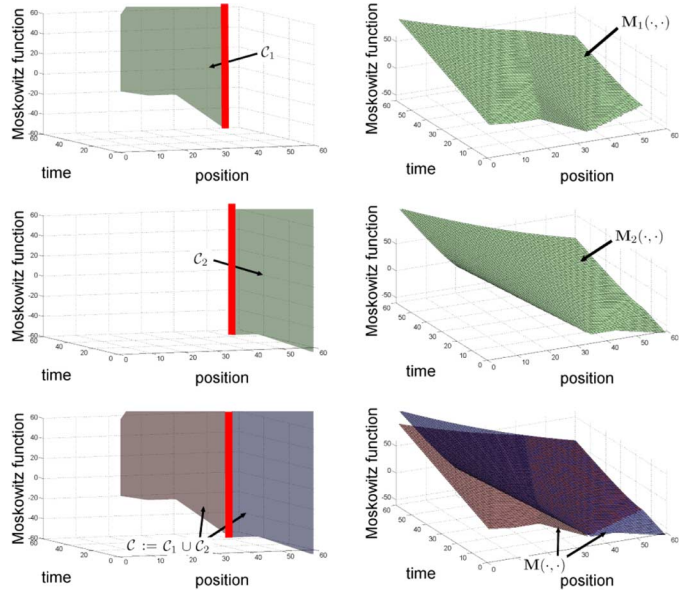


Fig. 5. Illustration of the inf-morphism property. Top: representation of the target  $\mathcal{C}_1 := \text{Epi}(\mathbf{c}_1)$  (left), representation of the corresponding episolution  $\mathbf{M}_1(\cdot, \cdot)$  (right). Center: representation of the target  $\mathcal{C}_2 := \text{Epi}(\mathbf{c}_2)$  (left), representation of the corresponding episolution  $\mathbf{M}_2(\cdot, \cdot)$  (right). Bottom: Representation of the target  $\mathcal{C} := \mathcal{C}_1 \cup \mathcal{C}_2$  (left). The episolution  $\mathbf{M}(\cdot, \cdot)$  associated with the target  $\mathcal{C}$  (right) is the minimum of the episolutions  $\mathbf{M}_1(\cdot, \cdot)$  and  $\mathbf{M}_2(\cdot, \cdot)$  associated with  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .

associated with a target  $\mathbf{c} := \inf_{i \in I} \mathbf{c}_i$  by taking the infimum of all episolutions  $\mathbf{M}_{\mathbf{c}_i}$  associated with the target functions  $\mathbf{c}_i$ , as illustrated in Fig. 5. From a practical perspective, this property is a significant breakthrough, as it not only enables us to divide the problem into independent subproblems, but it also enables us to update a solution with new data without recomputing the entire solution: it just suffices to augment the set  $I$  by new targets to be added to the set of existing targets.

##### B. Components of the Moskowitz Function

The inf-morphism property presented in the previous section can be used for the construction of the respective components of this problem, which we now define.

*Definition 4.2: Components of the Moskowitz Function:* The component  $\mathbf{M}_{\mathbf{c}_i}$  associated with the HJ PDE (4) is defined as the episolution associated with a target function  $\mathbf{c}_i$ :

$$\mathbf{M}_{\mathbf{c}_i}(t, x) := \inf_{(t, x, y) \in \text{Capt}_F(\mathcal{K}, \text{Epi}(\mathbf{c}_i))} y \quad (26)$$

where  $F$  is defined by (7) in Definition 2.3, and  $\mathcal{K}$  by Definition 2.4.

*Example 4.3: Initial and Boundary Condition Components:* We consider three given functions  $\mathcal{M}_0(\cdot, \cdot)$ ,  $\gamma(\cdot, \cdot)$  and  $\beta(\cdot, \cdot)$ , satisfying the following properties:

**Initial condition:**

$$\mathcal{M}_0(t, x) := \begin{cases} \mathbf{M}_0(x) (\text{given}) & \text{for } t = 0 \text{ and } x \in X \\ +\infty & \forall t \neq 0 \text{ or } \forall x \notin X \end{cases} \quad (27)$$

**Upstream boundary condition:**

$$\gamma(t, x) := \begin{cases} \mathbf{M}_\gamma(t) (\text{given}) & \text{for } x = \xi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \xi \text{ or } \forall t < 0 \end{cases} \quad (28)$$

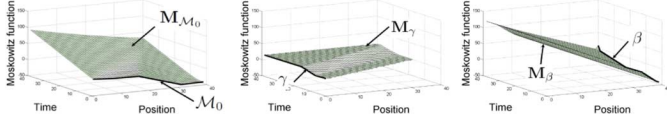


Fig. 6. Illustration of the initial, upstream and downstream boundary condition components. For each component, the associated target function is highlighted by a solid line. Left: representation of the initial condition component  $\mathbf{M}_{\mathcal{M}_0}(\cdot, \cdot)$  obtained by applying (26) to the function  $\mathcal{M}_0(\cdot, \cdot)$ . Center: representation of the upstream boundary condition component  $\mathbf{M}_\gamma(\cdot, \cdot)$  obtained by applying (26) to the function  $\gamma(\cdot, \cdot)$ . Right: representation of the downstream boundary condition component  $\mathbf{M}_\beta(\cdot, \cdot)$  obtained by applying (26) to the function  $\beta(\cdot, \cdot)$ .

### Downstream boundary condition:

$$\beta(t, x) := \begin{cases} \mathbf{M}_\beta(t) (\text{given}) & \text{for } x = \chi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \xi \text{ or } \forall t < 0 \end{cases} \quad (29)$$

The dependency of  $\mathcal{M}_0$ ,  $\gamma$  and  $\beta$  on two arguments has been added for notational consistency. Note that  $\mathcal{M}_0(t, x)$  is only defined when  $t = 0$  and  $x \in X$ , that  $\gamma(t, x)$  is only defined when  $t \geq 0$  and  $x = \xi$ , and that  $\beta(t, x)$  is only defined when  $t \geq 0$  and  $x = \chi$ . In the context of traffic, the initial condition function defined by (27) describes the distribution of the vehicles on the highway section, at the initial time  $t = 0$ . The upstream (respectively downstream) boundary condition function defined by (28) (respectively (29)) describe the inflow (respectively outflow) of vehicles on the highway section. The initial condition component  $\mathbf{M}_{\mathcal{M}_0}$ , upstream boundary condition component  $\mathbf{M}_\gamma$ , and downstream boundary condition component  $\mathbf{M}_\beta$  associated with the target functions  $\mathcal{M}_0$  and  $\gamma$  respectively, are defined by the following formulas:

$$\begin{cases} \mathbf{M}_{\mathcal{M}_0}(t, x) := \inf_{(t, x, y) \in \text{Capt}_F(\mathcal{K}, \text{Epi}(\mathcal{M}_0))} y \\ \mathbf{M}_\gamma(t, x) := \inf_{(t, x, y) \in \text{Capt}_F(\mathcal{K}, \text{Epi}(\gamma))} y \\ \mathbf{M}_\beta(t, x) := \inf_{(t, x, y) \in \text{Capt}_F(\mathcal{K}, \text{Epi}(\beta))} y \end{cases} \quad (30)$$

Fig. 6 shows an illustration of the initial, upstream and downstream boundary condition components for given functions  $\mathbf{M}_0(\cdot)$ ,  $\mathbf{M}_\gamma(\cdot)$  and  $\mathbf{M}_\beta(\cdot)$ .

The component  $\mathbf{M}_c$  associated with the target function  $c$  can be computed using the Lax Hopf formula (22), where  $S_c(t, x, u)$  is defined by (21), see the equation shown at the bottom of the page.

In the case of initial and boundary conditions, the following corollary of Theorem 3.1 provides the semi-explicit Lax-Hopf formulas. Note that it is in general not possible to derive explicit versions of these formulas, except for specific classes of  $\mathcal{M}_0(\cdot, \cdot)$ ,  $\gamma(\cdot, \cdot)$  and  $\beta(\cdot, \cdot)$ , as is shown in a companion article [21].

*Corollary 4.4: Lax Hopf Formulas Associated With the Initial and Boundary Condition Components:* The initial condition  $\mathbf{M}_{\mathcal{M}_0}$ , upstream boundary condition  $\mathbf{M}_\gamma$  and downstream

boundary condition  $\mathbf{M}_\beta$  components can be computed using the following formulas:

$$\mathbf{M}_{\mathcal{M}_0}(t, x) = \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(0, x + tu) + t\varphi^*(u))$$

$$\mathbf{M}_\gamma(t, x) = \begin{cases} \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \leq 0} \left( \gamma \left( t - \frac{\xi - x}{u}, \xi \right) + \frac{\xi - x}{u} \varphi^*(u) \right) & \text{if } x > \xi \\ \inf_{T \in [0, t]} (\gamma(t - T, \xi) + T\varphi^*(0)) & \text{if } x = \xi \end{cases}$$

$$\mathbf{M}_\beta(t, x) = \begin{cases} \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \geq 0} \left( \beta \left( t - \frac{\chi - x}{u}, \chi \right) + \frac{\chi - x}{u} \varphi^*(u) \right) & \text{if } x < \chi \\ \inf_{T \in [0, t]} (\beta(t - T, \chi) + T\varphi^*(0)) & \text{if } x = \chi. \end{cases} \quad (31)$$

*Proof:* The elements of the set  $S_{\mathcal{M}_0}(t, x, u)$  associated with the initial condition component  $\mathbf{M}_{\mathcal{M}_0}$  are given by the following formula:

$$S_{\mathcal{M}_0}(t, x, u) = \begin{cases} \{t\} & \text{if } \xi \leq x + tu \leq \chi \\ \emptyset & \text{otherwise.} \end{cases} \quad (32)$$

We can thus express formula (22) for the particular target function  $\mathcal{M}_0$  as:

$$\mathbf{M}_{\mathcal{M}_0}(t, x) = \inf_{u \in \text{Dom}(\varphi^*) \cap [(\xi - x)/t, (\chi - x)/t]} \left( \mathcal{M}_0(0, x + tu) + t\varphi^*(u) \right).$$

Since the function  $\mathcal{M}_0(0, x)$  is infinite when  $x \notin X$ , the constraint  $(\xi - x)/t \leq u \leq (\chi - x)/t$  is already implicitly expressed in the argument  $\mathcal{M}_0(0, x + tu)$ . We can thus rewrite the above equation as the first line in formula (31).

The set  $S_\gamma(t, x, u)$  associated with the boundary condition component  $\mathbf{M}_\gamma$  can be computed using the following relation:

$$S_\gamma(t, x, u) = \{T \in \mathbb{R}_+ \text{ such that } (t - T, x + Tu) \in \mathbb{R}_+ \times \{\xi\}\}. \quad (33)$$

Equation (33)  $S_\gamma(t, x, u)$  can be rewritten as

$$S_\gamma(t, x, u) = \mathbb{R}_+ \cap ] - \infty, t] \cap \{T \in \mathbb{R} \text{ such that } x + Tu = \xi\}. \quad (34)$$

The set  $\{T \in \mathbb{R} \text{ such that } x + Tu = \xi\}$  can be explicitated as

$$\begin{aligned} & \{T \in \mathbb{R} \text{ such that } x + Tu = \xi\} \\ &= \begin{cases} \left\{ \frac{\xi - x}{u} \right\} & \text{if } u \neq 0 \\ \mathbb{R} & \text{if } u = 0 \text{ and } x = \xi \\ \emptyset & \text{otherwise.} \end{cases} \end{aligned} \quad (35)$$

$$\begin{cases} S_c(t, x, u) := \{s \in \mathbb{R}_+ \text{ such that } (t - s, x + su) \in \text{Dom}(c)\} \\ \mathbf{M}_c(t, x) = \inf_{\{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+ \text{ such that } T \in S_c(t, x, u)\}} (c(t - T, x + Tu) + T\varphi^*(u)). \end{cases}$$

Hence, (34) and (35) yield the following formula:

$$S_\gamma(t, x, u) = \begin{cases} \{\frac{\xi-x}{u}\} & \text{if } u \neq 0 \text{ and } 0 \leq \frac{\xi-x}{u} \leq t \\ [0, t] & \text{if } u = 0 \text{ and } x = \xi \\ \emptyset & \text{otherwise.} \end{cases} \quad (36)$$

We can thus express formula (22) for the particular target function  $M_\gamma$  as (31), observing that if  $x = \xi$  and  $u \neq 0$ , then  $S_\gamma(t, x, u) = \{0\}$ . Note that the constraint  $0 \leq (\xi - x)/u$  is equivalent to  $u \leq 0$  since  $x > \xi$ . Note also that the constraint  $(\xi - x)/u \leq t$  can be omitted in (31), since the corresponding value of  $\gamma$  is infinite if this constraint is violated.

The computation of  $M_\beta(\cdot, \cdot)$  is similar to the computation of  $M_\gamma(\cdot, \cdot)$ . ■

As illustrated graphically in Fig. 6, for example for  $\gamma(\cdot, \cdot)$  and  $M_\gamma(\cdot, \cdot)$ , a given component is not necessarily defined for all  $(t, x) \in \mathbb{R}_+ \times X$ . Indeed, the set of points  $(t, x)$  which are influenced by the condition  $\mathbf{c}(\cdot, \cdot)$  are given by  $\text{Capt}_F(\mathcal{K}, \text{Epi}(\mathbf{c}))$ , and correspond intuitively to a form of reachable set from the condition  $\mathbf{c}(\cdot, \cdot)$  [41], [52]. This can be formalized using the notion of *domain of influence*, which is the domain of definition of a given component.

*Proposition 4.5: Domain of Influence:* For a given  $\mathbf{c}_i$ , the domain of definition of  $M_{\mathbf{c}_i}$  is also called *domain of influence of the component*  $M_{\mathbf{c}_i}$ . It is given by the following formula:

$$\text{Dom}(M_{\mathbf{c}_i}) = \bigcup_{(t,x) \in \text{Dom}(\mathbf{c}_i)} \left( \bigcup_{T \in \mathbb{R}_+} \{t + T\} \times [x - \nu^\# T, x + \nu^b T] \right). \quad (37)$$

*Proof:* The generalized Lax Hopf formula (22) implies that

$$\begin{aligned} \text{Dom}(M_{\mathbf{c}_i}) = \{ & (t, x) \in \mathbb{R}_+ \times X \text{ such that} \\ & \exists (T, u) \in \mathbb{R}_+ \times \text{Dom}(\varphi^*) \\ & \text{and } (t - T, x + Tu) \in \text{Dom}(\mathbf{c}_i) \} \end{aligned}$$

Equation (37) is derived from the previous formula, observing that  $u$  ranges in  $\text{Dom}(\varphi^*) := [-\nu^b, \nu^\#]$ . ■

The domain of influence of the component  $M_{\mathbf{c}_i}$  is the union of the cones originating at  $(t, x) \in \text{Dom}(\mathbf{c}_i)$  and limited by the minimal  $-\nu^b$  and maximal  $\nu^\#$  slopes of the Hamiltonian. This property is illustrated in Fig. 7.

The notion of component is essential to this work, since it is used to build the solution to the corresponding problem. However, one cannot prescribe components arbitrarily, which could lead to an ill-posed problem. In order to prevent ill-posedness, one needs to introduce the notion of proper formulation.

*Definition 4.6: Proper Formulation of a Component:* The component  $M_{\mathbf{c}_i}$  associated with a target function  $\mathbf{c}_i$  is said to be *properly formulated* if the following condition is satisfied:

$$\forall (t, x) \in \text{Dom}(\mathbf{c}_i), \quad M_{\mathbf{c}_i}(t, x) = \mathbf{c}_i(t, x). \quad (38)$$

This condition intuitively means that whenever the target function  $\mathbf{c}_i(\cdot, \cdot)$  is imposed, its corresponding component  $M_{\mathbf{c}_i}(\cdot, \cdot)$  will reflect the prescribed condition. Note that it is well known [2] that for any environment  $\mathcal{K}$  and target  $\mathcal{C}$ ,  $\mathcal{C} \subset \text{Capt}_F(\mathcal{K}, \mathcal{C})$ , which implies the following inequality:

$$\forall (t, x) \in \text{Dom}(\mathbf{c}_i), \quad M_{\mathbf{c}_i}(t, x) \leq \mathbf{c}_i(t, x). \quad (39)$$

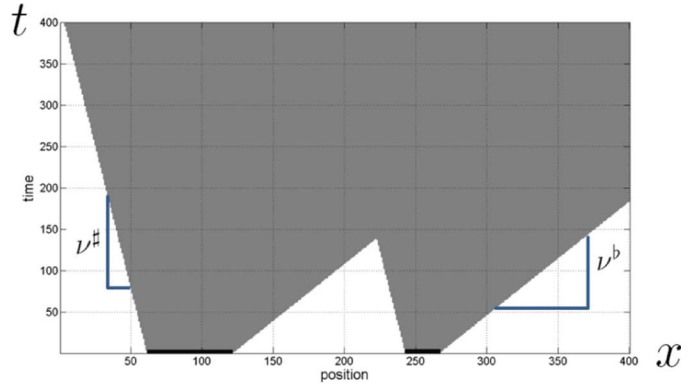


Fig. 7. Illustration of the domain of influence of a component. We define a target function  $\mathbf{c}_i$  on a domain represented by two black segments at  $t = 0$ . The domain of influence of the associated component  $M_{\mathbf{c}_i}$  is highlighted in gray.

However the reverse inequality is not necessarily true. A component is thus properly formulated if and only if the reverse inequality is true in the domain of  $\mathbf{c}_i$ .

*Example 4.7: Proper Formulation of the Initial and Boundary Condition Components:* The initial condition and boundary condition components can be computed using formulas (31). The initial condition component is unconditionally properly formulated, since  $\forall x \in X, M_{\mathcal{M}_0}(0, x) = \mathcal{M}_0(0, x)$ . However, the boundary condition components are not properly formulated for general lower semicontinuous functions  $\gamma(\cdot, \cdot)$  and  $\beta(\cdot, \cdot)$ . These components are properly formulated if and only if

$$\begin{aligned} \forall t \in \mathbb{R}_+, \quad \forall T \in [0, t], \\ \gamma(t - T, \xi) + T\varphi^*(0) &\geq \gamma(t, \xi) \\ \forall t \in \mathbb{R}_+, \quad \forall T \in [0, t], \\ \beta(t - T, \chi) + T\varphi^*(0) &\geq \beta(t, \chi). \end{aligned} \quad (40)$$

The previous conditions are satisfied if and only if  $\gamma(\cdot, \xi)$  and  $\beta(\cdot, \chi)$  satisfy the growth conditions  $(\gamma(t + T, \xi) - \gamma(t, \xi))/T \leq \varphi^*(0)$  and  $(\beta(t + T, \chi) - \beta(t, \chi))/T \leq \varphi^*(0)$  for all  $(t, T) \in \mathbb{R}_+^2$ . In the context of traffic, since  $\gamma$  and  $\beta$  denote the label of entering vehicles, these conditions amount to saying that the number of vehicles entering the highway during a time interval of length  $T$  must always be less than  $T\varphi^*(0)$ . Since  $\varphi^*(0) := \sup_{p \in \text{Dom}(\psi)} [\psi(p)]$  is the maximal flow that can circulate through the highway section (according to the model), (40) implies that the prescribed inflow and outflows must not exceed the maximal possible inflow or outflow predicted by the model.

We now finally have the necessary results to assemble the components into the proper weak solution of the HJ PDE (4), which was our initial goal. We formulate the capture basins in terms of Barron/Jensen-Frankowska solutions [10], [31], a weaker concept of viscosity solutions which only requires lower semicontinuity of the solution instead of its continuity [4].

We recall some standard concepts of convex analysis [13], [50] and set-valued analysis [2], [5], [6]. Full mathematic definitions are available in the references following each definition of a new symbol below (omitted here for brevity). Let  $T_{\mathcal{Z}}(z)$  denote the contingent cone to a set  $\mathcal{Z}$  at a point  $z$  (see [6]),



and let  $\sigma(\cdot, \cdot)$  represent the support function (see [2], [5], [6]) of a set, which is defined by  $\sigma(A, v) := \sup_{u \in A} \langle u, v \rangle$ . For a space  $Y$ , and a set  $P \subset Y$ , let  $P^+$  denote the polar cone of  $P$ , defined by  $P^+ = \{p \in Y^* | \forall y \in P, \langle p, y \rangle \leq 0\}$ , where  $Y^*$  denotes the dual of  $Y$ . Let the subdifferential  $\partial_-$  of a function  $u : Y \rightarrow \mathbb{R} \cup \{+\infty\}$  be defined by  $\partial_- u(x) = \{p \in Y^* | \forall y \in Y, \langle p, y \rangle \leq D_\uparrow u(x)(y)\}$ , where the epiderivative  $D_\uparrow$  is defined [4] by its epigraph as  $\text{Epi}(D_\uparrow Z(z)) := T_{\text{Epi}(Z)}(z, Z(z))$ .

The work [4] defines the Barron/Jensen-Frankowska solution in hypographical form for the function  $\mathbf{N}(\cdot, \cdot)$ . The following theorem is identical to the main existence and uniqueness theorem of [4] modulo the variable change (3), the translation of hypographs into epigraphs and the corresponding change on epi/hypo derivatives and differentials.

**Theorem 4.8: Barron-Jensen/Frankowska Solution:** For any lower semicontinuous target function  $\mathbf{c}_i$ , the associated component  $\mathbf{M}_{\mathbf{c}_i}$  is the **unique** lower semicontinuous function lower than  $\mathbf{c}_i$  satisfying

$$\left\{ \begin{array}{l} (i) \quad \forall (t, x) \in \text{Dom}(\mathbf{M}_{\mathbf{c}_i}) \setminus \text{Dom}(\mathbf{c}_i) \quad \forall (p_t, p_x) \in \partial_- \mathbf{M}_{\mathbf{c}_i}(t, x), \\ \quad p_t - \psi(-p_x) = 0 \\ (ii) \quad \forall (t, x) \in \text{Dom}(\mathbf{M}_{\mathbf{c}_i}) \setminus \text{Dom}(\mathbf{c}_i) \\ \quad \forall (p_t, p_x) \in (\text{Dom}(D_\uparrow \mathbf{M}_{\mathbf{c}_i}(t, x)))^+, \\ \quad p_t - \sigma(\text{Dom}(\varphi^*), p_x) = 0. \end{array} \right. \quad (41)$$

Theorem 4.8 results from [4, Theorem 9.1] with the aforementioned modifications. It expresses the fact that the component  $\mathbf{M}_{\mathbf{c}_i}$  is a solution to the HJ PDE (4) in the Barron–Jensen/Frankowska sense. In particular, since  $\partial_- \mathbf{M}_{\mathbf{c}_i}(t, x) = \{((\partial \mathbf{M}_{\mathbf{c}_i}(t, x)/\partial t), (\partial \mathbf{M}_{\mathbf{c}_i}(t, x)/\partial x))\}$  whenever  $\mathbf{M}_{\mathbf{c}_i}(t, x)$  is differentiable, (41) implies

$$\begin{aligned} & \forall (t, x) \in \text{Dom}(\mathbf{M}_{\mathbf{c}_i}) \setminus \text{Dom}(\mathbf{c}_i), \\ & \frac{\partial \mathbf{M}_{\mathbf{c}_i}(t, x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}_{\mathbf{c}_i}(t, x)}{\partial x}\right) = 0 \\ & \text{(whenever } \mathbf{M}_{\mathbf{c}_i} \text{ is differentiable.)} \end{aligned} \quad (42)$$

## V. DEFINITION AND CHARACTERIZATION OF THE INTERNAL COMPONENTS

The major contribution of this article is the extension of a mathematical framework of Sections III and IV based on Theorem 4.8, which enables us to add *internal boundary conditions* [32], [38] to the mixed initial boundary conditions problem as was presented until now. This section makes use of the framework developed in all the previous sections to integrate these conditions. In the context of traffic, internal boundary conditions encode the trajectory information that we want to integrate from probe vehicle measurements. In this section, we construct the target functions associated with the internal boundary conditions. We also derive the specific Lax-Hopf formula used to compute the resulting internal boundary condition components which are integrated into the HJ PDE in Section VI. The specific Lax-Hopf formula also enables us to characterize the proper formulation conditions and interpret them physically in terms of traffic flow parameters for transportation problems.

### A. Formulation of the Problem

**Definition 5.1: Trajectory Label History:** We consider a trajectory function  $\bar{x}_A(\cdot)$  defined in the time interval  $[\bar{t}_{\min}, \bar{t}_{\max}]$ . The *trajectory label history*  $S_A$  of  $A$  is defined as the following set:

$$S_A = \{(t, \bar{x}_A(t), l_A(t)), t \in [\bar{t}_{\min}, \bar{t}_{\max}]\}$$

where  $\bar{x}_A(\cdot)$  and  $l_A(\cdot)$  correspond respectively to the trajectory function and label function of  $A$ .

The trajectory label history  $S_A$  of  $A$  contains information regarding the trajectory and the evolution of the order of  $A$  (with respect to the surrounding points). In the following work, we assume *once for all* that  $\bar{x}_A(\cdot)$  is a continuous function defined in  $[\bar{t}_{\min}, \bar{t}_{\max}]$ , where  $[\bar{t}_{\min}, \bar{t}_{\max}] \subset \mathbb{R}_+$ . The label function can be computed and interpreted as follows.

**Definition 5.2: Computation of the Label Function:** We consider a given point  $A$  labeled  $\bar{M}$  at time  $\bar{t}_{\min}$ , for which we prescribe a change in value of  $R_A(t)$  per unit time at time  $t$ . The function  $R_A(\cdot)$  is a measurable integrable function belonging to  $[\bar{t}_{\min}, \bar{t}_{\max}]^{\mathbb{R}}$ . Under these assumptions, the label function  $l_A(t)$  of  $A$  is given by the following formula:

$$l_A(t) = \bar{M} + \int_{\bar{t}_{\min}}^t R_A(\tau) d\tau.$$

In the context of traffic, the point  $A$  represents a given probe vehicle, and the quantity  $R_A(t)$  represents the number of surrounding vehicles that overtake  $A$  per time unit, at time  $t$ . Inspired by traffic, the function  $R_A(\cdot)$  is denoted as the *overtaking rate function* (in the remainder of the article).

### B. Characterization of the Internal Component

The novelty of the definition below is the introduction of data in the HJ PDE at points  $(t, x)$ , not on the boundary of the domain  $\mathbb{R}_+ \times X$  as was the case for  $\mathcal{M}_0(\cdot, \cdot)$ ,  $\gamma(\cdot, \cdot)$  and  $\beta(\cdot, \cdot)$ , but in its interior.

**Definition 5.3: Target Function:** We define following target function  $\mu_A(\cdot, \cdot)$  associated with the internal  $\mathbf{M}_{\mu_A}$  as:

$$\mu_A(t, x) := \begin{cases} \bar{M} + \int_{\bar{t}_{\min}}^t R_A(\tau) d\tau & \text{if } (t, x) \in \text{Graph}(\bar{x}_A) \\ +\infty & \text{otherwise.} \end{cases} \quad (43)$$

These internal conditions lead to the definition of internal condition components, which we now characterize by their Lax-Hopf formula. In order to define this formula, we first need to characterize the set  $S_{\mu_A}$  of capture times as defined in (21).

**Proposition 5.4: Characterization of the Set of Capture Times:** For any trajectory function  $\bar{x}_A(\cdot)$ , and any element  $(t, x, u) \in \mathbb{R}_+ \times X \times \text{Dom}(\varphi^*)$ , the set  $S_{\mathbf{c}_A}(t, x, u)$  is defined by

$$S_{\mathbf{c}_A}(t, x, u) = \{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max}, t - \bar{t}_{\min}] | Tu = \bar{x}_A(t - T) - x\}. \quad (44)$$

*Proof:* Equation (21) and (43) imply that elements  $T$  of  $S_{\mathbf{c}_A}(t, x, u)$  satisfy  $(t - T, x + Tu) \in \text{Graph}(\bar{x}_A)$ . Hence, the elements  $T$  of  $S_{\mathbf{c}_A}(t, x, u)$  are positive solutions to the equation  $x + Tu = \bar{x}_A(t - T)$  satisfying  $t - T \in [\bar{t}_{\min}, \bar{t}_{\max}]$ . ■

Equation (44) thus implies the following characterization of  $S_{c_A}(t, x, u)$ :

$$T \in S_{c_A}(t, x, u) \Rightarrow \begin{cases} u = \frac{\bar{x}_A(t-T) - x}{T} \\ t - T \in [\bar{t}_{\min}, \bar{t}_{\max}], \end{cases} \quad (45)$$

This directly leads into the following Lax–Hopf formula for internal conditions:

*Proposition 5.5: Lax Hopf Formula:* We define  $S_{c_A}(t, x, u)$  as in (44). The viability episolution  $\mathbf{M}_{c_A}(t, x)$  associated with the target function (43) can be computed using the Lax–Hopf formula (46)

$$\mathbf{M}_{c_A}(t, x) = \inf_{\{(u, T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+ \text{ such that } T \in S_{c_A}(t, x, u)\}} \left( \bar{M} + \int_{\bar{t}_{\min}}^{t-T} R_A(\tau) d\tau + T\varphi^*(u) \right). \quad (46)$$

For any trajectory function  $\bar{x}_A(\cdot)$  and any element  $(t, x, u) \in \mathbb{R}_+ \times X \times \text{Dom}(\varphi^*)$ , the set  $S_{c_A}(t, x, u)$  is defined by (44). We can thus express formula (46) in terms of  $T$  only

$$\mathbf{M}_{c_A}(t, x) = \inf_{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max}, t - \bar{t}_{\min}]} \left( \bar{M} + \int_{\bar{t}_{\min}}^{t-T} R_A(\tau) d\tau + T\varphi^* \left( \frac{\bar{x}_A(t-T) - x}{T} \right) \right). \quad (47)$$

The components obtained from internal conditions have a domain of definition and a proper formulation condition which can be derived following the previous method:

*Proposition 5.6: Domain of Definition of an Internal Component:* The domain of definition of  $\mathbf{M}_{c_A}$  is a consequence of formula (37)

$$\text{Dom}(\mathbf{M}_{c_A}) = \bigcup_{t \in [\bar{t}_{\min}, \bar{t}_{\max}]} \left( \bigcup_{T \in \mathbb{R}_+} \{t + T\} \times \left[ \bar{x}_A(t) - \nu^\sharp T, \bar{x}_A(t) + \nu^\flat T \right] \right).$$

Note that when  $\bar{x}_A(\cdot)$  satisfies the growth condition  $-\nu^\sharp(t - \bar{t}_{\min}) \leq \bar{x}_A(t) - \bar{x}_A(\bar{t}_{\min}) \leq \nu^\flat(t - \bar{t}_{\min})$  for all  $t$  in  $[\bar{t}_{\min}, \bar{t}_{\max}]$ , the above formula becomes

$$\text{Dom}(\mathbf{M}_{c_A}) = \bigcup_{t \in \mathbb{R}_+} \left( \{t\} \times \left[ \bar{x}_A(\bar{t}_{\min}) - \nu^\sharp t, \bar{x}_A(\bar{t}_{\min}) + \nu^\flat t \right] \right).$$

As previously, the internal component  $\mathbf{M}_{c_A}$  is properly formulated if and only if it satisfies

$$\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}], \inf_{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max}, t - \bar{t}_{\min}]} \left( \bar{M} + \int_{\bar{t}_{\min}}^{t-T} R_A(\tau) d\tau + T\varphi^* \left( \frac{\bar{x}_A(t-T) - \bar{x}_A(t)}{T} \right) \right) \geq \bar{M} + \int_{\bar{t}_{\min}}^t R_A(\tau) d\tau. \quad (48)$$

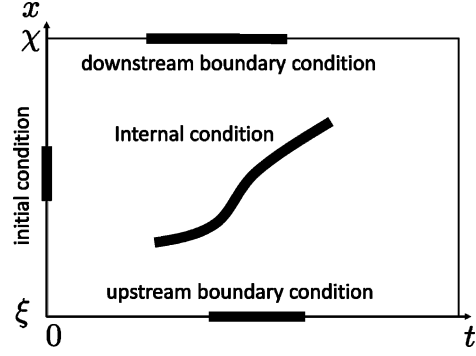


Fig. 8. Illustration of the domains of the possible value conditions used to construct the solution of the Moskowitz HJ PDE. The time is represented by the horizontal axis, while the location is represented by the vertical axis. The parameters  $\xi$  and  $\chi$  represent respectively the upstream and downstream boundaries of the highway segment of interest.

*Proposition 5.7: Proper Formulation of the Internal Component:* The internal component  $\mathbf{M}_{c_A}$  is properly formulated if and only if the trajectory function  $\bar{x}_A(\cdot)$  and the overtaking rate function  $R_A(\cdot)$  satisfy

$$\forall t \in [\bar{t}_{\min}, \bar{t}_{\max}], \inf_{T \in [0, t - \bar{t}_{\min}]} \left( T\varphi^* \left( \frac{\bar{x}_A(t-T) - \bar{x}_A(t)}{T} \right) - \int_{t-T}^t R_A(\tau) d\tau \right) \geq 0. \quad (49)$$

*Proof:* Equation (49) is a direct consequence of (48). ■

In the literature, in particular in [29], the transform  $\varphi^*(-v)$  is regarded in the context of traffic as the relative capacity of the highway for an observer traveling with velocity  $v$ . Hence, in the context of traffic flow, the internal component is properly formulated if and only if for all time intervals  $[t - T, t]$ , the average overtaking rate rate  $(1/T) \int_{t-T}^t R_A(\tau) d\tau$  does not exceed the relative capacity of the highway corresponding to the average velocity  $(\bar{x}_A(t) - \bar{x}_A(t - T))/T$ .

## VI. COMPUTATION OF THE MOSKOWITZ FUNCTION

We now have all the necessary results to solve the problem of interest: characterization of the solution of a HJ PDE with internal conditions, and corresponding Lax–Hopf formula computation. This section first derives the solution to the classical Cauchy problem (Section VI-A) as a reference and link to existing results, and then proceeds with the general case investigated here (Section VI-B) which is one of the novelties of this article.

Fig. 8 represents the possible conditions that can be imposed on the solution using the proposed method.

### A. The Mixed Initial-Boundary Conditions Problem

*Definition 6.1: Mixed Initial-Boundary Conditions Problem:* We consider an initial condition function  $\mathcal{M}_0$  as defined in (27), an upstream boundary condition function  $\gamma$  as defined in (28), and a downstream boundary condition function  $\beta$  as defined in

(29). The solution  $\mathbf{M}$  to the associated *mixed initial boundary conditions problem* is defined as

$$\left\{ \begin{array}{l} \mathbf{M} \text{ is a solution to equation (4) in the Barron/Jensen Frankowska} \\ \text{sense} \\ \mathbf{M}(0, x) = \mathcal{M}_0(0, x) \quad \forall x \in X \\ \mathbf{M}(t, \xi) = \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}(t, \chi) = \beta(t, \chi) \quad \forall t \in \mathbb{R}_+. \end{array} \right. \quad (50)$$

Assembling all the tools derived before, we now construct the solution using the following target function:

$$\mathbf{c} = \min(\mathcal{M}_0, \gamma, \beta). \quad (51)$$

The function  $\mathbf{c}$  defined by (51) is lower semicontinuous since it is the inf of lower semicontinuous functions. Theorem 4.8 states that the episolution  $\mathbf{M}_c$  associated with target  $\mathbf{c}$  defined by (51) is a Barron-Jensen/Frankowska solution to the Moskowitz HJ PDE (4).

We assume that condition (40) is satisfied (this condition implies that the boundary condition components are properly formulated). Since  $\mathbf{M}_c = \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_\gamma, \mathbf{M}_\beta)$ , we have the following equalities:

$$\left\{ \begin{array}{l} \mathbf{M}_c(0, x) = \min(\mathcal{M}_0(0, x), \mathbf{M}_\gamma(0, x), \mathbf{M}_\beta(0, x)) \quad \forall x \in X \\ \mathbf{M}_c(t, \xi) = \min(\mathbf{M}_{\mathcal{M}_0}(t, \xi), \gamma(t, \xi), \mathbf{M}_\beta(t, \xi)) \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}_c(t, \chi) = \min(\mathbf{M}_{\mathcal{M}_0}(t, \chi), \mathbf{M}_\gamma(t, \chi), \beta(t, \chi)) \quad \forall t \in \mathbb{R}_+. \end{array} \right. \quad (52)$$

In (52), we have used the fact that  $\forall x \in X$ ,  $\mathbf{M}_{\mathcal{M}_0}(0, x) = \mathcal{M}_0(0, x)$  by the proper formulation of the initial component, and similarly for the boundary condition components. Note that the initial condition component is always properly formulated as illustrated in example 4.7. The previous considerations lead to the following expression of the consistency conditions:

*Theorem 6.2: Consistency Conditions:* The episolution  $\mathbf{M}_c$  associated with target  $\mathbf{c}$  defined by (51) is solution to the mixed initial boundary condition problem (50) if and only if the following conditions are satisfied:

$$\left\{ \begin{array}{l} (i) \quad \mathcal{M}_0(0, \xi) = \gamma(0, \xi) \\ (ii) \quad \mathcal{M}_0(0, \chi) = \beta(0, \chi) \\ (iii) \quad \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(0, \xi + tu) + t\varphi^*(u)) \geq \gamma(t, \xi) \\ \quad \forall t \in \mathbb{R}_+ \\ (iv) \quad \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(0, \chi + tu) + t\varphi^*(u)) \geq \beta(t, \chi) \\ \quad \forall t \in \mathbb{R}_+ \\ (v) \quad \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \leq 0} \left( \gamma \left( t - \frac{\xi - \chi}{u}, \xi \right) \right. \\ \quad \left. + \frac{\xi - \chi}{u} \varphi^*(u) \right) \geq \beta(t, \chi) \\ \quad \forall t \in \mathbb{R}_+ \\ (vi) \quad \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \geq 0} \left( \beta \left( t - \frac{\chi - \xi}{u}, \chi \right) \right. \\ \quad \left. + \frac{\chi - \xi}{u} \varphi^*(u) \right) \geq \gamma(t, \xi) \\ \quad \forall t \in \mathbb{R}_+. \end{array} \right. \quad (53)$$

*Proof:* Equation (52) implies that  $\mathbf{M}_c$  is solution to the mixed initial boundary conditions problem (50) if the following conditions are satisfied:

$$\left. \begin{array}{l} (i) \quad \mathbf{M}_\gamma(0, x) \geq \mathcal{M}_0(0, x) \quad \forall x \in X \\ (ii) \quad \mathbf{M}_{\mathcal{M}_0}(t, \xi) \geq \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+ \\ (iii) \quad \mathbf{M}_\beta(0, x) \geq \mathcal{M}_0(0, x) \quad \forall x \in X \\ (iv) \quad \mathbf{M}_{\mathcal{M}_0}(t, \chi) \geq \beta(t, \chi) \quad \forall t \in \mathbb{R}_+ \\ (v) \quad \mathbf{M}_\gamma(t, \chi) \geq \beta(t, \chi) \quad \forall t \in \mathbb{R}_+ \\ (vi) \quad \mathbf{M}_\beta(t, \xi) \geq \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+. \end{array} \right\} \quad (54)$$

The conditions (54) (ii), (iv), (v) and (vi) are expressed by conditions (53) (iii), (iv), (v) and (vi) respectively. The conditions (54) (i) and (iii) are equivalent to  $\gamma(0, \xi) \geq \mathcal{M}_0(0, \xi)$  and  $\beta(0, \chi) \geq \mathcal{M}_0(0, \chi)$ , since  $\gamma(0, x) = +\infty$  if  $x \neq \xi$  and  $\beta(0, x) = +\infty$  if  $x \neq \chi$ . The conditions (54) (i), (ii), (iii) and (iv) thus imply the conditions (53) (i) and (ii). ■

Note that while conditions (53) do not seem to have appeared in the literature prior to this work, they find a counterpart in the literature on scalar hyperbolic PDEs. Indeed, the conditions (54) must be satisfied to impose at the same time the initial and boundary conditions in the strong sense [9], [39], [52]. When these conditions are not satisfied, the episolution associated with target defined by (51) will not satisfy at the same time all the prescribed conditions (*i.e.*, at least one of the prescribed conditions will be violated).

## B. The Mixed Initial-Boundary-Internal Conditions Problem

Finally, the main contribution of this work is the solution to the problem outlined below.

*Definition 6.3: Mixed Initial-Boundary-Internal Boundary Conditions Problem:* We consider a initial condition function  $\mathcal{M}_0$  as defined in (27), an upstream boundary condition function  $\gamma$  as defined in (28), a downstream boundary condition function  $\beta$  as defined in (29), a set of continuous trajectory functions  $\bar{x}_i(\cdot)$ , and a set of integrable overtaking rate functions  $R_i(\cdot)$  for  $i \in I$ . We assume that  $I$  is a finite set, and that the trajectory and overtaking rate functions  $\bar{x}_i(\cdot)$  and  $R_i(\cdot)$  are both defined in the time intervals  $[\bar{t}_{\min_i}, \bar{t}_{\max_i}]$ , and associated with the vehicle labeled  $\bar{M}_i$  at time  $\bar{t}_{\min_i}$ . Note that the corresponding internal condition is defined for all  $i \in I$  by definition 5.2. The solution  $\mathbf{M}$  to the associated *mixed initial-boundary-internal conditions problem* is defined

$$\left\{ \begin{array}{l} \mathbf{M} \text{ is a solution to equation (4) in the Barron/Jensen} \\ \text{Frankowska sense} \\ \mathbf{M}(0, x) = \mathcal{M}_0(x) \quad \forall x \in [\xi, \chi] \\ \mathbf{M}(t, \xi) = \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}(t, \chi) = \beta(t, \chi) \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}(t, \bar{x}_i(t)) = \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\ \quad \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \forall i \in I. \end{array} \right. \quad (55)$$

As for the previous case, the proper target is obtained by assembling the targets corresponding to all conditions. We define a set of target functions  $\mu_i(\cdot, \cdot)$ ,  $i \in I$  corresponding to the internal constraints as in (43). For the mixed initial-

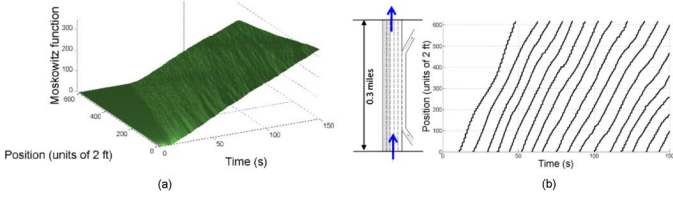


Fig. 9. NGSIM experimental data. Left: representation of the experimental Moskowitz surface obtained from the NGSIM data. Right: decimated representation of the associated trajectories. We represent only 5% of the trajectories for the sake of clarity of the figure.

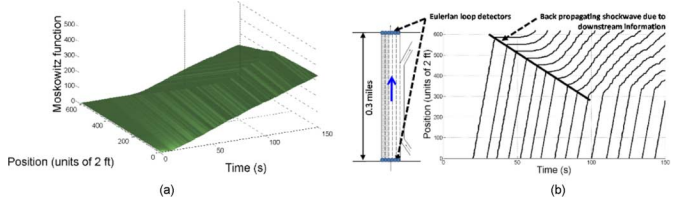


Fig. 10. Forward simulation using mixed initial and boundary conditions. Left: representation of the viability episolution solution to the mixed initial boundary conditions problem. Right: representation of the associated trajectories.

boundary-internal conditions problem, we define the target function  $\mathbf{c}(\cdot, \cdot)$

$$\mathbf{c} = \min \left( \mathcal{M}_0, \gamma, \beta, \min_{i \in I} (\mu_i) \right). \quad (56)$$

As previously, the function  $\mathbf{c}$  defined by (56) is lower semicontinuous since it is the infimum of lower semicontinuous functions. Theorem 4.8 states that the episolution  $\mathbf{M}_{\mathbf{c}}$  associated with the target  $\mathbf{c}$  defined by (56) is a Barron-Jensen/Frankowska solution to the Moskowitz HJ PDE. We assume that condition (40) is satisfied for both  $\gamma$  and  $\beta$  (this condition implies that the boundary condition components are properly formulated). We also assume that condition (48) is satisfied for each  $\bar{x}_i(\cdot)$  and  $R_i(\cdot)$  (this condition implies that the internal component  $i$  is properly formulated), and for all  $i$  in the set  $I$ . Since  $\mathbf{M}_{\mathbf{c}} = \min(\mathbf{M}_{\mathcal{M}_0}, \mathbf{M}_{\gamma}, \mathbf{M}_{\beta}, \min_{i \in I}(\mathbf{M}_{\mu_i}))$ , we have the following equalities:

$$\begin{cases} \mathbf{M}_{\mathbf{c}}(0, x) = \min(\mathcal{M}_0(0, x), \mathbf{M}_{\gamma}(0, x), \mathbf{M}_{\beta}(0, x), \min_{i \in I}(\mathbf{M}_{\mu_i}(0, x))) \\ \quad \forall x \in X \\ \mathbf{M}_{\mathbf{c}}(t, \xi) = \min(\mathbf{M}_{\mathcal{M}_0}(t, \xi), \gamma(t, \xi), \mathbf{M}_{\beta}(t, \xi), \min_{i \in I}(\mathbf{M}_{\mu_i}(t, \xi))) \\ \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}_{\mathbf{c}}(t, \chi) = \min(\mathbf{M}_{\mathcal{M}_0}(t, \chi), \mathbf{M}_{\gamma}(t, \chi), \beta(t, \chi), \min_{i \in I}(\mathbf{M}_{\mu_i}(t, \chi))) \\ \quad \forall t \in \mathbb{R}_+ \\ \mathbf{M}_{\mathbf{c}}(t, \bar{x}_i(t)) = \min(\mathbf{M}_{\mathcal{M}_0}(t, \bar{x}_i(t)), \mathbf{M}_{\gamma}(t, \bar{x}_i(t)), \mathbf{M}_{\beta}(t, \bar{x}_i(t)), \bar{M}_i \\ \quad + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau, \min_{j \in I \setminus \{i\}}(\mathbf{M}_{\mu_j}(t, \bar{x}_j(t))) \\ \quad \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \forall i \in I. \end{cases} \quad (57)$$

Similarly as in (52), we have used in (57) the fact that the initial, boundary and internal condition components are properly

formulated. The previous considerations lead to the following expression of the consistency conditions:

*Theorem 6.4: Consistency Conditions for the Mixed Initial-Boundary-Internal Conditions Problem:* We assume that  $\forall i \in I, \bar{t}_{\min_i} > 0$ . The episolution  $\mathbf{M}_{\mathbf{c}}$  associated with target  $\mathbf{c}$  is solution to the mixed initial-boundary-internal conditions problem (55) if and only if the following conditions are satisfied:

$$\left. \begin{cases} (i) & \mathcal{M}_0(0, \xi) = \gamma(0, \xi) \\ (ii) & \mathcal{M}_0(0, \chi) = \beta(0, \chi) \\ (iii) & \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(0, \xi + tu) + t\varphi^*(u)) \geq \gamma(t, \xi) \\ & \forall t \in \mathbb{R}_+ \\ (iv) & \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(t, \chi + tu) \\ & \quad + t\varphi^*(u)) \geq \beta(t, \chi) \\ & \forall t \in \mathbb{R}_+ \\ (v) & \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \leq 0} \left( \gamma \left( t - \frac{\xi - \chi}{u}, \xi \right) \right. \\ & \quad \left. + \frac{\xi - \chi}{u} \varphi^*(u) \right) \geq \beta(t, \chi) \\ & \forall t \in \mathbb{R}_+ \\ (vi) & \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \geq 0} \left( \beta \left( t - \frac{\chi - \xi}{u}, \chi \right) \right. \\ & \quad \left. + \frac{\chi - \xi}{u} \varphi^*(u) \right) \geq \gamma(t, \xi) \\ & \forall t \in \mathbb{R}_+ \\ (vii) & \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \leq 0} \left( \gamma \left( t - \frac{\xi - \bar{x}_i(t)}{u}, \xi \right) \right. \\ & \quad \left. + \frac{\xi - \bar{x}_i(t)}{u} \varphi^*(u) \right) \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\ & \forall i \in I, \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}] \\ (viii) & \inf_{u \in \text{Dom}(\varphi^*) \text{ such that } u \geq 0} \left( \beta \left( t - \frac{\chi - \bar{x}_A(t)}{u}, \chi \right) \right. \\ & \quad \left. + \frac{\chi - \bar{x}_A(t)}{u} \varphi^*(u) \right) \\ & \quad \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\ & \forall i \in I, \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}] \\ (ix) & \inf_{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max_i}, t - \bar{t}_{\min_i}]} \left( \bar{M}_i + \int_{\bar{t}_{\min_i}}^{t-T} R_i(\tau) d\tau \right. \\ & \quad \left. + T\varphi^* \left( \frac{\bar{x}_i(t-T) - \xi}{T} \right) \right) \geq \gamma(t, \xi) \\ & \forall i \in I, \forall t \in \mathbb{R}_+ \\ (x) & \inf_{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max_i}, t - \bar{t}_{\min_i}]} \left( \bar{M}_i + \int_{\bar{t}_{\min_i}}^{t-T} R_i(\tau) d\tau \right. \\ & \quad \left. + T\varphi^* \left( \frac{\bar{x}_i(t-T) - \chi}{T} \right) \right) \geq \beta(t, \chi) \\ & \forall i \in I, \forall t \in \mathbb{R}_+ \\ (xi) & \inf_{u \in \text{Dom}(\varphi^*)} (\mathcal{M}_0(0, \bar{x}_i(t) + tu) + t\varphi^*(u)) \geq \bar{M}_i \\ & \quad + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau, \forall i \in I, \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}] \\ (xii) & \inf_{T \in \mathbb{R}_+ \cap [t - \bar{t}_{\max_j}, t - \bar{t}_{\min_j}]} \left( \bar{M}_j + \int_{\bar{t}_{\min_j}}^{t-T} R_j(\tau) d\tau \right. \\ & \quad \left. + T\varphi^* \left( \frac{\bar{x}_j(t-T) - \bar{x}_i(t)}{T} \right) \right) \geq \bar{M}_i \\ & \quad + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\ & \forall i \in I, \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \quad \forall j \in I \setminus \{i\}. \end{cases} \quad (58)$$

*Proof:* Equation (57) implies that  $\mathbf{M}_{\mathbf{c}}$  associated with the target (56) is solution to the mixed initial boundary conditions

problem with general trajectory constraint (55) if and only if the following conditions are satisfied:

$$\begin{aligned}
(i) \quad & \mathbf{M}_\gamma(0, x) \geq \mathcal{M}_0(0, x) \quad \forall x \in X \\
(ii) \quad & \mathbf{M}_{\mathcal{M}_0}(t, \xi) \geq \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+ \\
(iii) \quad & \mathbf{M}_\beta(0, x) \geq \mathcal{M}_0(0, x) \quad \forall x \in X \\
(iv) \quad & \mathbf{M}_{\mathcal{M}_0}(t, \chi) \geq \beta(t, \chi) \quad \forall t \in \mathbb{R}_+ \\
(v) \quad & \mathbf{M}_\gamma(t, \chi) \geq \beta(t, \chi) \quad \forall t \in \mathbb{R}_+ \\
(vi) \quad & \mathbf{M}_\beta(t, \xi) \geq \gamma(t, \xi) \quad \forall t \in \mathbb{R}_+ \\
(vii) \quad & \mathbf{M}_{\mathcal{M}_0}(t, \bar{x}_i(t)) \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\
& \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \quad \forall i \in I \\
(viii) \quad & \mathbf{M}_{\mu_i}(0, x) \geq \mathcal{M}_0(0, x) \\
& \forall x \in X, \quad \forall i \in I \\
(ix) \quad & \mathbf{M}_\gamma(t, \bar{x}_i(t)) \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\
& \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \quad \forall i \in I \\
(x) \quad & \mathbf{M}_{\mu_i}(t, \xi) \geq \gamma(t, \xi) \\
& \forall t \in \mathbb{R}_+, \quad \forall i \in I \\
(xi) \quad & \mathbf{M}_\beta(t, \bar{x}_i(t)) \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\
& \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \quad \forall i \in I \\
(xii) \quad & \mathbf{M}_{\mu_i}(t, \chi) \geq \beta(t, \chi) \\
& \forall t \in \mathbb{R}_+, \quad \forall i \in I \\
(xiii) \quad & \mathbf{M}_{\mu_j}(t, \bar{x}_i(t)) \geq \bar{M}_i + \int_{\bar{t}_{\min_i}}^t R_i(\tau) d\tau \\
& \forall i \in I, \quad \forall t \in [\bar{t}_{\min_i}, \bar{t}_{\max_i}], \quad \forall j \in I \setminus \{i\}. \quad (60)
\end{aligned}$$

The conditions (ii), (iv), (v), (vi), (vii), (ix), (x), (xi), (xii) and (xiii) of (59) and (60) are expressed by conditions (58) (iii), (iv), (v), (vi), (xi), (vii), (ix), (viii), (x) and (xii) respectively. The conditions (59) (ii), (iv), (v) and (vi) are expressed by conditions (58) (iii), (iv), (v) and (vi) respectively. The conditions (59) (i) and (iii) are equivalent to  $\gamma(0, \xi) \geq \mathcal{M}_0(0, \xi)$  and  $\beta(0, \chi) \geq \mathcal{M}_0(0, \chi)$ , since  $\gamma(0, x) = +\infty$  if  $x \neq \xi$  and  $\beta(0, x) = +\infty$  if  $x \neq \chi$ . The conditions (59) (i), (ii), (iii) and (iv) thus imply the conditions (58) (i) and (ii).

Since we have  $\bar{t}_{\min_i} > 0$  for all  $i$  in the set  $I$ , we have that  $\forall x \in X, \forall i \in I, \mathbf{M}_{\mu_i}(0, x) = +\infty$ , and the condition (59) (viii) is always satisfied. ■

When inequalities (58) are not all satisfied, then the episolution  $\mathbf{M}_c(\cdot, \cdot)$  associated with (56) is still a Barron-Jensen/Frankowska of the HJ PDE (4). However, at least one of the conditions (55) will be violated.

While formula (58) seems very complex, it is actually computationally easy to check that these conditions are satisfied. Indeed, each condition requires at most a two dimensional search, and the number of constraints grows polynomially (indeed quadratically) with the number of internal conditions. In a companion article [21], we also show that the components can be computed explicitly when the initial, boundary and internal conditions are piecewise affine, which further simplifies the computation in this specific case.

The inequalities (58) can be used in the context of traffic flow data assimilation [22] to construct a *linear program* (LP) estimating bounds on some traffic flow parameters such as the travel time between two locations.

## VII. APPLICATION TO MACROSCOPIC TRAFFIC FLOW RECONSTRUCTION

We now illustrate the power of the previous results on a practical example: data assimilation using Lagrangian measurements. In an ongoing project with Nokia [34], [35], [55]–[57] and the US and California Departments of Transportation called *Mobile Millennium*, we are currently implementing prototype algorithms to use GPS equipped cellular phones traveling onboard vehicles on the highway as probe sensors to monitor the state of traffic in real time. This type of sensing is referred to as *Lagrangian* because the sensing device (the GPS) provides measurements along the trajectory it follows. This is in contrast to *Eulerian* sensing, which refers to fixed sensors monitoring physical quantities in a determined control volume, for example fixed loop detectors and the *Freeway Performance Measurement System* (PeMS) in California [54]. With the progressive penetration of GPS equipped phones in the market, Lagrangian sensors have the potential to become a fundamental data source in the context of highway systems, because of the prohibitive costs of deployment and maintenance of Eulerian sensors, and the dedicated infrastructure they require (in contrast, the cellular phone infrastructure is market driven and does not require any maintenance from state or federal agencies).

The example below uses *Next Generation Simulation* (NGSIM) data [58] from a section of Interstate I80 in Emeryville, CA as our main benchmark scenario for this study. This data set contains video extracted trajectories of all vehicles traversing a 0.4 mile long highway section during a period of 45 minutes. Given the accuracy of the video, this set of data can be considered as *ground truth*, i.e., it provides the exact location of vehicles to an accuracy of a few centimeters at a 10Hz rate. We degrade the quality of the data to model the inherent uncertainty linked with sensor measurement accuracy, and use the corresponding dataset for verification purposes. The corresponding data is represented in Fig. 9.

We use this data as follows:

- We record the initial state of traffic on the highway. This provides us with  $\mathcal{M}_0$ , i.e., the initial conditions of the problem.
- We create loop detector-like (Eulerian) data from this NGSIM data, following a standard procedure used in traffic engineering [34]. This provides us with traffic data similar to what the PeMS loop detector system would record in real life. This yields the boundary condition functions  $\gamma(t, \xi)$  and  $\beta(t, \chi)$  defined in Section IV-B for all positive times  $t$ .
- We extract some trajectories representative of measurements produced by GPS equipped Nokia N95 cellular phones traveling onboard of the selected vehicles. This can be done by sampling the NGSIM data along trajectories, and exporting the corresponding data in a format similar to what a GPS would produce. Noise is also added to the data, to make it similar to what GPS measurements would produce. This provides us with the trajectories  $x_A(\cdot)$  defined in Section V. We assume that the corresponding

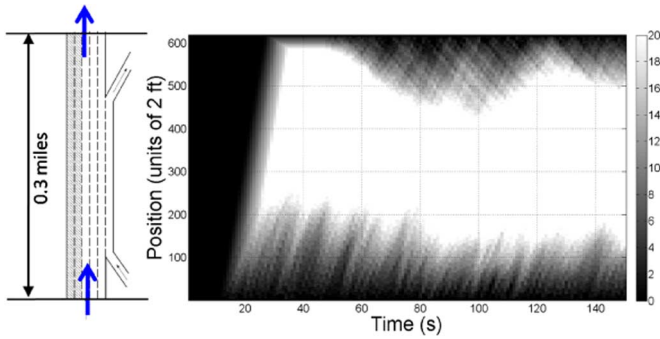


Fig. 11. Computation of the error between the simulation and the ground truth. This plot represents the error between the Moskowitz function estimated using initial and boundary conditions only, and the actual Moskowitz function. The error is expressed in total number of vehicles. In this Figure, the error is above 20 vehicles in white areas, and zero in black areas. The initial and boundary conditions correspond to areas with zero error, since we assume that these functions are perfectly known. The error increases away from these areas of perfect knowledge (which physically makes sense).

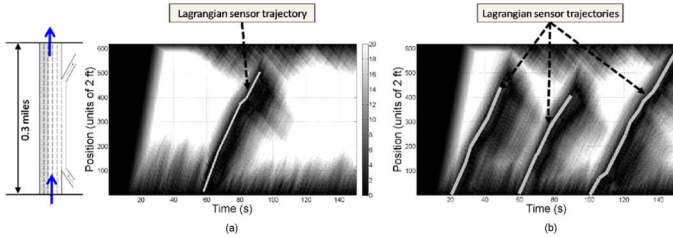


Fig. 12. Traffic flow reconstruction. For all plots, the error is low for dark areas, and high for white areas. Left: representation of the error associated with the solution to the initial boundary conditions problem with one trajectory constraint corresponding to the vehicle labeled 110. The constrained trajectory is represented by a gray line. Right: representation of the error associated with the solution to the initial boundary conditions problem with three trajectory constraints corresponding to the vehicles labeled 20, 120 and 220 respectively. The constrained trajectories are represented by gray lines. This situation represents a situation where 1% of the vehicles would be equipped with GPS sensors.

target functions  $c_A$  are given by (44) with  $R_A(\cdot) = 0$ : for the specific traffic flow situation described here, the overtaking events only marginally influence the solution. Note however that this last hypothesis does not hold for general traffic flow situations such as lane shearing situations.

Note that the two first sets of data are typical measurements obtainable from Eulerian sensing (see [34] for a full description of the procedure), while the third data set (the trajectory) is a typical Lagrangian data set, obtainable from a probe vehicle.

Note also that in this forward simulation, we know the label  $\bar{M}$  of the vehicle from which we extract the trajectory  $\bar{x}_A(\cdot)$ , since we have the full knowledge of the experimental Moskowitz function. Although the trajectory  $x_A(\cdot)$  is easily obtained using GPS measurements, there is no simple method to obtain the label  $\bar{M}$  of a probe vehicle without any other information. However, if we are able to obtain the function  $\gamma$  (from loop detector measurements for instance), the label  $\bar{M}$  of a vehicle entering the highway at time  $t$  is given by  $\bar{M} := \gamma(t, \xi)$ .

We next solve two problems.

*Problem 1:* First, we compute the solution of (4) satisfying the initial and boundary conditions. This yields the solution to the mixed initial boundary conditions problem (50), which corresponds to the estimate one could obtain without the probe data (GPS data). The corresponding results are shown in Fig. 10.

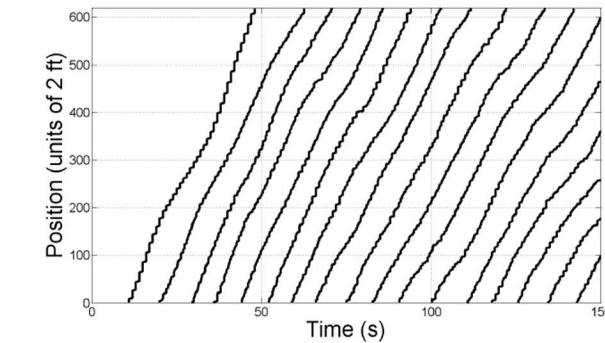
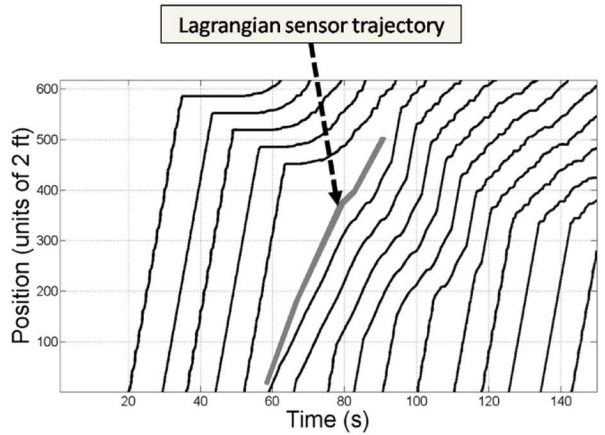
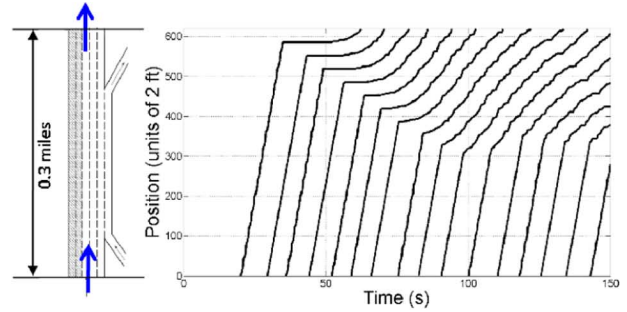


Fig. 13. Trajectories reconstruction. Top: representation of the trajectories associated with the solution to the mixed initial boundary conditions problem. Center: representation of the error associated with the solution to the mixed initial boundary conditions problem with trajectory constraint (the constrained trajectory is highlighted in gray). Bottom: representation of the ground truth trajectories corresponding to the NGSIM dataset.

As can be seen by visual comparison with Fig. 9, there is a significant mismatch between the estimates of the simulation and the ground truth. This is due to the fact that traffic does not obey the LWR model as soon as exogenous phenomena start to influence traffic, such as a driver randomly braking because of external disturbances. Fig. 11 displays the level of error between the estimates of the forward simulation and the ground truth.

*Problem 2:* In order to illustrate how the knowledge of the Moskowitz function can be improved from the incorporation of trajectories in the solution, we extract a real trajectories  $x_i(\cdot)$  from the NGSIM data. Using this new information, we solve the mixed initial-boundary-internal conditions problem (55), and construct the solution using internal components. As can be seen from Fig. 12, the solution models the ground truth more accurately after incorporating this Lagrangian data into the problem.

The corresponding trajectories are represented in Fig. 13. In this figure, one can clearly see that the solution to the mixed

initial boundary conditions problem with trajectory constraint now captures the trajectories associated with the vehicles downstream of the constrained trajectory.

On February 8, 2008, we successfully demonstrated the use of GPS equipped mobile phones for highway monitoring. Nicknamed the *Mobile Century* experiment, 100 vehicles drove loops on I-880 in California for 10 hours with GPS equipped mobile phones on board [57]. The data was collected and processed using a privacy aware architecture developed for this experiment, and new inverse models were implemented to enable accurate estimation of the velocity fields on the highway with 2%–5% of the traffic stream carrying the devices [55]. Estimates were broadcast live over the Internet to demonstrate the ability to integrate the data into flow models. The next phase of this project, nicknamed *Mobile Millennium* consists in the deployment of this technology for the general public in the Bay Area, and is now underway with more than 4000 users at the time of this submission. The solution method presented in this article was implemented in *Mobile Millennium* [57], and is an example of fusion of Eulerian and Lagrangian data in flow models. The numerical schemes used in the *Mobile Millennium* system are presented in a companion article [21].

### VIII. CONCLUSION

This article presents a technique based on control theory to include internal boundary conditions into Hamilton-Jacobi equations which can be used to model highway traffic. The proposed method relies on earlier work, which defines solutions in the sense of Barron/Jensen—Frankowska. This framework allows the use of epigraphical techniques, which in the present article allowed us to formulate the internal boundary conditions (Lagrangian data) in epigraphical form, and consequently pose the problem as a target problem. The corresponding target for each Lagrangian data is called component. Using an inf-morphism property, we were able to construct a solution to a problem containing multiple components. A semi explicit solution of the problem was proposed with a generalized Lax-Hopf formula, which was extended to the case of multiple components. In order to be applicable, the method requires consistency conditions to be satisfied, which are given explicitly. The method is implemented using one of the most accurate existing databases of highway traffic, the Next Generation Simulation data. The value provided by the data is demonstrated by adding measurements to simulations of the data where exogenous disturbances violate the flow model (i.e., the Hamilton-Jacobi equation). Numerical results show great improvement of the accuracy of traffic flow prediction due to this fusion of Lagrangian data into the flow model. Future work will include the development of numerical schemes to compute numerical solutions of this problem (beyond the Lax Hopf formula), presented in a companion article [21]. Current implementations of the techniques demonstrated in this article include the *Mobile Millennium* field operational test in the Bay Area, which was launched in November 2008.

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