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Stackelberg Thresholds on Parallel Networks with Horizontal Queues

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Abstract— **We study Stackelberg routing games on parallel networks with horizontal queues, in which a coordinator (leader) controls a fraction α of the total flow on the network, and the remaining players (followers) choose their routes selfishly. The objective of the coordinator is to minimize a system-wide cost function, the total travel-time, while anticipating the response of the followers.**

Nash equilibria of the routing game (with zero control) are known to be inefficient in the sense that the total travel-time is sub-optimal. Increasing the *compliance rate* α improves the cost of the equilibrium, and we are interested in particular in the *Stackelberg threshold*, i.e. the minimal compliance rate that achieves a *strict* improvement. In this work, we derive the optimal Stackelberg cost as a function of the compliance rate α, and obtain, in particular, the expression of the Stackelberg threshold.

I. Introduction

A. Motivation and related work

Non-cooperative network routing games model the interaction of selfish network users. Each player chooses a route that minimizes their individual travel-time. A Nash equilibrium, or Wardrop equilibrium [11], is a route assignment in which each player cannot improve their individual travel-time by unilaterally changing his/her route. The system-wide cost of a Nash equilibrium is, in general, sub-optimal, i.e. worse than the cost of the system optimum where a central coordinator assigns routes to every player in order to minimize the total cost [8].

In order to cope with selfishness, i.e. to reduce the cost of Nash equilibria, different tools have been studied, including congestion pricing [6], capacity allocation [3] and Stackelberg routing [7], [1], [10], [2]. In the Stackelberg routing game, a fraction α of the players are assumed to be controlled by a central coordinator. This may be the case in several situations, for instance when some players are not selfish and care about the system-wide efficiency, or when they have external incentive to do so. The total flow of these players will be referred to as *compliant flow*, and their routes are assigned by the central coordinator. The objective of the coordinator is to minimize the total travel-time, while anticipating the response of the remaining players, referred to as *non-compliant*. The solution to this game is a Stackelberg equilibrium.

In a Stackelberg routing game, the system-wide cost is a non-increasing function of the compliance rate α. When $\alpha = 0$, the coordinator has no control, and the equilibrium is simply a Nash equilibrium. The cost is then maximal. When $\alpha = 1$, the coordinator has total control, the cost is minimal, and the equilibrium is by definition, the system optimum.

Although the cost of the equilibrium is a non-increasing function of α, it may not be *strictly decreasing*. In particular, if the fraction of controlled players is too small, there may be no improvement. This leads to the following question: what is the minimal compliance rate[1] needed in order to achieve strict improvement in the total cost? This minimal fraction is called *Stackelberg threshold* [9]. Computing Stackelberg thresholds is of practical importance in several situations, such as traffic planning and control [4], [9].

In this article, we consider the same setting as in [5], [4], i.e. parallel networks with horizontal queues. In this setting, the latency of each link is given by a function that satisfies the assumptions of the class of latencies in horizontal queues, singled-valued in free-flow (HQSF). This class is useful in modeling congestion due to horizontal queues, e.g. in a transportation network, as opposed to vertical queues, e.g. in a communication network.

The contributions of the article are as follows: we derive the expression of optimal Stackelberg cost for the HQSF class on parallel networks. In particular, we obtain an expression for Stackelberg thresholds. We then illustrate the results on an example network by computing the optimal Stackelberg cost and the corresponding Stackelberg thresholds.

B. Organization of the article

In Section II, we define the Stackelberg routing game, present the assumptions of the model and review previous results. In Section III, we characterize the supports of Nash equilibria and Stackelberg equilibria, then derive in Section IV the general expression of the optimal Stackelberg cost. This leads in particular to the expression of Stackelberg thresholds, given in Section V. Finally, we present numerical results in Section VI.

II. Definitions and previous results

A. Routing game on a parallel link with horizontal queues

We consider a non-atomic routing game on a network of N parallel links, subject to flow demand r (see Figure 1). Each

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[1]The Stackelberg threshold is only defined when the cost of the social optimum is strictly less than the cost of a Nash equilibrium.

non-atomic player chooses a link $n \in \{1, \dots, N\}$, and he/she suffers a loss, or latency $\ell_n(x_n, m_n)$ that depends on the total flow on that link, $x_n \in [0, x_n^{\max}]$ and the congestion state of the link, $m_n \in \{0, 1\}$. By definition, the congestion state specifies whether the link is in free-flow $(m_n = 0)$ or is congested $(m_n = 1)$. The latency functions are assumed to

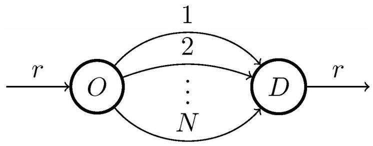

Fig. 1: Parallel network with N links and demand r.

be in the HQSF class introduced in [5], [4], i.e. satisfies the following assumptions:

1) The latency in free-flow $\ell_n(\cdot, 0) : [0, x_n^{\max}] \to \mathbb{R}_+$ is single-valued. We will denote by a_n its value, called the *free-flow latency*.
2) The latency in congestion $\ell_n(\cdot, 1) : (0, x_n^{\max}) \to (a_n, +\infty)$ is continuous decreasing and surjective.
3) Continuity: $\lim_{x_n \to x_n^{\max}} \ell(x_n, 1) = \ell_n(x_n^{\max}, 0) = a_n$

We also assume that the free-flow latencies are distinct, and that the links are ordered by increasing free-flow latency, i.e.

$$a_1 < a_2 < \cdots < a_N \tag{1}$$

Let (N, r) denote an instance of the routing game, $\boldsymbol{x} \in \mathbb{R}_+^N$ the vector of flows, $\boldsymbol{x}^{\max} \in \mathbb{R}_+^N$ the vector of capacities, and $\boldsymbol{m} \in \{0, 1\}^N$ the vector of congestion states on the network. The assignment $(\boldsymbol{x}, \boldsymbol{m})$ is said to be a feasible assignment if for every link n, the flow x_n is admissible $(x_n \le x_n^{\max})$ and the total flow is conserved, i.e. $\sum_{n=1}^N x_n = r$. For a feasible assignment $(\boldsymbol{x}, \boldsymbol{m})$, we define the total cost $C(\boldsymbol{x}, \boldsymbol{m})$ as the sum of the latencies experienced by all users on all links

$$C(\boldsymbol{x}, \boldsymbol{m}) = \sum_{n=1}^{N} \ell_n(x_n, m_n) x_n$$

Definition 1: Nash equilibrium
A feasible assignment $(\boldsymbol{x}, \boldsymbol{m})$ for the instance (N, r) is a Nash equilibrium if there exists a positive latency $\ell_0 > 0$ such that

$$\begin{aligned} n \in \text{supp}\,(\boldsymbol{x}) &\Rightarrow \ell_n(x_n, m_n) = \ell_0 \\ n \notin \text{supp}\,(\boldsymbol{x}) &\Rightarrow \ell_n(x_n, m_n) \ge \ell_0 \end{aligned} \tag{2}$$

Here $\text{supp}\,(\boldsymbol{x}) = \{n \in \{1, \dots, N\} | x_n > 0\}$ denotes the support of the flow vector. We will denote by $\text{NE}(N, r)$ the set of Nash equilibria of the instance (N, r).

Definition 2: Single-link free-flow equilibria and congestion flows
Let $(\boldsymbol{x}, \boldsymbol{m}) \in \text{NE}(N, r)$ be a Nash equilibrium, and let $k = \max \text{supp}\,(\boldsymbol{x})$ be the last link (i.e. the one with the largest free-flow latency) in the support of $\boldsymbol{x}$. If link k is in free-flow, then $(\boldsymbol{x}, \boldsymbol{m})$ is said to be a single-link-free-flow equilibrium, in which case the common latency on the support of $\boldsymbol{x}$ is a_k, and the flow on links $n \in \{1, \dots, k-1\}$ is given by the congestion flows $\hat{x}_n(k)$ defined by

$$\hat{x}_n(k) = \ell_n(\cdot, 1)^{-1}(a_k)$$

Therefore a single-link-free-flow equilibrium is of the form

$$\begin{aligned} \boldsymbol{m} &= (1, \quad \dots, 1, \qquad 0, \;\; 0, \dots, 0) \\ \boldsymbol{x} &= \big(\hat{x}_1(k), \dots, \hat{x}_{k-1}(k), x_k, 0, \dots, 0\big) \end{aligned}$$

with $x_k = r - \sum_{n=1}^{k-1} \hat{x}_n(k)$.

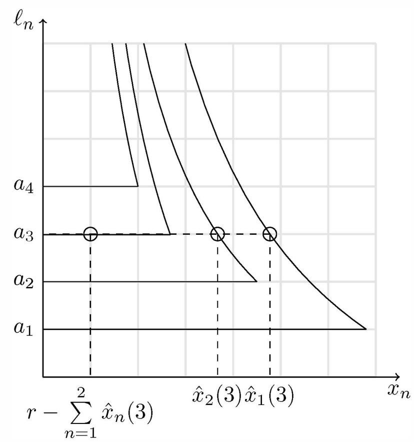

Fig. 2: Example of a single-link-free-flow equilibrium.

Figure 2 shows an example of a single-link-free-flow equilibrium on an instance with $N = 4$ links. We observe that the congestion flow $\hat{x}_n(k)$ is a decreasing function of k since $k \mapsto a_k$ is increasing by assumption (1) and $\ell_n(\cdot, 1)$ is decreasing.

Definition 3: For any $k \in \{1, \dots, N\}$, we denote by $r^{\text{NE}}(k)$ the maximum demand such that the set of Nash equilibria on the first k links, $\text{NE}(k, r)$, is non-empty. It is given by

$$r^{\text{NE}}(k) = \max_{j \in \{1, \dots, k\}} \{x_j^{\max} + \sum_{n=1}^{j-1} \hat{x}_n(j)\} \tag{3}$$

Remark 1: We also have the following property: a single-link-free-flow equilibrium exists for the instance (k, r) if and only if $r \le r^{\text{NE}}(k)$. See [5], [4].

By definition, we have $\forall k \in \{2, \dots, N\}$, $r^{\text{NE}}(k) \ge r^{\text{NE}}(k-1)$. We will be interested, in particular, in links that strictly increase the maximum demand, i.e. such that $r^{\text{NE}}(k) > r^{\text{NE}}(k-1)$. We denote these links by $k_1, \dots, k_c$, defined by induction as follows:

$$k_1 = 1 \tag{4}$$

$$\forall i \in \{2, \dots, c\},\; k_i = \min\{n \le N | r^{\text{NE}}(n) > r^{\text{NE}}(k_{i-1})\} \tag{5}$$

Therefore we have

$$r^{\text{NE}}(1) = r^{\text{NE}}(k_1) < \cdots < r^{\text{NE}}(k_c) = r^{\text{NE}}(N)$$

Definition 4: Best Nash equilibrium
The set of best Nash equilibria is the set of Nash equilibria

that minimize the system-wide latency.

$$\text{BNE}(N, r) = \underset{(\boldsymbol{x},\boldsymbol{m})\in \text{NE}(N,r)}{\arg\min} \quad C(\boldsymbol{x}, \boldsymbol{m}) \tag{6}$$

Remark 2: It is shown in [5] that the best Nash equilibrium is unique, and that it is equal to the single-link-free-flow equilibrium with smallest support. With a slight abuse of notation, we will use $\text{BNE}(N, r)$ to denote the unique best Nash equilibrium (identifying the set with its unique element).

Proposition 1: Last link in the support of a best Nash equilibrium
Let $(\boldsymbol{x}, \boldsymbol{m})$ be the best Nash equilibrium for the instance (N, r), then the last link in the support of $\boldsymbol{x}$ is given by

$$\max \text{supp}(\boldsymbol{x}) = \min \left\{k : r \le r^{\text{NE}}(k)\right\} \tag{7}$$

Proof: Let $b = \max \text{supp}(\boldsymbol{x})$. Since an equilibrium exists for the instance (b, r), then $r \le r^{\text{NE}}(b)$ by Definition 3 of the maximum demand. And for all k such that $r \le r^{\text{NE}}(k)$, by Remark 1, there exists a single-link-free-flow equilibrium supported on $\{1, \ldots, k\}$, thus by Remark 2, $k \ge b$. ∎

B. Stackelberg routing game

In the Stackelberg routing game, a central coordinator controls a fixed fraction of the total flow. This *compliant flow* corresponds to players who are either altruistic and care about the system-wide latency, or who may have an external incentive to be controlled by the coordinator. First, the coordinator (the leader) chooses the routes of the compliant flow. The resulting vector of flows is called a Stackelberg strategy and denoted by $\boldsymbol{s}$. It satisfies $\sum_{n=1}^{N} s_n = \alpha r$. Then the strategy $\boldsymbol{s}$ of the leader is revealed, and the remaining players (the followers, corresponding to non-compliant flow $(1 - \alpha)r$) choose their routes selfishly. The resulting non-compliant assignment, *induced* by Stackelberg strategy $\boldsymbol{s}$, is denoted by $(\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{m}(\boldsymbol{s}))$. It is assumed to be the best Nash equilibrium [5], [4] and satisfies the following: there exists a common latency ℓ_0 on the support of $\boldsymbol{t}(\boldsymbol{s})$ such that

$$\begin{aligned} n \in \text{supp}(\boldsymbol{t}(\boldsymbol{s})) &\Rightarrow \ell_n(t_n(\boldsymbol{s}) + s_n, m_n(\boldsymbol{s})) = \ell_0 \\ n \notin \text{supp}(\boldsymbol{t}(\boldsymbol{s})) &\Rightarrow \ell_n(s_n, m_n(\boldsymbol{s})) \ge \ell_0 \end{aligned} \tag{8}$$

We will denote by (N, r, α) an instance of the Stackelberg game played on a network with N parallel links, demand r and compliance rate α. The leader seeks to minimize the system-wide latency, or total cost, induced by the Stackelberg strategy $\boldsymbol{s}$, and given by $C(\boldsymbol{s}+\boldsymbol{t}(\boldsymbol{s}), \boldsymbol{m}(\boldsymbol{s}))$. The total assignment $(\boldsymbol{s} + \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{m}(\boldsymbol{s}))$ is called the Stackelberg equilibrium induced by $\boldsymbol{s}$.

Definition 5: Optimal Stackelberg strategies
The set of optimal Stackelberg strategies is

$$\text{S}^\star(N, r, \alpha) = \underset{\boldsymbol{s}\in \text{S}(N,r,\alpha)}{\arg\min} \; C(\boldsymbol{s} + \boldsymbol{t}(\boldsymbol{s}), \boldsymbol{m}(\boldsymbol{s})) \tag{9}$$

We will focus on one particular optimal Stackelberg strategy, the *non-compliant first* strategy (NCF), see [4]. It is defined as follows:

Definition 6: The non-compliant first strategy
Consider the Stackelberg instance (N, r, α). Let $(\boldsymbol{t}^{(\alpha)}, \boldsymbol{m}^{(\alpha)}) = \text{BNE}(N, (1 - \alpha)r)$ be the unique best Nash equilibrium of the non-compliant flow $(1 - \alpha)r$, and $k(\alpha) = \max \text{supp}(\boldsymbol{t}^{(\alpha)})$ be the last link in its support. Then the non-compliant first strategy is the Stackelberg strategy given by

$$\text{NCF}(N, r, \alpha) = \Big(0, \ldots, 0, \underbrace{x^{\max}_{k(\alpha)} - t^{(\alpha)}_{k(\alpha)}}_{k(\alpha)}, x^{\max}_{k(\alpha)+1}, \ldots, x^{\max}_{l(\alpha)-1}, \alpha r - \Big(\sum_{n=k(\alpha)}^{l(\alpha)-1} x^{\max}_n - t^{(\alpha)}_{k(\alpha)}\Big), 0, \ldots, 0\Big) \tag{10}$$

We will also use $\boldsymbol{s}^{(\alpha)}$ as a shorthand for $\text{NCF}(N, r, \alpha)$.[2]

The NCF strategy saturates links one by one starting from $k(\alpha)$, until the compliant flow αr is completely assigned. In expression (10), $l(\alpha)$ is the last link in the support of $\text{NCF}(N, r, \alpha)$, and can be defined as the maximal index l that satisfies $\alpha r - \left(\sum_{n=k(\alpha)}^{l-1} x^{\max}_n - t^{(\alpha)}_{k(\alpha)}\right) > 0$. Fig. 3 gives an illustration of the NCF strategy.

We note that the induced non-compliant equilibrium $(\boldsymbol{t}^{(\alpha)}, \boldsymbol{m}^{(\alpha)})$ is given by

$$\begin{aligned} \boldsymbol{m}^{(\alpha)} &= \big(1, \ldots, 1, 0, 0, \ldots, 0\big) \\ \boldsymbol{t}^{(\alpha)} &= \big(\hat{x}_1(k(\alpha)), \ldots, \hat{x}_{k(\alpha)-1}(k(\alpha)), t^{(\alpha)}_{k(\alpha)}, 0, \ldots, 0\big) \end{aligned} \tag{11}$$

the total flow $\boldsymbol{x}^{(\alpha)} = \boldsymbol{s}^{(\alpha)} + \boldsymbol{t}^{(\alpha)}$ is given by

$$\begin{aligned} \boldsymbol{x}^{(\alpha)} = \big(&\hat{x}_1(k(\alpha)), \ldots, \hat{x}_{k(\alpha)-1}(k(\alpha)), \\ &x^{\max}_{k(\alpha)}, \ldots, x^{\max}_{l(\alpha)-1}, x_{l(\alpha)}, 0, \ldots, 0\big) \end{aligned} \tag{12}$$

These results are summarized in Fig. 3.

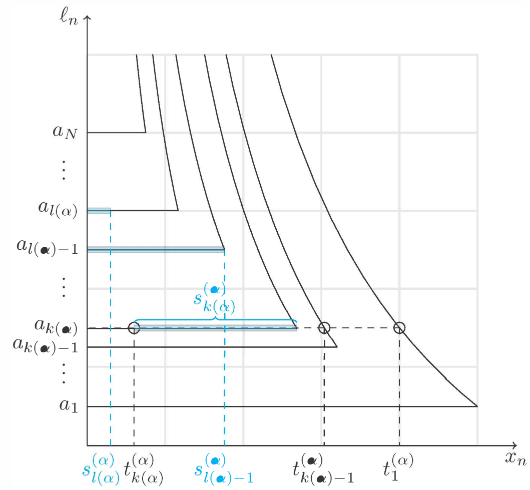

Fig. 3: Illustration of the NCF strategy. The best Nash equilibrium of the non-compliant flow is given by $(\boldsymbol{t}^{(\alpha)}, \boldsymbol{m}^{(\alpha)})$ (circles) and the NCF strategy $\boldsymbol{s}^{(\alpha)}$ is highlighted.

[2]Since we will consider instances with fixed demand and fixed number of links, we use superscript α to emphasize the dependency on the compliance rate.

III. Supports of equilibria induced by the NCF strategy

In this section, we show some properties of the supports of the non-compliant equilibrium and the Stackelberg equilibrium induced by the NCF strategy. We consider Stackelberg instances with a fixed number of links N, a fixed demand r, and a variable compliance rate α. Let $\boldsymbol{s}^{(\alpha)} = \text{NCF}(N, r, \alpha)$ be the NCF strategy as defined in Equation (10), $(\boldsymbol{t}^{(\alpha)}, \boldsymbol{m}^{(\alpha)}) = (\boldsymbol{t}(\boldsymbol{s}^{(\alpha)}), \boldsymbol{m}(\boldsymbol{s}^{(\alpha)}))$ the induced equilibrium of the non-compliant flow as defined in Equation (11), and $\boldsymbol{x}^{(\alpha)} = \boldsymbol{s}^{(\alpha)} + \boldsymbol{t}^{(\alpha)}$ the total flow of the Stackelberg equilibrium as defined in Equation (12).

Definition 7: We denote by $l(\alpha)$ the last link in the support of the Stackelberg equilibrium induced by the NCF strategy, i.e.

$$l(\alpha) = \max \operatorname{supp}\left(\boldsymbol{x}^{(\alpha)}\right) \tag{13}$$

Definition 8: We denote by $k(\alpha)$ the last link in the support of the non-compliant equilibrium induced by the NCF strategy, i.e.

$$k(\alpha) = \max \operatorname{supp}\left(\boldsymbol{t}^{(\alpha)}\right) \tag{14}$$

A. Properties of $k(\alpha)$

By definition, $(\boldsymbol{t}^{(\alpha)}, \boldsymbol{m}^{(\alpha)})$ is the best Nash equilibrium for the instance $(N, (1-\alpha)r)$. Thus by Proposition 1, the last link in its support is also given by

$$k(\alpha) = \min\left\{k : (1-\alpha)r \leq r^{\text{NE}}(k)\right\} \tag{15}$$

Remark 3: We observe that $r^{\text{NE}}(k(\alpha)) > r^{\text{NE}}(k(\alpha)-1)$ (otherwise $k(\alpha)$ would not be minimal and this would contradict Equation (15)), therefore we also have

$$r^{\text{NE}}(k(\alpha)) = x^{\max}_{k(\alpha)} + \sum_{n=1}^{k(\alpha)-1} \hat{x}_n(k(\alpha)) \tag{16}$$

Proposition 2: For all compliance rates $\alpha_1 \leq \alpha_2$, the best Nash equilibrium of $(N, (1-\alpha_2)r)$ uses at most as many links as the best Nash equilibrium of $(N, (1-\alpha_1)r)$. In other words, $\alpha \mapsto k(\alpha)$ is non-increasing.

Proof: Let $\alpha_1 \leq \alpha_2$. For all k such that $(1-\alpha_1)r \leq r^{\text{NE}}(k)$, we have $(1-\alpha_2)r \leq (1-\alpha_1)r \leq r^{\text{NE}}(k)$. Thus

$$\left\{k : (1-\alpha_1)r \leq r^{\text{NE}}(k)\right\} \subseteq \left\{k : (1-\alpha_2)r \leq r^{\text{NE}}(k)\right\}$$

Therefore using characterization (15), we have $k(\alpha_2) \leq k(\alpha_1)$. ■

B. Properties of $l(\alpha)$

In the next two propositions, we show how the support $k(\alpha)$ of the non-compliant equilibrium affects the support $l(\alpha)$ of the NCF strategy.

Proposition 3: For two given compliance rates, α_1 and α_2, if the best Nash equilibria of the instances $(N, (1-\alpha_1)r)$ and $(N, (1-\alpha_2)r)$ have the same support, then the corresponding optimal Stackelberg equilibria have the same support. In other words,

$$k(\alpha_1) = k(\alpha_2) \Rightarrow l(\alpha_1) = l(\alpha_2)$$

In that case, we additionally have $\boldsymbol{x}^{(\alpha_1)} = \boldsymbol{x}^{(\alpha_2)}$.

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$ be two compliance rates such that $k(\alpha_1) = k(\alpha_2) = k$, and suppose by contradiction that $l(\alpha_1) \neq l(\alpha_2)$. We assume without loss of generality that $l(\alpha_2) > l(\alpha_1)$. The total flow assignments $\boldsymbol{x}^{(\alpha_1)}$ and $\boldsymbol{x}^{(\alpha_2)}$ both sum to r, thus we have from the expression (12) of the total flows

$$r = r^{\text{NE}}(k) + \sum_{n=k+1}^{l(\alpha_1)-1} x_n^{\max} + x_{l(\alpha_1)}^{(\alpha_1)} \tag{17}$$

$$= r^{\text{NE}}(k) + \sum_{n=k+1}^{l(\alpha_2)-1} x_n^{\max} + x_{l(\alpha_2)}^{(\alpha_2)} \tag{18}$$

Substracting (17) from (18), we have

$$\left(\sum_{n=l(\alpha_1)+1}^{l(\alpha_2)-1} x_n^{\max}\right) + \left(x_{l(\alpha_1)}^{\max} - x_{l(\alpha_1)}^{(\alpha_1)}\right) + x_{l(\alpha_2)}^{(\alpha_2)} = 0$$

Since every term in the sum is non-negative, all terms are zero. In particular, $x_{l(\alpha_2)}^{(\alpha_2)} = 0$ which contradicts the definition of $l(\alpha_2)$ as the last link in the support of the Stackelberg equilibrium. Therefore we have $l(\alpha_2) = l(\alpha_1)$. Finally, we observe from the expression (12) that $\boldsymbol{x}^{(\alpha)}$ is entirely determined by $k(\alpha)$ and $l(\alpha)$. This proves that $\boldsymbol{x}^{(\alpha_1)} = \boldsymbol{x}^{(\alpha_2)}$. ■

Proposition 4: Let $\alpha_1, \alpha_2 \in [0,1]$. Then we have

$$k(\alpha_1) > k(\alpha_2) \Rightarrow l(\alpha_1) \geq l(\alpha_2)$$

Proof: Let $\alpha_1, \alpha_2 \in [0,1]$ be two compliance rates such that $k(\alpha_1) > k(\alpha_2)$, and suppose by contradiction that $l(\alpha_1) < l(\alpha_2)$. The total flow assignments $\boldsymbol{x}^{(\alpha_1)}$ and $\boldsymbol{x}^{(\alpha_2)}$ are given by

$$\boldsymbol{x}^{(\alpha_1)} = \big(\hat{x}_1(k(\alpha_1)), \dots, \hat{x}_{k(\alpha_1)-1}(k(\alpha_1)), x_{k(\alpha_1)}^{\max}, \dots, x_{l(\alpha_1)-1}^{\max}, s_{l(\alpha_1)}, 0, \dots, 0\big)$$

$$\boldsymbol{x}^{(\alpha_2)} = \big(\hat{x}_1(k(\alpha_2)), \dots, \hat{x}_{k(\alpha_2)-1}(k(\alpha_2)), x_{k(\alpha_2)}^{\max}, \dots, x_{l(\alpha_2)-1}^{\max}, s_{l(\alpha_2)}, 0, \dots, 0\big)$$

Since the congestion flow $\hat{x}_n(k)$ is a decreasing function of k, and since $k(\alpha_1) > k(\alpha_2)$, we have

$$\forall n \in \{1, \dots, k(\alpha_2)-1\},\ x_n^{(\alpha_1)} < x_n^{(\alpha_2)} \tag{19}$$

we also have $\forall n \in \{k(\alpha_2), \dots, l(\alpha_1)\}$, $x_n^{(\alpha_2)} = x_n^{\max}$, thus by definition of the maximum flow,

$$\forall n \in \{k(\alpha_2), \dots, l(\alpha_1)\},\ x_n^{(\alpha_1)} \leq x_n^{(\alpha_2)} \tag{20}$$

Summing inequalities (19) and (20), we have $\sum_{n=1}^{l(\alpha_1)} x_n^{(\alpha_1)} < \sum_{n=1}^{l(\alpha_1)} x_n^{(\alpha_2)}$, but $\sum_{n=1}^{l(\alpha_1)} x_n^{(\alpha_1)} = r$, and $\sum_{n=1}^{l(\alpha_1)} x_n^{(\alpha_2)} \leq \sum_{n=1}^{l(\alpha_2)} x_n^{(\alpha_2)} = r$. This leads to a contradiction and completes the proof. ■

Lemma 1: For all compliance rates $\alpha_1 \leq \alpha_2$, the Stackelberg equilibrium induced by $\boldsymbol{s}^{(\alpha_2)}$ uses at most as many links as the Stackelberg equilibrium induced by $\boldsymbol{s}^{(\alpha_1)}$. In other words, $\alpha \mapsto l(\alpha)$ is non-increasing.

Proof: This follows from Propositions 2, 3, and 4. ■

Corollary 1: The best Stackelberg assignment uses at most as many links as the Best Nash equilibrium, i.e. $l(\alpha) \leq k(0)$, for any $\alpha \in [0, 1]$.

Proof: In Stackelberg instance $(N, r, 0)$, since there is no compliant flow to assign, $l(0) = k(0)$, and since $\alpha \geq 0$, we have $l(\alpha) \leq l(0)$ by Lemma 1. This completes the proof. ■

This corollary states that increasing the compliance rate not only improves the system-wide cost, but it may also allow the central coordinator to use a smaller support for the total flow (i.e. less infrastructure).

IV. The cost of Stackelberg equilirbia

As in the previous section, we consider Stackelberg instances with a fixed number of links, fixed demand, and variable compliance rate. We derive the analytical expression of the optimal Stackelberg cost, which we will denote by $C_{\text{NCF}}(\alpha)$, as a function of the compliance rate $\alpha \in [0, 1)$.[3]

$$C_{\text{NCF}}(\alpha) = \sum_{n=1}^{l(\alpha)} x^{(\alpha)} \ell_n(x^{(\alpha)}, m^{(\alpha)}) \tag{21}$$

The main result is that $C_{\text{NCF}}(\alpha)$ is a non-increasing, piecewise-constant function of α with discontinuities exactly at the points $\left\{1 - \frac{r^{\text{NE}}(k_j)}{r}\right\}_{1 \leq j < j_\bullet}$ where k_j are the links that strictly increase the maximum demand, as defined in Section II-A, and j_0 is such that the last link in the support of the best Nash equilibrium $\text{BNE}(N, r)$ is $k(0) = k_{j_\bullet}$.

We define intervals $I_1, \dots, I_{j_\bullet}$ as follows:

- $I_1 = \left[1 - \frac{r^{\text{NE}}(k_1)}{r}, 1\right)$
- For $1 < j \leq j_0$, $I_j = \left[1 - \frac{r^{\text{NE}}(k_j)}{r}, 1 - \frac{r^{\text{NE}}(k_{j-1})}{r}\right)$

Proposition 5: The interval $I_{j_\bullet}$ satisfies $0 \in I_{j_\bullet}$.

Proof: By (15), we have

$$k_{j_\bullet} = k(0) = \min\{k : r \leq r^{\text{NE}}(k)\}$$

thus $r \leq r^{\text{NE}}(k_{j_0})$ and $r > r^{\text{NE}}(k_{j_0 - 1})$, i.e. $1 - \frac{r^{\text{NE}}(k_{j_0})}{r} \leq 0$ and $1 - \frac{r^{\text{NE}}(k_{j_0-1})}{r} > 0$. ■

Note that the intervals are disjoint by definition, and by Proposition 5, $[0, 1] \subseteq I_{j_\bullet} \cup \cdots \cup I_1$. See Fig. 4 for an illustration of the intervals $\{I_j\}_{1 \leq j \leq j_\bullet}$.

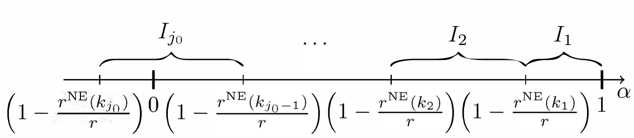

Fig. 4: Intervals $\{I_j\}_{1 \leq j \leq j_\bullet}$.

First, we prove that on each interval I_j, the optimal Stackelberg cost is constant.

[3]We exclude the case where the coordinator has total control ($\alpha = 1$) to simplify the discussion: in this case the non-compliant flow is zero and the last link in its support, $k(1)$ is not defined.

From the expression (12) of the total flow $\boldsymbol{x}^{(\alpha)}$ and the expression (11) of the congestion states, the optimal Stackelberg cost is given by

$$C_{\text{NCF}}(\alpha) = \left(\sum_{n=1}^{k(\alpha)-1} \hat{x}_n(k(\alpha))\right) a_{k(\alpha)} + \left(\sum_{n=k(\alpha)}^{l(\alpha)-1} x_n^{\max} a_n\right) + x_{l(\alpha)}^{(\alpha)} a_{l(\alpha)} \tag{22}$$

In this expression, several terms appear to depend on α: $k(\alpha)$, $l(\alpha)$ and $x_{l(\alpha)}^{(\alpha)}$. However, we show that when $\alpha \in I_j$, these terms are constant.

Lemma 2: Let $j \in \{1, \dots, j_0\}$. Then $\forall \alpha \in I_j$, $k(\alpha)$ is constant and equal to k_j, $l(\alpha)$ is constant, and the optimal Stackelberg cost $C_{\text{NCF}}(\alpha)$ is constant.

Proof: Let $j \in \{1, \dots, j_0\}$ and let $\alpha \in I_j$. We first show that $k(\alpha) = k_j$.

For $j \in \{1, \dots, j_0\}$, we have by definition of I_j

$$\alpha \in I_j \Leftrightarrow r^{\text{NE}}(k_{j-1}) < (1 - \alpha) r \leq r^{\text{NE}}(k_j)$$

(by convention, we let $k_0 = 0$ and $r^{\text{NE}}(0) = 0$ so that this statement is true for $j = 1$). By the inductive definition (5) of k_j, we have $\forall n < k_j$, $r^{\text{NE}}(n) \leq r^{\text{NE}}(k_{j-1})$, thus $\forall n < k_j$, $r^{\text{NE}}(n) < (1 - \alpha) r$. Therefore k_j is the minimal index such that $(1 - \alpha) r \leq r^{\text{NE}}(k_j)$, i.e. $k_j = k(\alpha)$ by characterization (14).

Next, since $k(\alpha)$ is constant, so are $l(\alpha)$ and $\boldsymbol{x}^{(\alpha)}$ by Proposition 3. Finally, from (22) the optimal Stackelberg cost is constant since all terms are constant. ■

For $\alpha \in I_j$, we will denote by l_j the constant value of $l(\alpha)$, and by C_j the constant value of $C_{\text{NCF}}(\alpha)$.

As a consequence of the previous Lemma, the optimal Stackelberg cost is piecewise constant as a function of the compliance rate α. The next theorem shows that it is a non-increasing function and specifies points of discontinuity.

Theorem 1: Optimal Stackelberg cost
The optimal Stackelberg cost $C_{\text{NCF}}(\alpha)$ is a non-increasing, piecewise-constant function of $\alpha \in [0, 1)$ with discontinuities exactly at the points $\left\{1 - \frac{r^{\text{NE}}(k_j)}{r}\right\}_{1 \leq j < j_0}$. On each I_j, $1 \leq j \leq j_0$, its constant value C_j is given by

$$C_j = \left(\sum_{n=1}^{k_j - 1} \hat{x}_n(k_j)\right) a_{k_j} + \left(\sum_{n=k_j}^{l_j - 1} x_n^{\max} a_n\right) + \left[r - \sum_{n=1}^{k_j - 1} \hat{x}_n(k_j) - \sum_{n=k_j}^{l_j - 1} x_n^{\max}\right] a_{l_j} \tag{23}$$

Proof: We need to prove that if $j > i$, then $C_j > C_i$. Let $i, j \in \{1, \dots, j_0 - 1\}$, such that $j > i$ and let $\alpha_i \in I_i$ and $\alpha_j \in I_j$.

We have by Lemma 2, $k(\alpha_i) = k_i$ and $k(\alpha_j) = k_j$. We also have

- $k_i < k_j$ (since $i < j$),
- $l_i \leq l_j$ (by Lemma 1, using $\alpha_i > \alpha_j$),

- $l_i > k_i$ (we have, by definition of the NCF strategy, $l_i \geq k_i$. If we have equality, then we have a single-link-free-flow equilibrium supported on $\{1, \dots, k_i\}$, thus $r \leq r^{\text{NE}}(k_i)$, but $k_i < k_j \leq k_{j_\bullet} = \min\{k | r \leq r^{\text{NE}}(k)\}$, this contradicts minimality of $k_{j_\bullet}$).

We now use the expression (12) to compare the flows $\boldsymbol{x}^{(\alpha_i)}$ and $\boldsymbol{x}^{(\alpha_j)}$. First, we have $\forall n \in \{1, \dots, k_i - 1\}$, $x_n^{(\alpha_i)} = \hat{x}_n(k_i)$ and $x_n^{(\alpha_j)} = \hat{x}_n(k_j)$ and since $k_i < k_j$, we have $\hat{x}_n(k_i) > \hat{x}_n(k_j)$ ($\hat{x}_n(\cdot)$ is decreasing). The latencies are given by $\ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) = a_{k_j}$ and $\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) = a_{k_i}$. Thus,

$$\forall n \in \{1, \dots, k_i - 1\}, x_n^{(\alpha_i)} > x_n^{(\alpha_j)} > 0 \text{ and } \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) > \ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) \quad (24)$$

Second, we have for $n = k_i$, $x_{k_i}^{(\alpha_i)} = x_{k_i}^{\max}$ (since $l_i > k_i$) and $x_{k_i}^{(\alpha_j)} = \hat{x}_n(k_j)$ (since $k_i < k_j$). Therefore,

$$x_{k_i}^{(\alpha_i)} > x_{k_i}^{(\alpha_j)} > 0 \quad (25)$$

Third, we have $\forall n \in \{k_i, \dots, l_i - 1\}$, $x_n^{(\alpha_i)} = x_n^{\max}$, and $\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) = a_n$. By definition of the maximum flow, we have $x_n^{(\alpha_j)} \leq x_n^{\max}$, and by definition of the free-flow latency a_n, $\ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \geq a_n$. Thus,

$$\forall n \in \{k_i, \dots, l_i - 1\}, x_n^{(\alpha_i)} \geq x_n^{(\alpha_j)} \text{ and } \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \geq \ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) \quad (26)$$

Finally, we have $\forall n \in \{l_i, \dots, l_j\}$, $\ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \geq a_n$ by definition of the latency function, and $a_n \geq a_{l_i}$ (by the ordering of the links). Thus

$$\forall n \in \{l_i, \dots, l_j\}, \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \geq a_{l_i} \quad (27)$$

Using the expression (21) of the optimal Stackelberg cost, we have

$$\begin{aligned} C_j &= \sum_{n=1}^{l_i-1} x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) + \sum_{n=l_i}^{l_j} x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \\ &\geq \sum_{n=1}^{l_i-1} x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) + \left(\sum_{n=l_i}^{l_j} x_n^{(\alpha_j)}\right) a_{l_i} \quad (28) \\ &> \sum_{n=1}^{l_i-1} x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) + \left(\sum_{n=l_i}^{l_j} x_n^{(\alpha_j)}\right) a_{l_i} \quad (29) \end{aligned}$$

where inequality (28) follows from (27), and inequality (29) follows from the fact that $\forall n \in \{1, \dots, l_i - 1\}$, $x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_j)}, m_n^{(\alpha_j)}) \geq x_n^{(\alpha_j)} \ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)})$, with strict inequality for $n \leq k_i$ by (24), (25) and (26).

We then use the fact that l_j is the last link in the support of $\boldsymbol{x}^{(\alpha_j)}$, thus $r = \sum_{n=1}^{l_j} x_n^{(\alpha_j)}$, i.e. $\sum_{n=l_i}^{l_j} x_n^{(\alpha_j)} = r - \sum_{n=1}^{l_i-1} x_n^{(\alpha_j)}$. Plugging this in the previous inequality, we have

$$C_j > \left(\sum_{n=1}^{l_i-1} x_n^{(\alpha_j)} \left(\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) - a_{l_i}\right)\right) + r a_{l_i}$$

We have $\forall n \in \{1, \dots, l_i - 1\}$, $\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) - a_{l_i} < 0$, and $x_n^{(\alpha_j)} - x_n^{(\alpha_i)} \leq 0$, thus

$$x_n^{(\alpha_j)} \left(\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) - a_{l_i}\right) \geq x_n^{(\alpha_i)} \left(\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) - a_{l_i}\right)$$

plugging this in the previous inequality and rearranging the terms, we obtain

$$\begin{aligned} C_j &> \left(\sum_{n=1}^{l_i-1} x_n^{(\alpha_i)} \left(\ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)}) - a_{l_i}\right)\right) + r a_{l_i} \\ &= \left(\sum_{n=1}^{l_i-1} x_n^{(\alpha_i)} \ell_n(x_n^{(\alpha_i)}, m_n^{(\alpha_i)})\right) + \left(r - \sum_{n=1}^{l_i-1} x_n^{(\alpha_i)}\right) a_{l_i} \\ &= C_i \end{aligned}$$

which competes the proof. ∎

V. Stackelberg threshold

Now that we have an analytical expression of the total cost of a Stackelberg equilibrium as a function of the compliance rate, we can express the Stackelberg threshold, or the minimum compliance rate the leader needs to control in order to achieve a strict improvement.

Proposition 6: The Stackelberg threshold is equal to $1 - \frac{r^{\text{NE}}(k_{j_0 - 1})}{r}$ where $k_{j_\bullet} = k(0) = \min\{k | r \leq r^{\text{NE}}(k)\}$.

Proof: Let $\alpha^\star$ be the Stackelberg threshold. By definition, $\alpha^\star = \inf\left\{\alpha : C_{\text{NCF}}(\alpha) < C_{\text{NCF}}(0)\right\}$.

By Proposition 5, we have $0 \in I_{j_\bullet}$, thus $C_{\text{NCF}}(0) = C_{j_\bullet}$. And if $\alpha \in I_j$, then $C_{\text{NCF}}(\alpha) = C_j$. Thus $\alpha^\star = \inf\{\alpha \in I_j : C_j < C_{j_\bullet}\}$ and by Theorem 1, $\alpha^\star = \inf\{\alpha \in I_{j_\bullet - 1}\}$. Therefore the Stackelberg threshold is simply given by the inf of interval $I_{j_\bullet - 1}$, i.e. $\alpha^\star = 1 - \frac{r^{\text{NE}}(k_{j_0 - 1})}{r}$ ∎

VI. Numerical results

In this section, we illustrate the results of Sections IV and V by numerically computing the NCF strategy and its cost on a example network. We generate a parallel network with $N = 5$ links, and randomly-generated latency functions, shown in Fig. 5. The optimal Stackelberg cost is

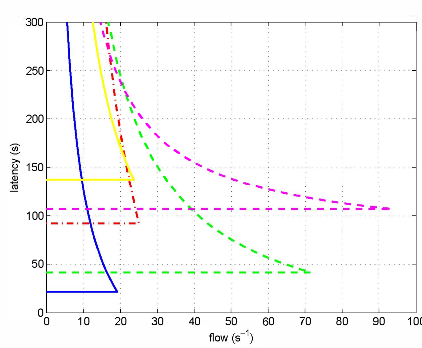

Fig. 5: Latency functions for the example network.

computed for $r \in [0, r^{\text{NE}}(N)]$ and $\alpha \in [0, 1]$. The results are shown in Fig. 6. For a fixed demand, the optimal cost is a piecewise constant function of α. This also illustrates the intervals $\{I_j\}_{1 \leq j \leq j_\bullet}$ discussed in the previous section. In this example, we have $r^{\text{NE}}(1) < r^{\text{NE}}(2) \leq r^{\text{NE}}(3) < r^{\text{NE}}(4) \leq r^{\text{NE}}(5)$, therefore the links that achieve a strict increase in the capacity are $k_1 = 1$, $k_2 = 2$ and $k_3 = 4$.

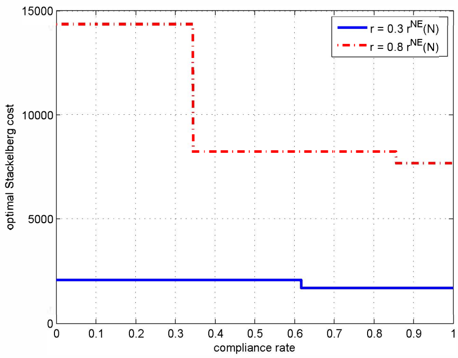

Fig. 6: $C_{\text{NCF}}(\alpha)$: optimal Stackelberg cost for a fixed demand r. The cost is a non-increasing, piecewise constant function of α. When the demand is $r = 0.8\ r^{\text{NE}}(N)$, we have $j_0 = 3$, therefore we have two discontinuities, at $\alpha = 1 - \frac{r^{\text{NE}}(k_1)}{r}$ and $\alpha = 1 - \frac{r^{\text{NE}}(k_2)}{r}$.

Finally, we numerically compute and plot the Stackelberg threshold for different values of the demand. The results, shown in Fig. 7, match the analytical expression given in Proposition 6.

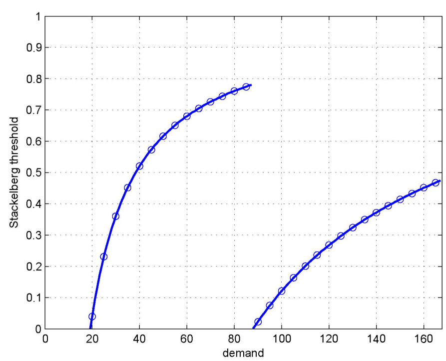

Fig. 7: Stackelberg thresholds as a function of the demand r. The solid line shows the analytical value of the Stackelberg threshold (given in Proposition 6). The circles show, for some values of r, the Stackelberg threshold computed numerically by a binary search to find the minimal α such that $C_{\text{NCF}}(\alpha) < C_{\text{NCF}}(0)$. The numerical value agrees with the analytical expression. The first branch corresponds to the range of demand $r \in (r^{\text{NE}}(1), r^{\text{NE}}(2)]$, and the second branch corresponds to the range of demands $(r^{\text{NE}}(2), r^{\text{NE}}(4)]$.

We observe that for low values of demand ($r \leq r^{\text{NE}}(1)$), the social optimum and the Nash equilibrium ($\alpha = 0$) are identical, therefore Stackelberg routing cannot strictly improve the cost. For $r > r^{\text{NE}}(1)$, we observe two branches: the first one corresponds to the range of demands $r \in (r^{\text{NE}}(1), r^{\text{NE}}(2)]$, for which the Stackelberg threshold is given by $1 - \frac{r^{\text{NE}}(k_1)}{r}$ ($j_0 = 2$). The second one corresponds to the range of demands $(r^{\text{NE}}(2), r^{\text{NE}}(4)]$, for which the Stackelberg threshold is given by $1 - \frac{r^{\text{NE}}(k_2)}{r}$ ($j_0 = 3$).

VII. Discussion and Concluding remarks

We studied Stackelberg routing games on parallel networks with the HQSF latency class, and we studied, in particular, how the optimal Stackelberg cost depends on the compliance rate α. We proved that it is a non-increasing, piecewise constant function, with discontinuities at specific points described in Theorem 1. As a consequence, we obtained an expression for the Stackelberg threshold, i.e. the minimal compliance rate needed to achieve a strict improvement in the cost. These results can be useful for efficient planning and control, for example on traffic networks. If a traffic planner can estimate the total demand on a parallel network, they can compute, given a model of latency on each route, the compliance rate needed to strictly improve the cost. This Stackelberg threshold can inform the planner whether Stackelberg routing is practical for the network considered.

While these results can be applicable in some scenarios of traffic, the simple topology of parallel networks limits applicability to a small subset of real networks. An immediate question is whether these results extend to more general topologies, and in particular, whether it is simple to characterize an optimal Stackelberg strategy for these topologies (similar to the NCF strategy in the parallel case). A second question is reachability of the equilibria: the analysis presented here gives existence results of static equilibria. Assuming one defines a dynamic model of response of the players to a Stackelberg strategy, a natural question is: which equilibria are reachable, and what are the optimal Stackelberg strategies in the dynamic case?

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