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Stability of Nash equilibria in the congestion game under Replicator dynamics

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***Abstract*— We consider the single commodity non-atomic congestion game, in which the player population is assumed to obey the replicator dynamics. We study the resulting rest points, and relate them to the Nash equilibria of the one-shot congestion game. The rest points of the replicator dynamics, also called *evolutionary stable* points, are known to coincide with a superset of Nash equilibria, called *restricted equilibria*. By studying the spectrum of the linearized system around rest points, we show that Nash equilibria are locally asymptotically stable stationary points. We also show that under the additional assumption of strictly increasing congestion functions, Nash equilibria are exactly the set of exponentially stable points. We illustrate these results on numerical examples.**

I. INTRODUCTION

Congestion games are a subclass of potential games that can be used to model the strategic interaction of a population of players who share resources, when the cost of a resource is increasing in the total mass of players utilizing it. An example of congestion game is the routing game, which models congestion on a transportation or communication network, in which every player chooses a route to be used for commute or to send packets between an origin node and a destination node. The total mass of players who are utilizing a given edge on the network is called the load of that edge, and it determines the *edge congestion*, that is, the cost incurred by the player for utilizing that edge. Routing games and their equilibria have been studied extensively, for example in [11], [1]. A particular attention was dedicated to studying the efficiency of Nash equilibria, for example in [9], and on developing schemes to improve the equilibria either through incentivization [6] or by controlling the route choice of a subset of the population [8].

The set of Nash equilibria of the congestion games is known to coincide with the set of minimizers of a convex potential function. This was proved by Rosenthal for the atomic routing game in [7], and later generalized. Thus computing the set of Nash equilibria can be done efficiently if one is given the exact formulation of the game, including the congestion functions of every resource. However, in realistic scenarios, it is unlikely that players have access to this information. A more natural model is that of adaptive play, which is of particular interest in evolutionary game theory, see for example [12] and the references therein. Evolutionary dynamics are usually specified using an ODE which describes the time-evolution of the population strategy profile, as a function of the loss profile. More precisely, if $\mu(t)$ is the strategy profile at time t, and $\ell(\mu(t))$ is the loss profile, an evolutionary dynamics is given by a vector field $\mu \mapsto F(\mu) = G(\ell(\mu), \mu)$, such that $\dot{\mu}(t) = F(\mu(t))$. Given the dynamics, a natural question is whether the strategy profile $\mu(t)$ converges to the set of Nash equilibria. This question has been studied for the congestion game under replicator dynamics in [3]. Fischer and Vöcking prove that the set of rest points of the replicator dynamics is a superset of Nash equilibria, which they call *restricted equilibria*.

In this paper, we study in more detail the stability of rest points of the replicator dynamics in the single-commodity case, by deriving the spectrum of the linearized dynamics around rest points. In particular, we relate the eigenvalues of the linearized system to the difference between the loss of a given bundle and the average loss. As a consequence, we show that Nash equilibria are exactly the set of stable rest points, and that under the additional assumption of strictly increasing congestion functions, Nash equilibria are exponentially stable.

Evolutionary stability of Nash equilibria has been studied in the more general context of potential games. In [10], Sandholm shows that if V is a potential function for the game, and the vector field F satisfies a positive correlation condition with respect to V, then Nash equilibria are rest points of the dynamics. If F further satisfies a non-complacency condition, then every rest point is a Nash equilibrium. The replicator dynamics only satisfy the positive correlation condition, therefore this result can only be used to conclude that Nash equilibria are rest points (but not the converse), and a more detailed analysis is needed to obtain stability or convergence rates.

We first define the congestion game in Section II, and give preliminary results on the replicator dynamics in Section III. Then, we study the linearization of the system about rest points, and show stability results in Section IV. Finally, we illustrate these results on a numerical example in Section V, and give a summary of results and some concluding remarks.

II. THE CONGESTION GAME

Consider a measurable set of players $(\mathcal{X}, \mathcal{M}, m)$, and assume that $m(\mathcal{X})$ is finite, and that m is a non-atomic measure, that is, $m(\{x\}) = 0$ for all $x \in \mathcal{X}$. Without loss of generality, we assume that $m(\mathcal{X}) = 1$. Let $\mathcal{R}$ be a finite set

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of resources, and let $\mathcal{P} \subset P(\mathcal{R})$ be a collection of subsets of $\mathcal{R}$, which we refer to as the set of bundles. We assume that $\mathcal{R}$ does not contain the empty bundle. The action set of every player is $\mathcal{P}$, that is, every player chooses a bundle.

The losses of the players are then determined as follows: $\forall r \in \mathcal{R}$, let c_r be the congestion function of resource r.

Assumption 1: Congestion functions c_r are assumed to be *continuously-differentiable, non-negative, non-decreasing*.

Then given the joint actions of all players, $A : \mathcal{X} \to \mathcal{P}(\mathcal{R})$ (assumed to be m-measurable), the loss of a player $x \in \mathcal{X}$ who chooses the bundle $p \in \mathcal{P}$ is

$$\sum_{r \in p} c_r(\phi_r(A))$$

where $\phi_r(A)$ is the resource load, determined by the joint action of all players, defined by $\phi_r(A) = \sum_{p \in \mathcal{P}: r \in p} m(\{x \in \mathcal{X} : A(x) = p\})$. In words, ϕ_r is the total mass of players utilizing resource r.

A. A macroscopic representation of the game

In order to use an evolutionary description of the strategy dynamics, we shall derive a macroscopic description of the game in terms of strategy distribution.

Let μ be the distribution of player strategies, that is, $\mu_p = m(\{x \in \mathcal{X} : A(x) = p\})$. Then μ is an element of $\Delta^{\mathcal{P}}$, the simplex on $\mathcal{P}$. Then we have

$$\phi_r = (M\mu)_r$$

where M is an incidence matrix such that $\forall r \in \mathcal{R}$ and $\forall p \in \mathcal{P}$,

$$M_{r,p} = \begin{cases} 1 & \text{if } r \in p \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Note that we use $\mathcal{P}$ and $\mathcal{R}$ as index sets instead of integers (but one could easily map each of these finite sets to a subset of integers). Column p in M, denoted M_p, is simply a binary representation of the resources in p. Then the loss of bundle $p \in \mathcal{P}$ is

$$\sum_{r \in p} c_r((M\mu)_r) = (M_p)^T c(M\mu)$$

where $c(M\mu)$ denotes the vector $\left(c_r((M\mu)_r)\right)_{r \in \mathcal{R}}$. Finally, we define the vector of losses, $\ell(\mu) = M^T c(M\mu)$. With this notation, we have a concise description of the game: the joint action of players determines the product distribution $\mu \in \Delta^{\mathcal{P}}$, which, in turn, determines the losses of every bundle, given by the vector $\ell(\mu) \in \mathbb{R}_+^{\mathcal{P}}$.

B. Nash equilibria

A Nash equilibrium of the one-shot congestion game is, by definition, a distribution $\mu \in \Delta$, such that for all $p \in \text{support}(\mu)$, $\ell_p(\mu)$ is the minimum of $\{\ell_{p'}(\mu), p' \in \mathcal{P}\}$. In other words, all bundles with positive mass have the same loss, and other bundles have greater or equal loss. In particular, no positive mass of players can decrease their loss by unilaterally switching to a different bundle. We will denote $\mathcal{N}$ the set of Nash equilibria.

The set of Nash equilibria is known to be the set of minimizers in Δ of the following convex problem:

$$\text{minimize}_{\mu \in \Delta} V(\mu) = \sum_{r \in \mathcal{R}} \int_0^{(M\mu)_r} c_r(\phi) d\phi \tag{2}$$

Convexity follows from the assumption that the congestion functions are non-negative non-decreasing. Optimality of $\mathcal{N}$ can be obtained by observing that $\mathcal{N}$ coincides with the set of points which satisfy the KKT conditions for problem (2). Additionally, the problem satisfies constraint qualification by Slater's condition since the relative interior of the product of simplexes, defined as

$$\mathring{\Delta} = \{\mu \in \Delta : \ \forall p \in \mathcal{P}, \ \mu_p > 0\} \tag{3}$$

is non-empty. Therefore KKT conditions are necessary and sufficient conditions for optimality (see [2] for example). For a detailed proof of optimality of $\mathcal{N}$, see [10].

III. Replicator dynamics

We now define the replicator dynamics. The strategy distribution $\mu(t)$ is assumed to obey the following dynamics: $\forall p \in \mathcal{P}$,

$$\begin{aligned} \dot{\mu}_p(t) &= F_p(\mu) \\ &= \frac{1}{\rho}\left(\bar{\ell}(\mu(t)) - \ell_p(\mu(t))\right)\mu_p(t) \end{aligned} \tag{4}$$

where

$$\bar{\ell}(\mu) = \langle \ell(\mu), \mu \rangle = \sum_{p \in \mathcal{P}} \ell_p \mu_p$$

is simply the average loss incurred by all players, and ρ is a positive parameter. Therefore, equation (4) states that the fraction of players who choose bundle p increases whenever the difference $\ell_p(\mu) - \bar{\ell}(\mu)$ is negative, that is, bundle p has lower loss than the average loss incurred by the entire population under distribution μ. For a more detailed motivation of the replicator equation, and a discussion of its properties, see [3] or Chapter III.29 in [5].

A. Properties of the solution trajectories

Equation (4) defines a vector field $F : \Delta \to \mathcal{H}_{\mathcal{P}}$, where

$$\mathcal{H}_{\mathcal{P}} = \left\{ v \in \mathbb{R}^{\mathcal{P}} : \sum_{p \in \mathcal{P}} v_p = 0 \right\}$$

is the linear hyperplane parallel to the simplex $\Delta^{\mathcal{P}}$. Indeed, we have for all $\mu \in \Delta$

$$\sum_{p \in \mathcal{P}} F_p(\mu) = \sum_{p \in \mathcal{P}} \ell_p(\mu)\mu_p - \bar{\ell}(\mu) \sum_{p \in \mathcal{P}} \mu_p = 0$$

This proves that the derivatives remain in the direction of the simplex.

Now, we consider the ODE system

$$\begin{aligned} &\dot{\mu} = F(\mu) \\ &\mu(0) \in \mathring{\Delta} \end{aligned} \tag{5}$$

Here, we require that the initial condition be in the relative interior of the simplex for the following reason: whenever

$\mu_p(0) = 0$, any solution trajectory will have $\mu_p(t) \equiv 0$ by equation (4). It is impossible for such trajectories to converge to a Nash equilibrium $\mu^\star$ when $\mu^\star_p > 0$. In other words, the replicator dynamics cannot expand the support of the strategies, and a bundle p which initially has zero mass will always have zero mass. Therefore it is natural to require that all bundles initially have positive mass.

The following proposition ensures that the solutions remain in the relative interior and are defined on all times.

Proposition 1: The ODE (5) has a unique solution $\mu(t)$ which remains in $\mathring{\Delta}$ and is defined on $\mathbb{R}_+$.

Proof: First, since the congestion functions c_r are assumed to be continuously differentiable, so is the vector field F. Thus we have existence and uniqueness of a solution by the Cauchy-Lipschitz theorem.

To show that the solution remains in the relative interior of Δ, we observe that:

- For all $\mu \in \Delta$, we have: $\frac{d}{dt}\sum_{p\in\mathcal{P}} \mu_p(t) = \sum_{p\in\mathcal{P}} F_p(\mu(t)) = 0$ by the previous observation. Therefore, $\sum_{p\in\mathcal{P}} \mu_p(t)$ is constant and equal to 1.
- To show that $\mu_p(t) > 0$ for all t in the solution domain, assume by contradiction that there exists $t_0 > 0$ and $p_0 \in \mathcal{P}$ such that $\mu_{p_0}(t_0) = 0$. Since the solution trajectories are continuous, we can assume, without loss of generality, that t_0 is the infimum of all such times (thus for all $t < t_0$, $\mu_{p_0}(t) > 0$). Now consider the new system given by

$$\begin{aligned} \dot{\tilde{\mu}}_p &= \frac{1}{\rho}(\bar{\ell}(\tilde{\mu}) - \ell_p(\tilde{\mu}))\tilde{\mu}_p \quad && \forall p \neq p_0 \\ \tilde{\mu}_p(t_0) &= \mu_p(t_0) && \forall p \neq p_0 \end{aligned}$$

  and $\tilde{\mu}_{p_0}(t)$ is constant equal to 0. Any solution of the new system, defined on $(t_0 - \delta, t_0]$, is also a solution of equation (5). Since $\mu(t_0) = \tilde{\mu}(t_0)$, we have $\mu \equiv \tilde{\mu}$ by uniqueness of the solution. This leads to a contradiction since by assumption, for all $t < t_0$, $\mu_p(t) > 0$ but $\tilde{\mu}_p(t) = 0$.

This proves that μ remains in $\mathring{\Delta}$. Furthermore, since Δ is compact, we have by Theorem 2.4 in [4] that the solution is defined on $\mathbb{R}_+$ (otherwise it would eventually leave any compact set). ∎

B. Stationary points

We now identify stationary points of the dynamics:

Proposition 2: μ is a stationary point of the system (5) if and only if the losses $(\ell_p(\mu), p \in \operatorname{support}(\mu))$ are equal. The set of all such stationary points will be denoted $\mathcal{RN}$.

Proof: From equation (4), we have

$$\begin{aligned} F_p(\mu) = 0 &\Leftrightarrow \mu_p = 0 \text{ or } \ell_p(\mu) = \bar{\ell}(\mu) \\ &\Leftrightarrow \forall p \in \operatorname{support}(\mu),\ \ell_p(\mu) = \bar{\ell}(\mu) \end{aligned}$$

∎

In particular, any Nash equilibrium satisfies this assumption, and is a rest point for the replicator dynamics (this also follows from the more general result given in [10]). However, a rest point is not necessarily a Nash equilibrium, since one may have a rest point with $\mu_{p_0} = 0$ and $\ell_{p_0}(\mu)$ strictly lower than $\bar{\ell}(\mu)$.

A rest point $\mu^\dagger \in \mathcal{RN}$ given by Proposition 2 is also called *restricted Nash equilibrium* (hence the notation $\mathcal{RN}$), since it is a Nash equilibrium for the congestion game if the action set is restricted to $\operatorname{support}(\mu^\dagger)$.

The set of equilibria can be partitioned into $\mathcal{RN} = \mathcal{N} \cup (\mathcal{RN} \setminus \mathcal{N})$. In the next section, we show that $\mathcal{N}$ is exactly the set of stable rest points, and that any point in $\mathcal{RN} \setminus \mathcal{N}$ is unstable.

IV. Stability of equilibria

First, we can write F in the form

$$\begin{aligned} F : \Delta &\to \mathcal{H}_\mathcal{P} \\ \mu &\mapsto -\operatorname{dg}(\ell(\mu))\mu + \mu\bar{\ell}(\mu) \end{aligned} \tag{7}$$

where $\operatorname{dg} : \mathbb{R}^\mathcal{P} \to \mathbb{R}^{\mathcal{P}\times\mathcal{P}}$ is the operator which maps a vector to the diagonal matrix whose diagonal elements are given by the the vector entries. Observing that $\bar{\ell}(\mu) = \ell(\mu)^T\mu = \mathbf{1}_\mathcal{P}^T \operatorname{dg}(\ell(\mu))\mu$, where $\mathbf{1}_\mathcal{P}$ is a vector in $\mathbb{R}^\mathcal{P}$ whose entries are all equal to one, we can write

$$\begin{aligned} F(\mu) &= -\operatorname{dg}(\ell(\mu))\mu + \mu\mathbf{1}_\mathcal{P}^T \operatorname{dg}(\ell(\mu))\mu \\ &= -\left[I_\mathcal{P} - \mu\mathbf{1}_\mathcal{P}^T\right]\operatorname{dg}(\ell(\mu))\mu \\ &= -\Psi(\mu)L(\mu)\mu \end{aligned} \tag{8}$$

where $\Psi(\mu) = I_\mathcal{P} - \mu\mathbf{1}_\mathcal{P}^T$ and $L(\mu) = \operatorname{dg}(\ell(\mu))$ are $\mathcal{P}\times\mathcal{P}$ matrices. This matrix form of F will be useful when we derive the Jacobian of the system.

A. Instability of non-Nash stationary points

Proposition 3: If μ is a stationary point of system (5) but not a Nash equilibrium, then μ is unstable.

To prove this proposition, we derive the eigenvalues $\mathfrak{S}$ of the Jacobian of the vector field at stationary points. Note that the vector field is continuously differentiable by assumption on the congestion functions, so the Jacobian exists and is continuous.

As observed in the previous section, F is defined on $\Delta^\mathcal{P}$ and has values in $\mathcal{H}_\mathcal{P}$, the linear hyperplane orthogonal to the unit vector $\mathbf{1}_\mathcal{P}$. But F can also be viewed as a function from $\mathbb{R}^\mathcal{P}$ to itself. We first derive the Jacobian of F viewed as function from $\mathbb{R}^\mathcal{P}$ to $\mathbb{R}^\mathcal{P}$, denoted $\nabla F(\mu)$, and then consider its restriction to $\mathcal{H}_\mathcal{P}$ to obtain $\mathfrak{S}$ as the eigenvalues of $\nabla F(\mu)_{|\mathcal{H}_\mathcal{P}}$.

Lemma 1: The Jacobian of F is given by

$$\nabla F(\mu) = \bar{\ell}(\mu)I_\mathcal{P} - \Psi(\mu)\operatorname{dg}(\mu)\nabla\ell(\mu) - \Psi(\mu)L(\mu) \tag{9}$$

Proof: Let $DF(\mu)$ be the differential of F at μ, and let e_p be a vector of the canonical basis. Then $DF(\mu)(e_p)$ is the directional derivative of F in the direction of e_p. From the matrix form of F given in equation (8), and using the product rule of differentials, we have the expression (6) of DF, given on the top of the next page, where we use the following differentials:

$$\begin{aligned} D\Psi(\mu)(e_p) &= -e_p\mathbf{1}_\mathcal{P}^T \\ DL(\mu)(e_p) &= \operatorname{dg}(\nabla\ell(\mu)e_p) \end{aligned}$$

$$DF(\mu)(e_p) = -D\Psi(\mu)(e_p)L(\mu)\mu - \Psi(\mu)DL(\mu)(e_p)\mu - \Psi(\mu)L(\mu)e_p \quad \text{by the product rule}$$
$$= e_p\mathbf{1}_{\mathcal{P}}^T L(\mu)\mu - \Psi(\mu)\,\mathrm{dg}\left(\nabla\ell(\mu)e_p\right)\mu - \Psi(\mu)L(\mu)e_p \quad \text{using the expressions of } D\Psi \text{ and } DL$$
$$= e_p\ell(\mu)^T\mu - \Psi(\mu)\,\mathrm{dg}(\mu)\nabla\ell(\mu)e_p - \Psi(\mu)L(\mu)e_p \quad \text{using that } \mathrm{dg}(u)v = \mathrm{dg}(v)u \tag{6}$$
$$= \left(\bar{\ell}(\mu)I_{\mathcal{P}} - \Psi(\mu)\,\mathrm{dg}(\mu)\nabla\ell(\mu) - \Psi(\mu)L(\mu)\right)e_p$$

This proves the claim. ■

Now that we have the expression of the Jacobian, we are ready to prove Proposition 3.

Proof: (of Proposition 3): Let μ be a stationary point of Equation (5). Let $\mathcal{P}^*$ be the support of μ and $\mathcal{P}^\diamond = \mathcal{P} \setminus \mathcal{P}^*$. Without loss of generality, we assume that in the vector representation of μ, the support corresponds to the first elements. Finally, for a vector $v \in \mathbb{R}^{\mathcal{P}}$, we write v^* as a shorthand for $(v_p)_{p\in\mathcal{P}^*}$ and $v^\diamond$ as a shorthand for $(v_p)_{p\in\mathcal{P}^\diamond}$. Finally, we write ∇_* and $\nabla_\diamond$ the gradients with respect to μ^* and $\mu^\diamond$, respectively. Then we can calculate the different terms in the expression (9) of the Jacobian. The derivation is given on the top of the next page in equation (10).

Combining these terms, we obtain that $\nabla F(\mu)$ is an upper block-triangular matrix of the form

$$\nabla F(\mu) = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$

with diagonal elements

$$A = (\bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T - \Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu))$$
$$B = \mathrm{dg}(\bar{\ell}(\mu)\mathbf{1}_{\mathcal{P}^\diamond} - \ell^\diamond(\mu))$$

and upper-right elements

$$C = (\mu^*\ell^\diamond(\mu)^T - \Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_\diamond\ell^*(\mu))$$

The stability of μ is determined by the restriction of $\nabla F(\mu)$ to $\mathcal{H}_{\mathcal{P}}$. Let $\alpha \in \mathcal{H}_{\mathcal{P}}$ and write $\alpha = \begin{pmatrix} \alpha^* \\ \alpha^\diamond \end{pmatrix}$. Then $\mathbf{1}^T\alpha^* + \mathbf{1}^T\alpha^\diamond = 0$. We calculate $\nabla F(\mu)\alpha$, and obtain expression (12) on the top of the next page.

Let us denote $\tilde{A} = -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu)$ and $\tilde{C} = -\bar{\ell}(\mu)\mu^*\mathbf{1}^T + C$, then the restriction of $\nabla F(\mu)$ to $\mathcal{H}_{\mathcal{P}}$ coincides with the restriction of $\tilde{\nabla} F(\mu)$ to $\mathcal{H}_{\mathcal{P}}$, where $\tilde{\nabla} F(\mu)$ is obtained by replacing, in equation (12), the block A by $\tilde{A}$ and C by $\tilde{C}$.

The benefit of the latter formulation is that the range of $\tilde{\nabla} F(\mu)$ is a subset of $\mathcal{H}_{\mathcal{P}}$ since $\mathbf{1}^T\tilde{\nabla} F(\mu) = 0$ (see equation (13), where we used the fact that $\mathbf{1}_{\mathcal{P}^*}^T\Psi^*(\mu) = 0$ and that $\mathbf{1}_{\mathcal{P}^*}^T\mu^* = 1$). Therefore, $\mathcal{H}_{\mathcal{P}}$ is an invariant subspace of $\tilde{\nabla} F(\mu)$, and $\mathfrak{S}$ is given by the spectrum of $\tilde{\nabla} F(\mu)$, from which we remove 1 zero (corresponding to the left eigenvector $\mathbf{1}_{\mathcal{P}}$). Next, since $\tilde{\nabla} F(\mu)$ is block-upper triangular, we have

$$Sp(\tilde{\nabla} F(\mu)) = Sp(\tilde{A}) \cup \{\bar{\ell}(\mu) - \ell_p^\diamond(\mu)\}_{p\in\mathcal{P}^\diamond}$$

Therefore,

$$\mathfrak{S} = Sp(\hat{A}) \cup \{\bar{\ell}(\mu) - \ell_p^\diamond(\mu)\}_{p\in\mathcal{P}^\diamond}$$

where $\hat{A}$ is the restriction of $\tilde{A}$ to $\mathcal{H}_{\mathcal{P}^*} = \mathbf{1}_{\mathcal{P}^*}^\perp$.

To conclude the proof, suppose that μ is a stationary point but not a Nash equilibrium, i.e. $\mu \in \mathcal{RP} \setminus \mathcal{N}$. Then there exists $p \in \mathcal{P}^\diamond$ such that $\bar{\ell}(\mu) - \ell_p^\diamond(\mu) > 0$, and it follows that $\mathfrak{S}$ contains at least one positive eigenvalue, therefore μ is unstable (by Theorem 3.7 in [4] for example). ■

B. Stability of Nash equilibria

In order to prove the converse of Proposition 3, we need to study the eigenvalues of $\hat{A}$, the restriction to $\mathcal{H}_{\mathcal{P}^*}$ of $\tilde{A} = -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu)$.

Lemma 2: The matrix $\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)$ is symmetric positive-semidefinite and its restriction to $\mathcal{H}_{\mathcal{P}^*}$ is positive-definite.

Proof: We have $\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*) = \mathrm{dg}(\mu^*) - \mu^*(\mu^*)^T$ is symmetric. We also have $\mathbf{1}_{\mathcal{P}^*}^T\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\mathbf{1}_{\mathcal{P}^*} = 0$, and for all $y \in \mathcal{H}_{\mathcal{P}^*}$

$$y^T\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)y = \sum_{p\in\mathcal{P}^*}\mu_p^* y_p^2 - \left(\sum_{p\in\mathcal{P}^*}\mu_p^* y_p\right)^2$$

which, by Jensen's inequality, is strictly positive except at $y = 0$. ■

Lemma 3: The gradient $\nabla_*\ell^*(\mu)$ is a symmetric positive-semidefinite matrix. Furthermore, if all congestion functions are strictly increasing and if the incidence matrix M is injective, then it is positive-definite.

Proof: We have $\ell^*(\mu) = M_*^T c(M\mu)$, where M_* is the submatrix with columns $p \in \mathcal{P}^*$, the support of μ. Thus

$$\nabla_*\ell^*(\mu) = M_*^T\nabla c(M\mu)M_*$$

where, by definition of c,

$$\nabla c(M\mu) = \mathrm{dg}(\{c_r'((M\mu)_r)\}_{r\in\mathcal{R}}).$$

Thus $\nabla_*\ell^*(\mu)$ is symmetric, and since c_r is non-increasing for all r, it is a positive semi-definite matrix. Furthermore, if all congestion functions are strictly increasing and M is assumed to be injective, then $\nabla^*\ell^*$ is positive definite. ■

Lemma 4: Let R and S be two symmetric matrices such that R is positive-definite and S is positive-semidefinite. Then the product RS is diagonalizable, has non-negative eigenvalues and has the same number of zero eigenvalues as S (with the same eigenvectors).

Proof: Since R is positive definite, there exists a positive definite matrix $\bar{R}$ such that $R = \bar{R}^2$. Then we have

$$\bar{R}^{-1}RS\bar{R} = \bar{R}S\bar{R}$$

thus RS is similar to the symmetric matrix $\bar{R}S\bar{R}$, and is diagonalizable.

$$
\begin{aligned}
\Psi(\mu) &= \begin{pmatrix} I_{\mathcal{P}^*} & 0 \\ 0 & I_{\mathcal{P}^\diamond} \end{pmatrix} - \begin{pmatrix} \mu^* \\ 0 \end{pmatrix} (\mathbf{1}_{\mathcal{P}^*}^T \ \mathbf{1}_{\mathcal{P}^\diamond}^T) = \begin{pmatrix} \Psi^*(\mu^*) & -\mu^* \mathbf{1}_{\mathcal{P}^\diamond}^T \\ 0 & I_{\mathcal{P}^\diamond} \end{pmatrix} \\
\mathrm{dg}(\mu)\nabla\ell(\mu) &= \begin{pmatrix} \mathrm{dg}(\mu^*) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \nabla_*\ell^*(\mu) & \nabla_\diamond\ell^*(\mu) \\ \nabla_*\ell^\diamond(\mu) & \nabla_\diamond\ell^\diamond(\mu) \end{pmatrix} = \begin{pmatrix} \mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) & \mathrm{dg}(\mu^*)\nabla_\diamond\ell^*(\mu) \\ 0 & 0 \end{pmatrix} \\
\Psi(\mu)\,\mathrm{dg}(\mu)\nabla\ell(\mu) &= \begin{pmatrix} \Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) & \Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_\diamond\ell^*(\mu) \\ 0 & 0 \end{pmatrix} \\
\Psi(\mu)L(\mu) &= \begin{pmatrix} \Psi^*(\mu^*) & -\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T \\ 0 & I_{\mathcal{P}^\diamond} \end{pmatrix} \begin{pmatrix} \mathrm{dg}(\ell^*(\mu)) & 0 \\ 0 & \mathrm{dg}(\ell^\diamond(\mu)) \end{pmatrix} \\
&= \begin{pmatrix} \bar{\ell}(\mu)I_{\mathcal{P}^*} - \bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T & -\mu^*\ell^\diamond(\mu)^T \\ 0 & \mathrm{dg}(\ell^\diamond(\mu)) \end{pmatrix}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
\nabla F(\mu) &= \bar{\ell}(\mu)I_{\mathcal{P}} - \Psi(\mu)\,\mathrm{dg}(\mu)\nabla\ell(\mu) - \Psi(\mu)L(\mu) \\
&= \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) + \bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T & -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_\diamond\ell^*(\mu) + \mu^*\ell^\diamond(\mu)^T \\ 0 & \bar{\ell}(\mu)I_{\mathcal{P}^\diamond} - \mathrm{dg}(\ell^\diamond(\mu)) \end{pmatrix}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
\nabla F(\mu) \begin{pmatrix} \alpha^* \\ \alpha^\diamond \end{pmatrix} &= \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) + \bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T & C \\ 0 & B \end{pmatrix} \begin{pmatrix} \alpha^* \\ \alpha^\diamond \end{pmatrix} \\
&= \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu)\alpha^* + \bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T\alpha^* + C\alpha^\diamond \\ B\alpha^\diamond \end{pmatrix} \\
&= \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu)\alpha^* - \bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T\alpha^\diamond + C\alpha^\diamond \\ B\alpha^\diamond \end{pmatrix} \\
&= \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) & -\bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T + C \\ 0 & B \end{pmatrix} \begin{pmatrix} \alpha^* \\ \alpha^\diamond \end{pmatrix}
\end{aligned} \tag{12}
$$

$$
\begin{aligned}
\mathbf{1}_{\mathcal{P}}^T \tilde{\nabla} F(\mu) &= (\mathbf{1}_{\mathcal{P}^*}^T \ \mathbf{1}_{\mathcal{P}^\diamond}^T) \begin{pmatrix} -\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu) & -\bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T - \Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_\diamond\ell^*(\mu) + \mu^*\ell^\diamond(\mu)^T \\ 0 & \bar{\ell}(\mu)I_{\mathcal{P}^\diamond} - \mathrm{dg}(\ell^\diamond(\mu)) \end{pmatrix} \\
&= \begin{pmatrix} 0 & \mathbf{1}_{\mathcal{P}^*}^T[-\bar{\ell}(\mu)\mu^*\mathbf{1}_{\mathcal{P}^\diamond}^T + \mu^*\ell^\diamond(\mu)^T] + \mathbf{1}_{\mathcal{P}^\diamond}^T[\bar{\ell}(\mu)I_{\mathcal{P}^\diamond} - \mathrm{dg}(\ell^\diamond(\mu))] \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\bar{\ell}(\mu)\mathbf{1}_{\mathcal{P}^\diamond}^T + \ell^\diamond(\mu)^T + \bar{\ell}(\mu)\mathbf{1}_{\mathcal{P}^\diamond}^T - \ell^\diamond(\mu)^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \end{pmatrix}
\end{aligned} \tag{13}
$$

Consider the function $h : x \mapsto RSx$ and the inner product $\langle x; y\rangle = x^T R^{-1} y$. We have

$$\langle h(x); y\rangle = x^T SRR^{-1} y = x^T S y$$

Thus if λ is an eigenvalue of h with eigenvector x, then

$$\langle h(x); x\rangle = \lambda \langle x; x\rangle$$

i.e. $\lambda = \frac{x^T S x}{x^T R^{-1} x}$ which is non-negative since $S \succeq 0$ and $R \succ 0$. Furthermore, $\lambda = 0$ if and only if $Sx = 0$, which proves the claim. ■

We can now show that the set of Nash equilibria is, in fact, exactly the set of asymptotically stable equilibria.

Proposition 4: Assume that the congestion functions are strictly increasing and that the incidence matrix M is injective. Then μ is a Nash equilibrium if and only if μ is a locally exponentially stable stationary point of the replicator dynamics (5).

Proof: Proposition 3 provides one direction of the proof: if μ is a locally exponentially stable stationary point, then μ asymptotically stable, and by Proposition 3, μ is a Nash equilibrium. To show the converse, suppose that μ is a Nash equilibrium. Then it is a stationary point of the system (5). To show that it is asymptotically stable, recall that the eigenvalues of the Jacobian are given by:

$$\mathfrak{S} = Sp(\hat{A}) \cup \{\bar{\ell}(\mu) - \ell_p^\diamond(\mu)\}_{p\in\mathcal{P}^\diamond}$$

where $\hat{A}$ is the restriction of $-\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)\nabla_*\ell^*(\mu)$ to $\mathcal{H}_{\mathcal{P}^*}$. By Lemma 2, $\Psi^*(\mu^*)\,\mathrm{dg}(\mu^*)$ has a positive definite restriction to $\mathcal{H}_{\mathcal{P}^*}$, and by Lemma 3, $\nabla\ell^*(\mu)$ is positive definite. Therefore applying Lemma 4, we have that all eigenvalues of $\hat{A}$ are real negative. Therefore μ is asymptotically stable (for example by Theorem 4.7 of [4]). ■

If the congestion functions are not strictly increasing or the incidence matrix is not injective, then the gradient $\nabla_*\ell^*(\mu) = M_*^T \nabla c(M\mu) M_*$ may have zero eigenvalues. By Lemma 4, we have that $\hat{A}$ is diagonalizable, has 0 as an eigenvalue with the same multiplicity as $\nabla_*\ell^*(\mu)$, but one may not conclude stability of μ in general.

Note that the incidence matrix may not be injective in general, since $M \in \{0,1\}^{\mathcal{R}\times\mathcal{P}}$, and $|\mathcal{P}| = 2^{|\mathcal{R}|}$ in the worst case. The expression of the spectrum suggests that if we could find a more concise representation of the game, by reducing the number of bundles, the reduced game may converge faster. This is discussed in the next section.

C. Reducing the size of the game

We observe that if a bundle p_0 is a conic combination of other bundles, then the congestion game without p_0 is equivalent, in a sense, to the original game, allowing us to reduce the size of the bundle set $\mathcal{P}$.

More precisely, consider a given bundle $p_0 \in \mathcal{P}$, and let $\bar{\mathcal{P}} = \mathcal{P} \setminus \{p_0\}$. Assume p_0 is a conic combination of bundles in $\bar{\mathcal{P}}$, that is, $M_{p_0} = \sum_{p \in \bar{\mathcal{P}}} \lambda_p M_p$ for some non-negative coefficients λ_p. First, we must have $\sum_{p \in \bar{\mathcal{P}}} \lambda_p \geq 1$: since p_0 is non-empty (by assumption, no bundle is empty), there exists r such that $M_{rp_0} = 1$. But

$$M_{rp_0} = \sum_{p \in \bar{\mathcal{P}}} \lambda_p M_{rp} \leq \sum_{p \in \bar{\mathcal{P}}} \lambda_p$$

since $M_{r,p} \in \{0, 1\}$, which proves the claim.

Proposition 5: If $\nu \in \Delta^{\bar{\mathcal{P}}}$ is a Nash equilibrium for the game without p_0, then ν (augmented with a 0 on p_0) is also a Nash equilibrium of the original game.

Proof: We have for all $p \in \text{support}(\nu)$, the loss $\ell_p(\nu)$ is equal to $\bar{\ell}(\nu)$ the minimum loss across all bundles in $\bar{\mathcal{P}}$. Then if c_ν is the vector of resource congestions, the loss of bundle p_0 under distribution ν is

$$\ell_{p_0}(\nu) = M_{p_0}^T c_\nu = \left(\sum_{p \in \bar{\mathcal{P}}} \lambda_p M_p \right)^T c_\nu = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \ell_p(\nu)$$
$$\geq \left(\sum_{p \in \bar{\mathcal{P}}} \lambda_p \right) \bar{\ell}(\nu) \geq \bar{\ell}(\nu)$$

Thus ν augmented by 0 on p_0 is an equilibrium of the original game. ■

Proposition 6: If $\mu \in \Delta^{\mathcal{P}}$ is a Nash equilibrium for the original game, then $\nu \in \Delta^{\bar{\mathcal{P}}}$ defined by

$$\nu_p = \mu_p + \frac{\lambda_p}{\sum_{q \in \bar{\mathcal{P}}}} \mu_{p_0}$$

is a Nash equilibrium for the game without p_0.

Proof: First, ν is, by definition, a distribution over $\bar{\mathcal{P}}$. To show that it is a Nash equilibrium of the reduced game, we argue that ν and μ induce the same resource loads, that is, $M\mu = \bar{M}\nu$. To show this, we observe that if $p_0 \in \text{support}(\mu)$, we must have $\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1$. Indeed, if $\mu_{p_0} > 0$, then by definition of a Nash equilibrium, $\ell_{p_0}(\mu) \leq \ell_p(\mu)$ for all p. But

$$\ell_{p_0}(\mu) = \sum_{p \in \bar{\mathcal{P}}} \lambda_p \ell_p(\mu) \geq \sum_{p \in \bar{\mathcal{P}}} \lambda_p \ell_{p_0}(\mu)$$

therefore $\sum_{p \in \bar{\mathcal{P}}} \lambda_p \leq 1$, which, combined with the previous observation that $\sum_{p \in \bar{\mathcal{P}}} \lambda_p \geq 1$, proves the claim.

Now we consider two cases: if $\mu_{p_0} = 0$, then we have immediately $\bar{M}\nu = M\mu$. If $\mu_{p_0} > 0$, then we have

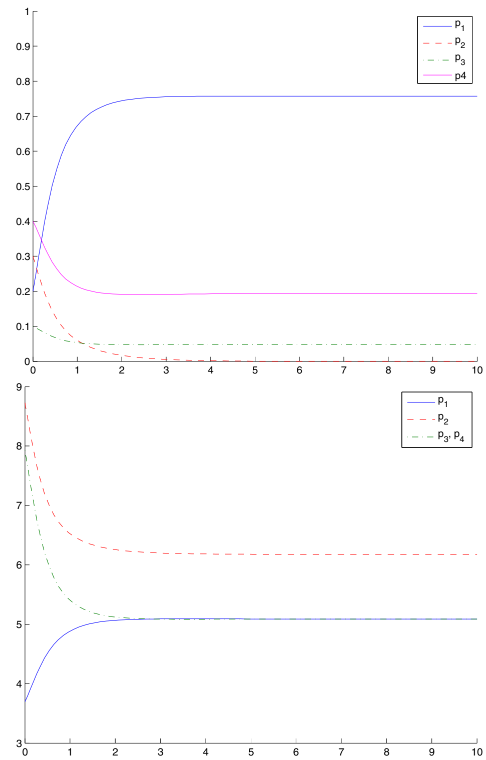

Fig. 1. Trajectory of $\mu(t)$ (top) and evolution of loss functions (bottom). We have convergence to the set of Nash equilibria: on the support of the limit distribution, all bundle losses are equal.

$\sum_{p \in \bar{\mathcal{P}}} \lambda_p = 1$ and

$$\bar{M}\nu = \sum_{p \in \bar{\mathcal{P}}} \nu_p M_p = \sum_{p \in \bar{\mathcal{P}}} \mu_p M_p + \mu_{p_0} \left(\sum_{p \in \bar{\mathcal{P}}} \lambda_p M_p \right)$$
$$\geq \sum_{p \in \bar{\mathcal{P}}} \mu_p M_p + \mu_{p_0} M_{p_0} = M\mu$$

Therefore the distribution ν induces the same resource loads as μ, hence the same bundle losses, and ν is a Nash equilibrium of the reduced game. ■

With the previous propositions, one can reduce the size of the game by removing p_0 from the set of bundles, and obtain an equivalent game. Applying this argument repeatedly, we can reduce $\mathcal{P}$ to a minimal set $\hat{\mathcal{P}}$. One way to compute such a minimal set is to find a Hilbert basis of the family $\{M_p\}_{p \in \mathcal{P}}$, $H = \{M_p\}_{p \in \bar{\mathcal{P}}}$, and use $\bar{\mathcal{P}}$ as the reduced set of bundles.

V. NUMERICAL EXAMPLE

Suppose we have 3 resources $\mathcal{R} = \{r_1, r_2, r_3\}$, with quadratic congestion functions

$$c_1(\phi_1) = \frac{1}{2}(\phi_1 + 1)^2$$
$$c_2(\phi_2) = (1 + \phi_2)^2$$
$$c_3(\phi_3) = 2(1 + \phi_3)^2$$

and consider the following bundles

$$p_1 = \{r_1, r_2\}, \; p_2 = \{r_2, r_3\}, \; p_3 = \{r_3, r_1\}, \; p_4 = \{r_3, r_1\}$$

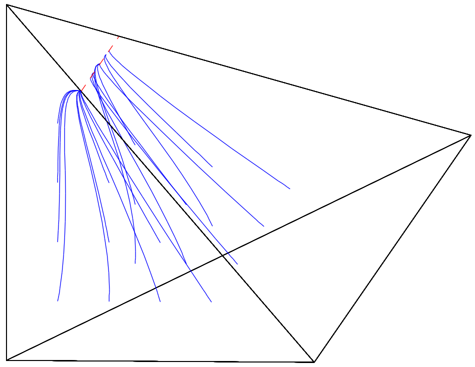

Fig. 2. Trajectory of $\mu(t)$ in the simplex $\Delta^{\mathcal{P}}$ (represented as a tetrahedron). Starting from different initial conditions, we have convergence to different points in $\mathcal{N}$ (represented with a red dashed line).

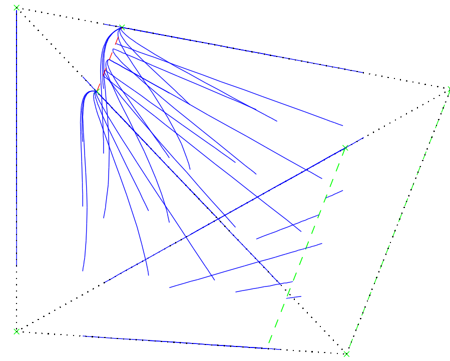

Fig. 3. Stationary points in the simplex. Nash equilibria are represented using a red dashed line, and other stationary points are in dashed green lines. Solid lines show sample trajectories. In particular, the entire facet $\{\mu : \mu_1 = \mu_2 = 0\}$ is stationary.

In particular, we have $p_4 = p_3$. This is a degenerate case to illustrate some of the phenomena. In particular, we do not have uniqueness of the Nash equilibrium in this case. The set of Nash equilibria is given by

$$\mathcal{N} = \{\mu : \mu_1 = .757, \mu_2 = 0, \mu_3 + \mu_4 = .2426\}$$

If we apply the replicator dynamics with $\rho = 1$ from the initial condition $\mu_0 = \begin{pmatrix} .2 & .3 & .1 & .4 \end{pmatrix}^T$, we obtain the trajectories shown in Figure 1.

Starting from different initial conditions in the interior of the simplex $\mathring{\Delta}$, we have convergence to different points in the set of Nash equilibria $\mathcal{N}$. This is illustrated in Figure 2.

If we start on the boundary of the simplex, we may have convergence to stationary points which are not Nash equilibria. Any facet of the simplex is invariant for the dynamics (and so is any intersection of facets). Therefore a stationary point which is unstable in the entire simplex may be stable if we restrict the dynamics to an invariant facet. This is illustrated in Figure 3.

VI. Conclusion

We studied the Jacobian of the vector field of the replicator dynamics applied to the single-commodity congestion game. We showed that the set of Nash equilibria is exactly the set of asymptotically stable stationary rest points of the dynamics. We also showed that if the congestion functions are strictly increasing and the incidence matrix is injective, then Nash equilibria are exponentially stable. These results were derived for the single-commodity congestion game, and a natural question is whether one can generalize to the multi-commodity case. Lyapunov arguments have been used to show that the trajectories will converge to the set of rest points (restricted Nash equilibria), however, explicitly studying the eigenvalues of the Jacobian seems challenging due to the coupling in the dynamics between populations.

References

[1] Martin J Beckmann, Charles B McGuire, and Christopher B Winsten. Studies in the economics of transportation. 1955.

[2] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*, volume 25. Cambridge University Press, 2010.

[3] Simon Fischer and Berthold Vöcking. On the evolution of selfish routing. In *Algorithms–ESA 2004*, pages 323–334. Springer, 2004.

[4] H.K. Khalil. *Nonlinear systems*. Macmillan Pub. Co., 1992.

[5] Noam Nisan, Tim Roughgarden, Eva Tardos, and Vijay V. Vazirani. *Algorithmic Game Theory*. Cambridge University Press, New York, NY, USA, 2007.

[6] Asuman Ozdaglar and R Srikant. Incentives and pricing in communication networks. *Algorithmic Game Theory*, pages 571–591, 2007.

[7] Robert W Rosenthal. A class of games possessing pure-strategy nash equilibria. *International Journal of Game Theory*, 2(1):65–67, 1973.

[8] Tim Roughgarden. Stackelberg scheduling strategies. *SIAM Journal on Computing*, 33(2):332–350, 2004.

[9] Tim Roughgarden and Éva Tardos. How bad is selfish routing? *Journal of the ACM (JACM)*, 49(2):236–259, 2002.

[10] William H Sandholm. Potential games with continuous player sets. *Journal of Economic Theory*, 97(1):81–108, 2001.

[11] John Glen Wardrop. Some theoretical aspects of road traffic research. In *ICE Proceedings: Engineering Divisions*, volume 1, pages 325–362. Thomas Telford, 1952.

[12] Jörgen W Weibull. *Evolutionary game theory*. MIT press, 1997.