Solving the user equilibrium departure time problem at an off-ramp with incentive compatible cost functions

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Abstract— We consider the equilibrium departure time problem for a set of vehicles that travel through a network with capacity restrictions and need to reach a destination at a fixed time. The vehicles incur a penalty for both any queuing delays and arriving at the destination early or late. In particular, we consider the case of a congested off-ramp, which is a common occurrence next to commercial hubs during the morning commute, and has the added negative effect of reducing the capacity on the freeway for through traffic. We study the use of incentives and tolls to manipulate the equilibrium departure times of the exiting vehicles and thereby mitigate the impact on through traffic. Our main result is to show the existence and uniqueness properties of the departure time equilibrium for a general class of delay and arrival time cost functions, which allows for discontinuities in the arrival cost function. This enables the use of step incentives or tolls, which are the mostly common strategies used in practice. Our results also apply to the Vickrey single bottleneck equilibrium, which is a special case of our network.

I. INTRODUCTION

We consider the equilibrium departure time problem when a set of vehicles has to travel through a network with capacity restrictions and reach the destination at a fixed time. The vehicles incur a penalty due to both queuing delays at the bottleneck and not arriving at the destination on time (being early or late). The case of a single bottleneck with homogeneous flow was studied by Vickrey [1] in his seminal paper from 1969. The Vickrey model is elegant and simple, and has been widely adopted in many settings from equilibrium analysis to time-dependent toll pricing. The key assumptions of the model are that: (1) travelers have identical and piecewise linear cost functions; (2) there is only a single bottleneck and a single route; (3) the adoption of user equilibrium assumes perfect information, rationality, and perfect decision; (4) the queue is vertical and spatial extent of the queue is not considered. Hendrikson and Kocur [2] showed the existence and uniqueness of the equilibrium with linear cost functions and subsequent studies have gradually relaxed the assumptions of the Vickrey model. For example, Smith [3] and Daganzo [4] proved the existence and uniqueness of equilibrium with convex cost functions, Newell [5] and Lindsey [6] relaxed the identical cost functions assump-

tion and introduced groups of commuters, and a numerical solution method was given by Zijpp and Koolstra [7]. Furthermore, the assumption of a single bottleneck and a single route was relaxed by Kuwahara [8] who studied the case of two tandem bottlenecks, and Arnott et al. [9] who studied the case of parallel routes with bottlenecks. Mahmassani and Chang [10], [11] studied the day to day variations of the equilibrium to relax assumption (3). Finally, Mahmassani and Herman [12] (with some corrections by Newell [13]) relaxed assumption (4) by studying the horizontal queuing both upstream and downstream of the bottleneck.

In our analysis, we generalize the single bottleneck single route assumption by considering a freeway network with an off-ramp exit[1]. A bottlenecked off-ramp is a common occurrence during the morning commute at commuter heavy exits close to large corporations, schools, industrial parks etc. An off-ramp bottleneck causes congestion to spill back onto the freeway, and thereby reduces the available capacity for vehicles that are continuing on the freeway, which can lead to undesirable side effect of additional delays for these vehicles. Solving for the equilibrium departure times in this network requires an explicit relationship between the flows and the delays in the network. In a parallel research effort [14], we present such an explicit relationship for single source multiple destination networks with point queue dynamics.

There have been limited efforts to study the equilibrium behavior in the case of junctions with bottlenecks such as the case of the congested off-ramp. Lago and Daganzo [15] considers the Vickrey equilibrium at a merge between two freeways, and Yperman et al. [16] consider a freeway diverge (although the source of congestion is not at the freeway diverge). Therefore, to the best of our knowledge, this is the first analysis of the departure time equilibrium for a congested network diverge.

As mentioned previously, the congested off-ramp can cause a capacity drop on the freeway and lead to additional delays for the non-exiting vehicles. This problem can be mitigated either by building additional capacity at the off-ramp to accommodate the peak flow or by altering the demand of vehicles that are exiting at the off-ramp during the peak period. Adding new capacity at a freeway off-ramp is extremely disruptive in the short term and incurs a large monetary cost. Furthermore, the appropriate capacity requirement needs to be known in advance and cannot be modified easily. Therefore, we study the use of incentives

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[1]This problem reduces to the single source and single route problem when all the flow exits the freeway at the off-ramp

and tolls to manipulate the equilibrium departure times of the exiting vehicles to mitigate their negative impact on the non-exiting vehicles. In a practical setting, the incentives and tolls that are applied will not be continuous functions and are most likely to be piece-wise constant, therefore the cost functions of the exiting vehicles can no longer be assumed to be convex. Our main result is to show the existence and uniqueness properties of the departure time equilibrium for a general class of delay and arrival time cost functions that allow for such discontinuities in the arrival cost function. Our results also apply to the standard Vickrey equilibrium, since it is a special case of our network, and therefore extends the existence and uniqueness results for the Vickrey equilibrium to discontinuous cost functions. Finally, we analyze a number of incentive and tolling strategies that can be used by a transportation planning authority to achieve different objectives.

The contributions of this article are as follows: 1) a framework for analyzing the equilibrium behavior of vehicles that exit a freeway at a congested off-ramp, 2) proof of existence and uniqueness of the equilibrium for a set of general cost functions that allow for discontinuities in the arrival time function and 3) equilibrium analysis under different incentive/tolling mechanisms and an algorithm for identifying system optimal tolls under different objectives.

II. NETWORK AND DEMAND MODEL

A. Network

We consider a freeway segment with an off-ramp, where the number of vehicles that exit at the off-ramp exceeds its capacity during a peak congestion period. The dynamics of the network are modeled using the point queue model described in [14] with a first-in-first-out (FIFO) junction model. The network contains two types of vehicles; a) vehicles that drive past the off-ramp and stay on the freeway that are called freeway vehicles (denoted with the subscript h) and b) vehicles that exit the freeway at the off-ramp are called exiting vehicles (denoted with the subscript e). The flow of each of these vehicles types is constrained by the capacity limitations of the network, and a queuing delay occurs when the inflow is greater than the bottleneck capacity.

Definition 1. *Flow*
The flow of freeway vehicles (and resp. exiting vehicles) entering the network at time is t is $\lambda_h(t)$ (and resp. $\lambda_e(t)$). The flow of freeway vehicles (and resp. exiting vehicles) that exit the network at time t is $\lambda_h^{out}(t)$ (and resp. $\lambda_e^{out}(t)$).

Definition 2. *Capacity*
The capacity $\mu_b(t)$ of a bottleneck b is the maximum flow that can enter the link from its input node at time t.

Definition 3. *Delay*
The queuing delay $\delta_q(t)$ at queue q at time t is the waiting time at queue q due to the capacity constraints of the outgoing links from the queue. The total delay $\Delta_g(t)$ for a vehicles of type g entering the network at time t is the

total delay that the vehicle experiences across all queues prior to exiting the network[2].
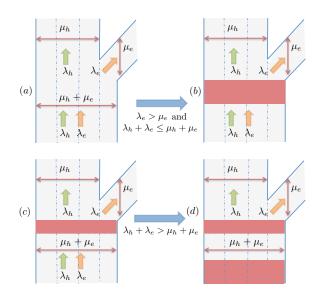

Fig. 1. Network model. There are two types of queues that can form; i) (a) → (b): a queue forms at the off-ramp entrance if the exiting flow is greater than the off-ramp capacity, ii) (c) → (d) a queue forms at the entrance to the network if the total demand is greater than the capacity freeway capacity at the entrance.

For analyzing the equilibrium departure flows, we need to quantify the delay characteristics of the off-ramp model under different boundary flows. Parmentier et al. [14] formally derive the delay characteristics of single source networks that are flow constrained[3], satisfy the first-in-first-out (FIFO) property, and where the flow through each junction is maximized subject to the FIFO constraint, which we use as the delay model in our analysis. Using this model, we can analytically express the delays observed by the exiting vehicles as they travel through the network.

Proposition 1. *Exiting vehicle delay*
If freeway flow is restricted to $\lambda_h \leq \mu_h$, i.e. there is no bottleneck purely due to the freeway vehicles, the delay seen by the exiting vehicles that enter the network at time t is given by the following differential equation:

$$\left.\frac{d\Delta_e}{dt}\right|_t = \begin{cases} \dfrac{\lambda_e(t)}{\mu_e} - 1 & \text{if there is an active queue} \\ 0 & \text{otherwise, i.e. no queing} \end{cases} \quad (1)$$

where queue q being active implies that $\Delta_q(t) > 0$ when the exiting vehicle that enters the network at time t reaches queue q. See [17] for proof.

B. Demand model

Assumption 1. *Exiting vehicle demand*
We assume that the exiting vehicles are free to choose their departure times and do so in a selfish manner to minimize a cost function C. Therefore, the demand for the exiting vehicles $\lambda_e(t)$ will form a Nash equilibrium (or user equilibrium) with respect to the cost function C.

Assumption 2. *freeway vehicle demand*
We assume that the freeway vehicle demand $\lambda_h(t)$ such that $\lambda_h(t) \leq \mu_h(t)$ is fixed (exogenous) and not a function of the exiting vehicle demand distribution.

The cost function C consists of a cost related to the queuing delay on the network and a cost related to the arrival time at the destination.

Definition 4. *Delay cost function*
The delay cost function C_δ assigns a cost $C_\delta(\Delta(t))$ corresponding to a queuing delay of $\Delta(t)$. The delay cost encountered by an exiting vehicle that enters the network at time t is given by $C_\delta(\Delta_e(t))$.

Definition 5. *Schedule time cost function*
The exiting vehicles have an expected arrival time at the destination and the schedule time cost function C_S assigns a penalty $C_S(t_a)$ corresponding to the actual arrival time t_a. The schedule time cost encountered by the an exiting vehicle that enter the network at time t is given by $C_S(t + \Delta_e(t))$.

The goal is to analyze the impact of incentive and tolling strategies on the departure time equilibrium of the exiting vehicles and the resulting impact on overall congestion, so we also define an incentive/toll cost function with a toll being modeled as a negative incentive.

Definition 6. *Incentive/toll cost*
The incentive/toll cost function C_I assigns a cost $C_I(t_a)$ corresponding to the incentive/toll for arriving at the destination at time t_a.
If $C_I \leq 0$, $|C_I(t_a)|$ represents the incentive or negative toll given to the vehicles that exit the network at time t_a.
If $C_I > 0$, $C_I(t_a)$ represents the toll or negative incentive charged to the vehicles that exit the network at time t_a.

Definition 7. *Arrival cost*
The arrival cost C_A is the total cost experienced by a vehicle due to its arrival time.

$$C_A(t_a) = C_S(t_a) + C_I(t_a) \tag{2}$$

The delay cost is a function of the queuing delay, while both the schedule time and incentive/toll costs are functions of the arrival time. The total cost can now be defined as follows.

Definition 8. *Total cost C*
The total cost $C(t)$ is the sum of the delay cost and the arrival cost for a vehicles that enters the network at time t.

$$C(t) = C_\delta(\Delta_e(t)) + C_A(t + \Delta_e(t)) \tag{3}$$

We will now model the behavior of the existing vehicles with respect to the network model and the cost functions.

Definition 9. *Exiting vehicle equilibrium*
Given a network with an exit and a fixed number of exiting vehicles N, cost functions (C_δ, C_A, C_I) and freeway vehicle demand $\lambda_h(t)$, λ_e is an exiting vehicle equilibrium if and only if

$$\begin{cases} \lambda_e(t) & \geq & 0 \text{ is piecewise continuous} \\ \int_{\mathbb{R}} \lambda_e(\tau)\, d\tau & = & N \\ \lambda_e(t) > 0 & \Rightarrow & C(t) \leq C(t'), \ \forall t' \end{cases} \tag{4}$$

where $C(t)$ is the total cost and $\Delta_e(t)$ is the total delay in the network given $\lambda_e(\cdot)$ and $\lambda_h(\cdot)$. The equilibrium cost for each vehicle is denoted by C_E.

III. EXISTENCE AND UNIQUENESS OF THE EXITING VEHICLE EQUILIBRIUM

In this section, we will prove the existence and uniqueness of the exiting vehicle equilibrium for a general class of cost functions within the dynamics of our network model. We first introduce the general class of cost functions that we consider.

A. Equilibrium compatible cost functions

The classical single route single bottleneck equilibrium departure time problem was first introduced by Vickrey [1] in 1969. Smith [3] proved the existence of an equilibrium for convex arrival cost functions, and Daganzo [4] proved the uniqueness of this solution. Convex cost functions imply that the marginal cost of earliness (or lateness) increases as commuters arrive earlier (or later), which is a reasonable assumption. However, convex cost functions by themselves are not adequate in our setting. To design time dependent incentives and tolls, we require the ability to use schedule cost functions $C_A = C_S + C_I$. This introduces a more complex set of cost functions we must be able to accommodate. The following generalizations are required:

1) Local maximums: An incentive/toll is intended for pushing commuters out of the peak congestion period. Thus, it is reasonable to envision incentives/tolls that are proportional to the peak congestion pattern and therefore inversely proportional to the schedule cost function C_S. This could result in an arrival time function C_A that admits local maximums, which we must be able to support.

2) Discontinuity: A fixed incentive/toll to encourage commuters to arrive before some time t_I could take the form

$$C_I(t) = \begin{cases} I < 0 & \text{if } t < t_I \\ 0 & \text{if } t \geq t_I \end{cases} \tag{5}$$

Thus a discontinuity $C_A(t_I^+) - C_A(t_I^-) = I$ will appear in the arrival cost function $C_A(t)$ at $t = t_I$, where I is the value of the incentive/toll.

Definition 10. *Equilibrium compatible cost functions*
The functions (C_A, C_δ) are equilibrium compatible cost functions if they satisfy the following requirements.

1) C_δ *is convex on $\mathbb{R}^+$, C^1 and admits a unique minimum at 0.*
2) C_A *is C^1 on the right and piecewise C^1, with a finite number of positive discontinuities such that $C_A(t^+) \geq$*
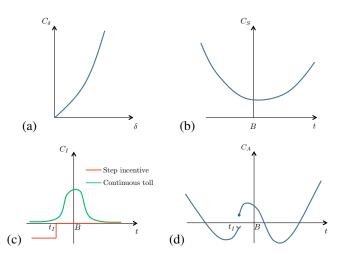
Fig. 2. Illustration of equilibrium compatible cost functions - (a) Equilibrium compatible delay cost functions C_δ are convex on $\mathbb{R}^+$, C^1 and admit a unique minimum at 0. (b) A classical convex schedule cost function C_S (c) A continuous toll that induces a local minimum and a step incentive/toll that induces a discontinuity in the arrival cost function d) The resulting arrival cost function C_A.

C_E and no negative discontinuities in the support of the solution.

3) C'_A has a finite number of sign changes $\Leftrightarrow C_A$ has a finite number of local maximums.
4) $\lim_{t \to \pm\infty} C_A(t) = +\infty$
5) $\exists t_0 : -\frac{dC_A(t)}{dt} < \frac{dC_\delta(0)}{d\delta}, \; \forall t > t_0$

The first condition ensures that C_δ penalizes queuing delay and that the marginal cost of delay is monotonically increasing. The second condition allows for a finite number of positive discontinuities in C_A due to step incentives or tolls. The restriction on a finite number discontinuities is not a practical limitation, since there will be finite number of incentives/tolls implemented in practice. The third and the fourth conditions replace the convexity assumption of the arrival cost function with something more general that allows for a finite number of local maximums in C_A, as long as very large early (or late) arrivals still result in large penalties. The last assumption ensures that marginal cost of delay is greater than the marginal cost of the arrival time. Figure 2 illustrates the different cost functions that our extended framework can accommodate.

B. Fixed cost equilibrium

Solving for an exiting vehicle equilibrium directly is difficult due to the flow conservation constraint $\int_{\mathbb{R}} \lambda_e(\tau) \, d\tau = N$. Therefore, we will first consider the simpler problem of finding the equilibrium for a fixed cost C_E, where the total number of exiting vehicles $\int_{\mathbb{R}} \lambda_e(\tau) \, d\tau > 0$ is not fixed and is a function of the cost C_E.

Definition 11. *Fixed cost equilibrium*
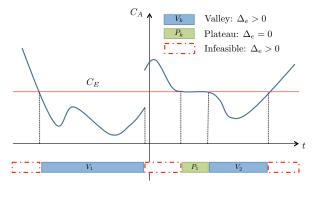*The fixed cost equilibrium $E(C_E)$ for a given cost C_E is*

Fig. 3. Set of windows and plateaus

given by $\lambda_e(t)$ that satisfies the following equations.

$$\begin{cases} \lambda_e(t) & \geq & 0 & \text{is piecewise continuous} \\ \lambda_e(t) > 0 & \Rightarrow & C(t) = C_E \\ \lambda_e(t) = 0 & \Rightarrow & C(t) \geq C_E \end{cases} \tag{6}$$

Proposition 2. *Exiting vehicle equilibrium for a fixed cost*
If $\lambda_e(t)$ is the solution to the fixed cost equilibrium $E(C_E)$, then $\lambda_e(t)$ is also an exiting vehicle equilibrium for $N = \int \lambda_e(\tau) \, d\tau$ exiting vehicles. See [17] for proof.

Definition 12. *Plateau*
A plateau P is an interval $[t_a, t_b)$ such that $C_A(t) = C_E, \forall t \in P$ and $|P| > 0$. The arrival time cost for all vehicles that arrive at the destination during this interval is C_E.

Definition 13. *Valley*
A valley V is an interval $[t_a, t_b)$ such that $C_A(t_a) = C_E$ and $C_A(t) < C_E, \; \forall t \in (t_a, t_b)$. The arrival time cost for a vehicle that arrives at the destination at time t_a is C_E and the cost is strictly less than C_E for all $t \in (t_a, t_b)$.

Figure 3 gives a graphical illustration of how an arrival time function is split into valleys and plateaus.

Proposition 3. *All valleys and plateaus are dominant*
A plateau or valley $[t_a, t_b)$ is dominant if $\Delta(t_a) = 0$. A vehicle that arrives at the beginning of a dominant plateau or valley has zero queuing delay. A plateau or valley that is not dominant is called dominated plateau or valley. See proposition 4 in [17] for proof.

Proposition 4. *Window of feasible arrival times for a fixed cost equilibrium*
The window of feasible arrival times F (i.e. times at which the exiting vehicles leave the network) for a fixed cost equilibrium with cost C_E is the union of a finite number of plateaus and valleys.

$$F = \{\cup_{i=1}^{n_P} P_i\} \cup \{\cup_{j=1}^{n_V} V_j\} \tag{7}$$

See proposition 5 in [17] for proof.

Proposition 5. *Vehicles only enter the network inside the window of feasible arrival times*
If $\lambda_e(t)$ is the solution to the fixed cost equilibrium $E(C_E)$),

the support of $\lambda_e(t)$ (i.e. times at which the exiting vehicles enter the network) is limited to the window of feasible arrival times, i.e. $t \notin F \Rightarrow \lambda_e(t) = 0$ See proposition 6 in [17] for proof.

From definition 11 for a fixed cost equilibrium, we know that $C(t)$ is constant for all t such that $\lambda_e(t) > 0$.

If $\frac{dC(t)}{dt} = 0$ and $C(t) = C_E$,

$$\frac{dC_\delta(t)}{dt}\bigg|_{\Delta_e(t)} \cdot \frac{d\Delta_e(t))}{dt} + \frac{dC_A(t)}{dt}\bigg|_{t+\Delta_e(t)} \cdot \left[1 + \frac{d\Delta_e(t))}{dt}\right] = 0$$

$$\frac{d\Delta_e(t)}{dt} = \frac{-\dfrac{dC_A(t)}{dt}\bigg|_{t+\Delta_e(t)}}{\dfrac{dC_\delta(t)}{dt}\bigg|_{\Delta_e(t)} + \dfrac{dC_A(t)}{dt}\bigg|_{t+\Delta_e(t)}} \tag{8}$$

Lemma 1. *Existence and uniqueness of the solution of fixed cost equilibrium on a valley*
Let $J = [t_a, t_b]$ be a valley. Given the boundary condition $\Delta_e(t_a)$ on the left of the valley J, equations (6) have a unique continuous solution on a dominant valley. A solution exists but is not unique for a dominated valley. See [17] for proof.

Proposition 6. *Feasible flow in plateaus*
On a plateau $P = (t_a, t_b]$, any flow $\lambda_e(t) \in [0, \mu_e]$ is a feasible flow. See proposition 7 in [17] for proof.

Proposition 7. *Unique flow in a valley*
$N(V) = \mu_e |V|$ is the unique number of vehicles that pass the exit during a valley $V = (t_a, t_b]$. See proposition 8 in [17] for proof.

Theorem 1. *Existence and uniqueness of fixed cost equilibrium*
If λ_h is piece-wise continuous, C_A and C_δ are equilibrium compatible cost functions, the fixed cost employee equilibrium $E(C_E)$ exists and the solution is unique if C_A does not contain any plateaus. See [17] for proof.

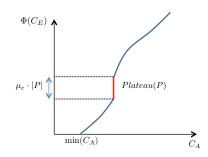*C. Existence and uniqueness of exiting vehicle equilibrium*

Fig. 4. Number of employees which can arrive as a continuous correspondence of fixed cost C_E

Lemma 2. *Number of exiting vehicles as a function of equilibrium cost*
If C_A does not contain any plateaus for $C_E \in$

(C_{min}, C_{max}), then $C_E \mapsto \Phi(C_E)$ is a continuous function for $C_E \in (C_{min}, C_{max})$. If C_A does contain a plateaus for $C_E \in (C_{min}, C_{max})$, then $C_E \mapsto \Phi(C_E)$ is a continuous correspondence for $C_E \in (C_{min}, C_{max})$. This property is illustrated in figure 4. See [17] for proof.*

Theorem 2. *Existence and uniqueness of exiting vehicle equilibrium*
For any total demand of exiting vehicles N, an exiting vehicle equilibrium $\lambda_e(t)$ that satisfies definition 10 exists. The equilibrium is unique if there are no plateaus at the equilibrium cost C_E. See [17] for proof.

Corollary 1. *Solution to the exiting vehicle equilibrium*
The solution to the exiting vehicle equilibrium with N vehicles can be found as follows.

1) *Find the equilibrium cost C_E, which is the minimum cost C such that the length of the support of arrival cost function $\{C_A : C_A \leq C_E\}$ is greater than or equal to $\frac{N}{\mu_e}$, i.e. $C_E = \min\{C : |C(t) \leq C| \cdot \mu_e \geq N\}$. The condition holds with equality if there are no plateaus at $C = \Phi^{-1}(N)$.*
2) *For each valley in $V \in V(C_E)$, solve equation (8) with the initial condition $\Delta_e(t_a) = 0$ and plug it into equation (1) to obtain the solution on V.*

The equilibrium departure flows are solved in practice (approximately) by numerically integrating equation (8). Algorithm 1 shows how this numerical integration can be done for each valley V.

Algorithm 1 Calculate λ_e

Require: $\{t_a, t_b\}$ $\forall V \in V(C_E)$ and unit time discretization[4]
optFlow$(\mathbf{t_a}, \mathbf{t_b})$
  for $V \in V(C_E)$ **do**
    $\Delta_e[t_a(V)] = 0$
    for $t = t_a(V)$ to $t_b(V)$ **do**
$$d\Delta_e[t] = \frac{-\dfrac{dC_A(t)}{dt}\bigg|_{t+\Delta(t)}}{\dfrac{dC_\delta(t)}{dt}\bigg|_{\Delta(t)} + \dfrac{dC_A(t)}{dt}\bigg|_{t+\Delta(t)}} \cdot \Delta t$$
$$\Delta_e[t+1] = \Delta_e[t] + d\Delta_e[t]$$
$$\lambda_e[t] = \mu_e \cdot \left(1 + \frac{d\Delta_e[t]}{\Delta t}\right)$$
    end for
  end for
  return λ_e

IV. ANALYSIS OF INCENTIVE/TOLLING FUNCTIONS

We will now analyze different incentive/tolling functions that reduce the freeway congestion caused by the bottleneck at the off-ramp with respect to congestion reduction, cost efficiency and robustness of the solution.

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[4]The problem can be normalized to achieve a unit time discretization without any loss of generality.

A. Zero-congestion incentives/tolls

Proposition 8. *Computing the freeway optimal incentive*
The freeway optimal incentive/toll is the incentive/toll required to eliminate congestion on the freeway due to the exiting vehicles during the exiting vehicle equilibrium. Let C_E be the equilibrium cost without any incentives for N exiting vehicles and $\mathrm{supp}(\lambda)$ be the support of the equilibrium flow of exiting vehicles. The freeway optimal incentive is:

$$C_I(t) = \begin{cases} \min(C_S) - C_S & \text{if } t \in \mathrm{supp}(\lambda) \\ 0 & \text{if } t \notin \mathrm{supp}(\lambda) \end{cases} \quad (9)$$

The equilibrium cost will be $\min(C_S)$. See proposition 9 in [17] for proof.

For each feasible incentive C_I, there is a corresponding toll C_I^T such that both the incentive and tolling function lead to the same equilibrium flow distribution.

Definition 14. *Complementary toll*
Given a bounded incentive C_I, the complementary toll for this incentive C_I^T is $C_I^T(t) = -\min(C_I(t)) + C_I(t) \geq 0$.

Proposition 9. *Incentives and tolls*
For a fixed schedule cost function C_S and delay function C_δ, both the incentive function C_I and the tolling function C_I^T lead to the same equilibrium. The exiting vehicle equilibrium only depends on the shape of the arrival cost function $C_A(t)$, i.e. the relative cost, so adding a constant $-\min(C_I)$ will not alter the equilibrium.

Note that the equilibrium cost for the exiting vehicles and who bares the cost of moving the equilibrium is different in the two cases. In the case of an incentive, the controlling agency will bear the entire cost of the demand shift, while in the case of a toll, the exiting vehicles will bear the entire cost of the demand shift.

Corollary 2. *Any equilibrium that is achieved via a incentive or toll can also be achieved via a combination of incentives and tolls. Figure 5 illustrates a incentive/toll combination that achieves an freeway optimal flow allocation for a simple schedule cost function.*

This allows the controlling agency to distribute the cost of the demand shift in an equitable manner. For example, the distribution can be such that the tolls charged to the exiting vehicles is equal to the incentive, which means that the controlling agency has no net gains or losses (i.e. the control strategy is revenue neutral).
A freeway optimal incentive/toll eliminates freeway congestion. However, a continuous time varying incentive/toll is extremely difficult to implement in practice and the precise arrival and delay cost function are not known. Therefore, it is unlikely that such an incentive/toll will be used in practice. However, these optimal strategies serve as a useful reference for implementing the simpler piecewise constant incentives/tolls that are most commonly used in practice.
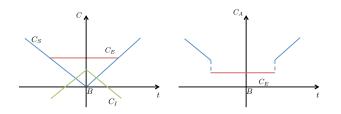

Fig. 5. A combined incentive and tolling strategy that achieves a freeway optimal flow allocation for a simple schedule cost function. **Left:** a simple schedule cost function C_S with linear earliness and lateness costs, an equilibrium cost C_E and corresponding freeway optimal incentive/toll C_I. **Right:** the corresponding arrival cost function C_A with the new equilibrium C_E that leads to no queuing. The vehicles that arrive within t_I of the scheduled arrival time B are tolled, while the vehicles that arrive outside this window are given an incentive.

B. Step incentive/toll

Definition 15. *Step incentive/toll*
A step incentive/toll is an incentive/tolling function with a constant value up to a given time t_I and zero after that.

$$C_I(t) = \begin{cases} I < 0 & \text{if } t < t_I \\ 0 & \text{if } t \geq t_I \end{cases} \quad (10)$$

Thus, since the schedule cost function C_S is continuous, a step incentive/toll will impose a discontinuity $C_A(t_I^+) - C_A(t_I^-) = I$ in the arrival cost function $C_A(t)$ at $t = t_I$, where I is the value of the incentive/toll.

From the definition of equilibrium compatible cost functions (definition 10), we know that the arrival time cost function C_A can admit positive discontinuities. Therefore, arrival time functions with step incentives/tolls still admit equilibrium solutions.

Proposition 10. *Demand shift with step incentives*
The exiting vehicle equilibrium can be shifted such that the support of the equilibrium flow is either to the left of some time t_{min} or the right of some time t_{max} using step incentives. See proposition 12 in [17] for the corresponding functions and proof.

Step incentives/tolls are inefficient for multiple reasons. As all the exiting vehicles in the incentive window must be given the same incentive, the vehicles that arrive close to the desired arrival are given a much larger incentive than needed. Consequently, the equilibrium solution requires that these vehicles occur a large queuing delay to compensate for the incentive. In fact, a step incentive can increase the total delay in the network, causing undesirable side effects such as increasing emissions in addition to the additional cost incurred. Furthermore, step incentives cannot move a congested equilibrium to a congestion-free equilibrium.

However, the efficiency of step incentives/tolls can be improved by combining step incentives/tolls. A sequence of step incentives can be used to approximate the freeway optimal incentive and obtain an equilibrium with a lower total incentive/toll cost. Figure 6 illustrates this. However, this still does not allow for a congestion free equilibrium. Furthermore, step incentives can also be mixed with step

tolls to shift the equilibrium cost between the vehicles and the controlling agency for any shift.
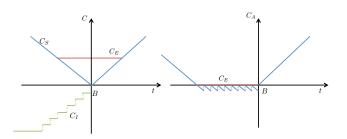

Fig. 6. A more efficient step incentive strategy that shifts the the exiting vehicle flow to the left of the scheduled arrival time B for a simple schedule cost function. **Left:** a simple schedule cost function C_S with linear earliness and lateness costs, an equilibrium cost C_E and piecewise constant step incentive/toll function C_I that corresponds the left shift of the equilibrium. **Right:** the corresponding arrival cost function C_A with the new flow distribution with equilibrium cost C_E.

The inefficiency of step incentives/tolls is a direct result of the assumption that the schedule time cost function C_S is continuous. However, in reality the actual schedule time cost incurred by commuters (imposed by employers) is likely to be discrete. In the event of discrete schedule time cost functions, a sequence of step incentives/tolls can be used to obtain a equilibrium that is congestion free for the freeway traffic. Figure 7 illustrates this.
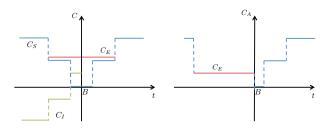

Fig. 7. A step incentive/toll strategy that shifts the the exiting vehicle flow to the left of the scheduled arrival time B for a simple schedule cost function. **Left:** a simple schedule cost function C_S with linear earliness and lateness costs, an equilibrium cost C_E and piecewise constant step incentive/toll function C_I that corresponds the left shift of the equilibrium. **Right:** the corresponding arrival cost function C_A with the new flow distribution with equilibrium cost C_E.

In conclusion, we can make the following observations on controlling the departure time equilibrium using incentives and tolls: 1) continuous incentives/tolls can be used to obtain a congestion free equilibrium, time shift the exiting vehicle demand and to allocate the cost of the control between the controlling agency and the drivers at any ratio, 2) step incentives/tolls can be used to time shift the exiting vehicle demand and control the cost allocation, but cannot be used to obtain a congestion free equilibrium for general schedule cost functions, and 3) if the schedule cost function is piecewise constant, then step incentives/tolls can be used to obtain a congestion free equilibrium.

V. CONCLUSION

This article considers the spill-back from a congested off-ramp and the resulting throughput loss on a freeway when the departure times of the exiting vehicles form an equilibrium with respect to their total cost. Existence and uniqueness properties are proved for a general class of cost functions that allow for local minima and discontinuities, which is a new result for the equilibrium departure time problem. We should how tolling and incentives can be used in tandem to achieve a wide variety of demand shifts for vehicles exiting a freeway and thereby increase the throughput on the freeway. The cost of the demand shift can be distributed in any ratio between the traffic management authority and the commuters by picking the appropriate incentive/tolling function. This allows for revenue neutral management strategies that are viewed more favorably with respect to public policy considerations.

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