Solutions to Estimation Problems for Scalar Hamilton–Jacobi Equations Using Linear Programming

Christian G. Claudel, *Member, IEEE*, Timothée Chamoin, and Alexandre M. Bayen, *Member, IEEE*

***Abstract*— This brief presents new convex formulations for solving estimation problems in systems modeled by scalar Hamilton–Jacobi (HJ) equations. Using a semi-analytic formula, we show that the constraints resulting from a HJ equation are convex, and can be written as a set of linear inequalities. We use this fact to pose various (and seemingly unrelated) estimation problems related to traffic flow-engineering as a set of linear programs. In particular, we solve data assimilation and data reconciliation problems for estimating the state of a system when the model and measurement constraints are incompatible. We also solve traffic estimation problems, such as travel time estimation or density estimation. For all these problems, a numerical implementation is performed using experimental data from the Mobile Century experiment. In the context of reproducible research, the code and data used to compute the results presented in this brief have been posted online and are accessible to regenerate the results.**

***Index Terms*— Linear programming, state estimation.**

I. Introduction

A. Background and Motivation

ESTIMATING or controlling the state of a distributed parameter system [7], [20], [24] is a very complex problem in general. It somehow requires the combination of data constraints, i.e., constraints on the possible trajectories of the system derived from measurement data, with model constraints, which means constraints on the possible trajectories of the system derived from the model. For the case in which the dynamic of the system is encoded by a partial differential equation (PDE), it is usually difficult to incorporate the model constraints into the estimation problem, since these constraints can be nonlinear, nonconvex, and even nonexplicit. Ultimately, the model constraints could theoretically be enforced through the use of Monte Carlo techniques, but these methods are impractical for large dimensional problems.

For the case in which the dynamic of the system is described by a Hamilton–Jacobi (HJ) PDE [12] for which the initial, boundary, or internal conditions are piecewise affine, we have proven [10] that the constraints of the model are convex. In addition, we showed that many data constraints could also be encoded as convex inequality constraints, yielding a convex optimization-based formulation for solving data assimilation and data reconciliation problems.

In this brief, we show that the same framework can be extended beyond data assimilation and data reconciliation, to solve a variety of estimation problems of interest for our main application, Lagrangian (mobile) traffic flow sensing [28]. For each of the estimation problems of interest, we show how the framework can be used to pose this particular problem as a linear program (LP), or as a set of LPs. We then solve these estimation problems using experimental data from the Mobile Century experiment [19].

The rest of this brief is organized as follows. Section II defines the solution to the HJ PDE investigated in this brief. Section III presents the general convex optimization framework used, and defines the linear or quadratic model and data inequality constraints used for solving the estimation problems. Section IV introduces three general estimation and data consistency problems that can be solved using the proposed framework. It also proves an important monotonicity property which ensures that the uncertainty on the estimates decreases as new data is added into the estimation problem. The following sections present specific applications for traffic-flow engineering problems. Section VI shows how the problem of data assimilation and data reconciliation can be posed as LPs. In Section VII, we estimate some functions of the state of the system, such as the travel time or the total number of vehicles present on a highway section using LPs. All of the above problems are illustrated by numerical computations performed using the Mobile Century data, freely available from [28].

II. HJ Equations

A. Definitions

We consider a spatial domain defined by $[\xi, \chi]$. We assume that the state of the system is described by a scalar function $\mathbf{M}(\cdot,\cdot)$ of both time and space, which satisfies a HJ PDE evolution equation

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0. \quad (1)$$

The function ψ used in (1) is called *Hamiltonian*.

From now on, we assume that the Hamiltonian is defined as follows:

$$\psi(\rho) = \begin{cases} v\rho, & \text{if } \rho \le k_c \\ w(\rho - k_m), & \text{otherwise.} \end{cases} \quad (2)$$

This form of Hamiltonian is known as triangular fundamental diagram in the context of traffic flow modeling, and is

Manuscript received April 25, 2012; accepted January 6, 2013. Manuscript received in final form January 7, 2013. Date of publication March 7, 2013; date of current version December 17, 2013. Recommended by Associate Editor C. N. Hadjicostis.

C. G. Claudel is with the Department of Electrical Engineering, King Abdullah University of Science and Technology, Thuwal 23955-6900, Saudi Arabia (e-mail: christian.claudel@kaust.edu.sa).

T. Chamoin is with the Department of Applied Mathematics, Ecole Polytechnique, Paris H3C 3A7, France (e-mail: timothee.chamoin@gmail.com).

A. M. Bayen is with the Department of Electrical Engineering and Computer Sciences, University of California, Berkeley, CA 94720-1710 USA (e-mail: bayen@berkeley.edu).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TCST.2013.2238940

widely used in the literature [15]. The parameters v, k_c, w, and k_m denote the free flow speed, critical density, congestion speed, and maximal density, respectively.

Several classes of weak solutions to (1) exist, such as viscosity solutions [3], [12], [13]. In this brief, we investigate a special class of viscosity solutions known as Barron-Jensen/Frankowska (B-J/F) solutions [6], [17]. The B-J/F solutions to (1) can be represented by a Hopf–Lax formula [1], [2], [8]. The Hopf–Lax solutions of HJ equations have been introduced independently by Hopf and Lax. The validity of Hopf–Lax formulas for weak solutions in the viscosity sense was first proved in [4]. The issue of their uniqueness is addressed in [1]. In this context, the Hopf–Lax formula [8], [9] was initially re-derived directly from viability theory, using a capture basin formulation of the problem presented in [2], which enabled Aubin *et al.* [2] to make the link between its formulation, and the B-J/F formulation (which itself provides the link with viscosity, existence, and uniqueness). Note that the equivalence with viscosity solutions holds only for initial value problems or initial/boundary problems. Our brief does not fall into these categories, as we investigate a problem in which the initial condition is not specified, and in which the value function is specified in the interior of the computational domain.

B. Hopf–Lax Formula for HJ Equations

In order to characterize the B-J/F solutions, we first need to define the Legendre–Fenchel transform of the Hamiltonian $\psi(\cdot)$ as follows.

Definition 2.1 (Legendre–Fenchel Transform): For an upper semicontinuous concave Hamiltonian $\psi(\cdot)$, the Legendre–Fenchel transform $\psi^*(\cdot)$ is given by

$$\psi^*(u) := \sup_{p \in \mathrm{Dom}(\psi)} [p \cdot u + \psi(p)]. \tag{3}$$

With our choice of Hamiltonian (2), the Legendre–Fenchel transform ψ^* is linear: $\psi^*(u) = k_c(u + v)$ on its effective domain [22] of definition $[-v, w]$.

Solving the HJ PDE (1) requires the definition of value conditions, which encode the traditional concepts of initial, boundary, and internal conditions.

Definition 2.2 (Value Condition): A value condition $\mathbf{c}(\cdot,\cdot)$ is a lower semicontinuous function defined on a subset of $[0, t_{\max}] \times [\xi, \chi]$.

By convention, a value condition $\mathbf{c}(\cdot,\cdot)$ as defined in Definition 2.2 satisfies $\mathbf{c}(t, x) = +\infty$ if $(t, x) \notin \mathrm{Dom}(\mathbf{c})$. Thus, the effective domain of a value condition represents the subset of the space time domain $\mathbb{R}_+ \times [\xi, \chi]$ in which we want the value condition to apply.

In the remainder of this brief, the solution $\mathbf{M_c}(\cdot,\cdot)$ to (1) associated with a value condition $\mathbf{c}(\cdot,\cdot)$ is given by the Hopf–Lax formula [2] and [8].

Proposition 2.3 (Hopf–Lax Formula): Let $\psi(\cdot)$ be a concave Hamiltonian, and let $\psi^*(\cdot)$ be its Legendre–Fenchel transform (3). Let $\mathbf{c}(\cdot,\cdot)$ be a lower value condition, as in Definition 2.2. The B-J/F solution $\mathbf{M_c}(\cdot,\cdot)$ to (1) associated with $\mathbf{c}(\cdot,\cdot)$ is given [2], [8] by

$$\mathbf{M_c}(t, x) = \inf_{(u,T) \in \mathrm{Dom}(\psi^*) \times \mathbb{R}_+} \big(\mathbf{c}(t - T, x + Tu) + T\psi^*(u)\big). \tag{4}$$

Equation (4) implies the existence of a B-J/F solution $\mathbf{M_c}(\cdot,\cdot)$ for any value condition function $\mathbf{c}(\cdot,\cdot)$. However, the solution itself may be incompatible with the value condition that we imposed on it, i.e., we do not necessarily have $\forall (t, x) \in \mathrm{Dom}(\mathbf{c}), \mathbf{M_c}(t, x) = \mathbf{c}(t, x)$.

The structure of the Hopf–Lax formula (4), implies the following important property, known as inf-morphism property. The inf-morphism property can be formally derived through capture basins, such as in [2].

Proposition 2.4 (Inf-Morphism Property): Let the value condition $\mathbf{c}(\cdot,\cdot)$ be minimum of a finite number of lower semicontinuous functions

$$\forall (t, x) \in [0, t_{\max}] \times [\xi, \chi], \quad \mathbf{c}(t, x) := \min_{j \in J} \mathbf{c}_j(t, x). \tag{5}$$

The solution $\mathbf{M_c}(\cdot,\cdot)$ associated with the above value condition can be decomposed [2], [8], [9] as

$$\forall (t, x) \in [0, t_{\max}] \times [\xi, \chi], \quad \mathbf{M_c}(t, x) = \min_{j \in J} \mathbf{M}_{\mathbf{c}_j}(t, x). \tag{6}$$

In the following section, we express the model constraints as a set of inequality constraints using the inf-morphism property.

C. Model Constraints

In the remainder of this brief, we decompose the value condition $\mathbf{c}(\cdot,\cdot)$ into block value conditions $\mathbf{c}_j,\ j \in J$. The relation between block value conditions and the physics of the problem is presented in Section III. The inf-morphism property and Hopf–Lax formula (4) imply the following compatibility property.

Proposition 2.5 (Model Compatibility of Block Value Conditions): Let $\mathbf{c}(\cdot,\cdot) = \min_{j \in J} \mathbf{c}_j(\cdot,\cdot)$ be given, and let $\mathbf{M_c}(\cdot,\cdot)$ be defined as in (4). The value condition $\mathbf{c}(\cdot,\cdot)$ satisfies $\forall (t, x) \in \mathrm{Dom}(\mathbf{c}), \mathbf{M_c}(t, x) = \mathbf{c}(t, x)$ if and only if the following inequality constraints are satisfied:

$$\mathbf{M}_{\mathbf{c}_j}(t, x) \geq \mathbf{c}_i(t, x) \qquad \forall (t, x) \in \mathrm{Dom}(\mathbf{c}_i) \ \ \forall (i, j) \in J^2. \tag{7}$$

The proof of this proposition is available in [10]. Note that the model constraints (7) depend upon the choice of the Hamiltonian $\psi(\cdot)$ through the Hopf–Lax formula (4).

III. Convex Formulation of Data and HJ PDE Model Constraints

A. Expression of the Upstream, Downstream, and Internal Boundary Conditions

In our specific application, the sensor data does not provide the initial condition of the problem, since this would require us instrumenting the whole spatial domain. Fixed traffic sensors traditionally measure the inflow and outflow of vehicles at the boundaries of the spatial domain, which are related to the upstream and downstream boundary conditions. In addition to fixed sensors, mobile sensors onboard vehicles track the vehicle trajectory and thus generate internal conditions [8].

The formal link between value condition blocks and measurable coefficients is shown in the following definition.

Definition 3.1 (Affine Upstream, Downstream, and Internal Conditions): Let us define $\mathcal{N} = \{0, \ldots, n_{\max}\}$ and $\mathcal{M} = \{0, \ldots, m_{\max}\}$. For all $n \in \mathcal{N}$ and $m \in \mathcal{M}$, we define the following functions, respectively, called upstream, downstream, and internal conditions:

$$\gamma_n(t,x) = \begin{cases} \sum_{i=0}^{n-1} q_{\text{in}}(i)\Delta t + q_{\text{in}}(n)(t - n\Delta t), \\ \quad \text{if } x = \xi \text{ and } t \in [n\Delta t, (n+1)\Delta t]; \\ +\infty, \quad \text{otherwise} \end{cases} \tag{8}$$

$$\beta_n(t,x) = \begin{cases} \sum_{i=0}^{n-1} q_{\text{out}}(i)\Delta t + q_{\text{out}}(n)(t - n\Delta t) - \Delta, \\ \quad \text{if } x = \chi \text{ and } t \in [n\Delta t, (n+1)\Delta t]; \\ +\infty, \quad \text{otherwise} \end{cases} \tag{9}$$

$$\mu_m(t,x) = \begin{cases} L_m + r_m(t - t_{\min}(m)), \\ \quad \text{if } x = x_{\min}(m) + \frac{x_{\max}(m) - x_{\min}(m)}{t_{\max}(m) - t_{\min}(m)}\left(t - t_{\min}(m)\right) \\ \quad \text{and } t \in [t_{\min}(m), t_{\max}(m)]; \\ +\infty, \quad \text{otherwise.} \end{cases} \tag{10}$$

In the above definition, m represents the label of an element of probe data, and is not related to the original Moskowitz function.

Given a boundary value problem given by a HJ equation (1) and a boundary condition $\mathbf{c}(\cdot,\cdot) := \min\left(\min_{n\in\mathcal{N}} \gamma_n(\cdot,\cdot), \min_{n\in\mathcal{N}} \beta_n(\cdot,\cdot), \min_{m\in\mathcal{M}} \mu_m(\cdot,\cdot)\right)$, a solution, in the B-J sense, is provided by the Hopf–Lax formula (4). In this brief, we show that the optimization problem resulting from (7) is a LP for many applied problems, including traffic estimation.

The variables used in (8)–(10) have the following physical interpretation:

$$\begin{cases} q_{\text{in}}(n), & \text{average inflow between times } n\Delta t \text{ and } (n+1)\Delta t \\ q_{\text{out}}(n), & \text{average outflow between times } n\Delta t \text{ and } (n+1)\Delta t \\ \Delta, & \text{initial number of vehicles on the highway section} \end{cases}$$

$$\begin{cases} t_{\min}(m), & \text{initial time at which the internal condition } m \text{ applies} \\ t_{\max}(m), & \text{final time at which the internal condition } m \text{ applies} \\ x_{\min}(m), & \text{initial location at which the internal condition } m \text{ applies} \\ x_{\max}(m), & \text{final location at which the internal condition } m \text{ applies} \\ L_m, & \text{label of the vehicle } m \text{ at time } t_{\min}(m) \\ r_m, & \text{rate of change of the label of vehicle } m. \end{cases} \tag{11}$$

Some of the coefficients used to define (8)–(10) can be estimated (with some error) through traffic measurement data. Inductive loop detectors [30] and speed radars located in ξ and χ can measure the inflow $q_{\text{in}}(n)$ and outflow $q_{\text{out}}(n)$ for all time intervals $[n\Delta t, (n+1)\Delta t]$. The coefficients $t_{\min}(m)$, $t_{\max}(m)$, $x_{\min}(m)$, and $x_{\max}(m)$ can be obtained using vehicle positioning systems, such as GPS-enabled cellphones onboard vehicles [28]. In contrast, the coefficients Δ, L_m, and r_m cannot be measured using conventional traffic sensors, but can sometimes be constrained by additional assumptions, see Section VII-B for instance.

B. Assumptions

In the remainder of this brief, we assume that the Hamiltonian $\psi(\cdot)$ is fixed, and given by (2) where v, w, and k_m are fixed.

We also assume that the coefficients $t_{\min}(m)$, $t_{\max}(m)$, $x_{\min}(m)$, and $x_{\max}(m)$ are fixed for all $m \in \mathcal{M}$. However, we do not assume that the coefficients $q_{\text{in}}(i)$ and $q_{\text{out}}(i)$ for all $i \in \mathcal{N}$ (boundary flows) are unknown. Similarly, the initial number of vehicles Δ, as well as the vehicle labels and passing rates L_m and r_m for $m \in \mathcal{M}$ are also variables of our problem.

Remark: The most general estimation problem would call for all coefficients used in (8)–(10) to be variables of the problem. However, using unknown coefficients for $t_{\min}(m)$, $t_{\max}(m)$, $x_{\min}(m)$, and $x_{\max}(m)$ would make the problem nonconvex. For our specific application, the assumption that $t_{\min}(m)$, $t_{\max}(m)$, $x_{\min}(m)$, and $x_{\max}(m)$ are all fixed does not significantly affect the results, since these coefficients are usually measured by GPS devices with an excellent accuracy (compared to the other measurements). ■

Given the above assumptions, we define a decision variable y fully characterizing a set of upstream (8), downstream (9), and internal conditions (10) as follows.

Definition 3.1 (Decision Variable): Let us define a finite set of upstream, downstream, and internal conditions blocks as in (8)–(10). The decision variable y associated with this set of value condition blocks is defined by

$$y := \Big(q_{\text{in}}(1), \ldots, q_{\text{in}}(n_{\max}), q_{\text{out}}(1), \ldots, q_{\text{out}}(n_{\max}), \\ L_1, r_1, \ldots, \ldots, L_{m_{\max}}, r_{m_{\max}}, \Delta\Big). \tag{12}$$

C. Model Constraints

Given the decision variable (12), the HJ PDE model constraints (7) for the Hamiltonian (2) can be written [10] as a finite set of linear inequalities, namely

$$A_{\text{model}}(\psi)y \le b_{\text{model}}(\psi). \tag{13}$$

The complete set of model constraints is available in [10].

D. Linear or Quadratic Data Constraints

Similar to the model constraints presented above, measurement data also restricts the possible values that the coefficients (12) can take. The values of $t_{\min}(\cdot)$, $t_{\max}(\cdot)$, $x_{\min}(\cdot)$, $x_{\max}(\cdot)$ are assumed to be perfectly known (i.e., measured without error). The measured values of $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$ are denoted by $q_{\text{in}}^{\text{meas}}(\cdot)$ and $q_{\text{out}}^{\text{meas}}(\cdot)$, respectively. In the remainder of this brief, we choose the following type of error model for $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$:

$$\left\| \frac{q_{\text{in}}(\cdot) - q_{\text{in}}^{\text{meas}}(\cdot)}{q_{\text{in}}^{\text{meas}}(\cdot)} \right\|_p \le e_{\max} \\ \left\| \frac{q_{\text{out}}(\cdot) - q_{\text{out}}^{\text{meas}}(\cdot)}{q_{\text{out}}^{\text{meas}}(\cdot)} \right\|_p \le e_{\max} \tag{14}$$

where $||\cdot||_p$ is the standard L_p norm

$$||f(\cdot)||_p = \left(\sum_{n=1}^{n_{\max}} |f(n)|^p\right)^{\frac{1}{p}}. \tag{15}$$

Different choices of norms are possible, and all choices $p \ge 1$ yield convex constraints by convexity of the norm.

In particular, the choices $p = 1$ and $p = +\infty$ yield linear constraints, which can be written as

$$A_{\text{data}} y \leq b_{\text{data}} \quad \text{for } p = 1 \text{ or } p = +\infty. \tag{16}$$

The choice $p = 2$ yields quadratic convex constraints, which can be written as

$$y^T Q(i) y + P(i)^T y + r(i) \leq 0, \qquad Q(i) \geq 0, \\ \text{for } \; i \in \{1, 2\} \; \text{ and } \; p = 2. \tag{17}$$

Note that the error model (14), for $p = +\infty$ is commonly used in practice. It corresponds to a situation in which we assume that the relative error on each measurement of the sensor is bounded by a constant value.

IV. Formulation of Estimation and Consistency Problems as Convex Programs

As shown earlier, the data constraints can be expressed as linear or quadratic inequalities in the decision variable, while the model constraints can be expressed as linear inequalities in the decision variable. This is precisely a contribution of this brief. For compactness, we choose to investigate problems associated with linear data constraints only (the extension to convex quadratic constraints is straightforward), obtained for $p = +\infty$ in (16).

We now define a fundamental convex feasibility problem, called data and model compatibility problem, which will play an important role in the following sections.

The objective of the data and model compatibility problem is to check if the model and data constraints are compatible, that is, if there exists a value of the decision variable (12) which satisfies both the model and data constraints. Hence, this requires us to check if the following LP is feasible:

$$\text{Find} \quad y \\ \text{such that} \begin{cases} A_{\text{model}}(\psi) y \leq b_{\text{model}}(\psi) \\ A_{\text{data}} y \leq b_{\text{data}}. \end{cases} \tag{18}$$

When the above problem is feasible, there exists some values of the decision variable y for which the model and data constraints are both satisfied, and this set is convex (intersection of convex sets). Hence, one can estimate the minimum (respectively, maximum) of a piecewise affine convex (respectively, concave) function of the decision variable using a LP. We apply this property in Section VII to estimate upper and lower bounds on functions of the decision variable.

In contrast, when (18) is infeasible, no set of value conditions satisfying both the model and data constraints can exist. However, by relaxing alternatively the model or data constraints, one can define two problems of interest [10]. The data assimilation problem consists in finding the set of value conditions satisfying the data constraints, that is as close as possible (in some norm sense) to satisfy the model constraints. The model reconciliation problem is the converse problem: it consists in finding the set of value conditions satisfying the model constraints, that is as close as possible (in some norm sense) to satisfy the data constraints. Both problems are solved simultaneously in Section VI.

A. Estimation Problems

A number of traffic-flow related quantities can be written as linear functions of the decision variable (12), and can be estimated using linear programming, as shown in the following proposition.

Proposition 4.1 (Estimation of Linear Functions of the Decision Variable): Let $f(\cdot)$ be a linear function of the decision variable y given by (12), and defined by $f(y) = d^T y$. The possible values that $f(\cdot)$ can take under the linear model (13) and data constraints (14) is the interval $[f_{\min}, f_{\max}]$, where $f_{\min}$ and $f_{\max}$ are solutions to the following LPs:

$$\text{Min (or Max)} \quad d^T y \\ \text{such that} \begin{cases} A_{\text{model}}(\psi) y \leq b_{\text{model}}(\psi) \\ A_{\text{data}} y \leq b_{\text{data}}. \end{cases} \tag{19}$$

Note that the above estimation problem only has a sense if the compatibility problem (18) is feasible.

B. Monotonicity Property of the Model Constraints With Respect to New Data

An important property of the estimation problems of the form (19) is monotonicity with respect to additional data, outlined in the following proposition.

Proposition 4.2 (Monotonicity Property): Let $m_{\max}$ and $n_{\max}$ be given positive integers, and let a set of block boundary and internal conditions be defined as in (8)–(10) for $1 \leq m \leq m_{\max}$ and $1 \leq n \leq n_{\max}$. Let the decision variable be defined as in (12), and let $f(\cdot)$ be a function of this decision variable. The upper bound on $f(\cdot)$ under the model (13) and data (16) constraints (for $p = +\infty$) decreases as new data is added into the estimation problem. The lower bound on $f(\cdot)$ under the model (13) and data (16) constraints (for $p = +\infty$) increases as new data is added into 347 the estimation problem.

Proof: Increasing the amount of data will add new model and data constraints to the estimation problem, reducing the feasible set, and thereby decreasing the upper bound and increasing the lower bound of a function of the decision variable. ∎

We illustrate the above property in Section VII in the cases of initial number of vehicles estimation and travel time estimation.

V. Experimental Setup and Implementation

In the following sections, we illustrate the power of the method with four different traffic flow estimation problems, which can all be formulated as LPs or sequences of LPs. For all of these estimation problems, we implement the method on the experimental data from the Mobile Century [19] field experiment.

In all numerical applications, we consider a 3.858-km long spatial domain, located between the PeMS [30] stations 400 536 and 400 284 on Highway $I880N$ in Hayward, California. The measurement data comes from two sources. The flow data $q_{\text{in}}^{\text{meas}}(\cdot)$ and $q_{\text{out}}^{\text{meas}}(\cdot)$ is generated by the PeMS stations 400 536 and 400 284, respectively. The probe location and timing data comes from GPS measurements generated by

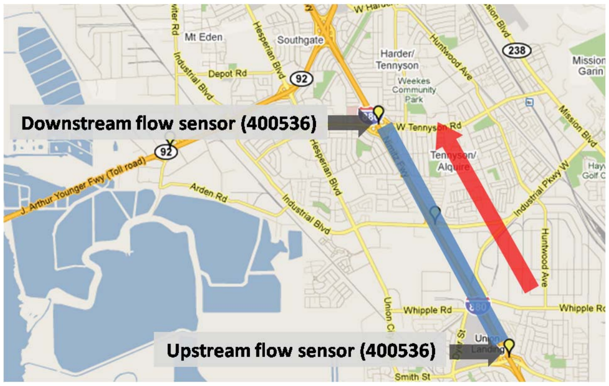

Fig. 1. Experiment site layout. The upstream and downstream PeMS stations are delimiting a 3.858-km spatial domain, outlined by a solid line. The direction of traffic flow is represented by an arrow.

Nokia $N95$ cellphones located onboard probe vehicles. The layout is illustrated in Fig. 1.

The complete experimental setting is described in [19]. The data set used in all numerical applications of this brief can be freely downloaded from [28].

All LPs have been implemented using the package CVX [18] of MATLAB. The problems solved in this brief are tractable: they typically involve thousands of variables and constraints, and can be solved numerically in a few seconds on a typical desktop computer.

VI. Data Assimilation and Reconciliation

A. Problem Definition

In the field of distributed parameters system estimation, the problems of data reconciliation [14] and data assimilation [16] are closely linked. The data assimilation process consists in finding the value of the state of the system that satisfies the observations, and that is the closest to being a solution to the evolution model. In contrast, the data reconciliation process consists in finding a solution to the evolution model that is the closest to the observations. Given the framework detailed above, the data assimilation and reconciliation problems are related to the solutions of the following convex optimization program

$$\begin{aligned} \text{Min} \quad & \|y_1 - y_2\|_q \\ \text{such that} \quad & \begin{cases} A_{\text{model}}(\psi) y_1 \le b_{\text{model}}(\psi) \\ A_{\text{data}} y_2 \le b_{\text{data}}. \end{cases} \end{aligned} \tag{20}$$

In the above optimization program, we have to choose $q = 1$ or $q = +\infty$ to obtain a linear objective. Two situations can arise.

1) If the optimal value of (20) is 0, the model and data constraints can be satisfied at the same time. In this situation, the data assimilation and data reconciliation problems coincide in a setting in which data and model are compatible. The solution is not unique.
2) If the optimal value of (20) is nonzero, the optimal solutions y_1^{optimal} and y_2^{optimal} enable us to compute the upstream, downstream, and internal conditions, respectively, associated with the data reconciliation and data assimilation problems. Note that these solutions may not be unique. The value conditions associated with y_1^{optimal} satisfy the model constraints by construction, i.e., all upstream boundary, downstream boundary, and internal conditions blocks apply in the strong sense [2], [5]. They, do not, however, satisfy the data constraints, but are as close as possible in the $\|\cdot\|_q$ sense to satisfy them. In contrast, the value conditions associated with y_2^{optimal} satisfy the data constraints by construction, but do not satisfy the model constraints (they are as close as possible to satisfy them in the $\|\cdot\|_q$ sense).

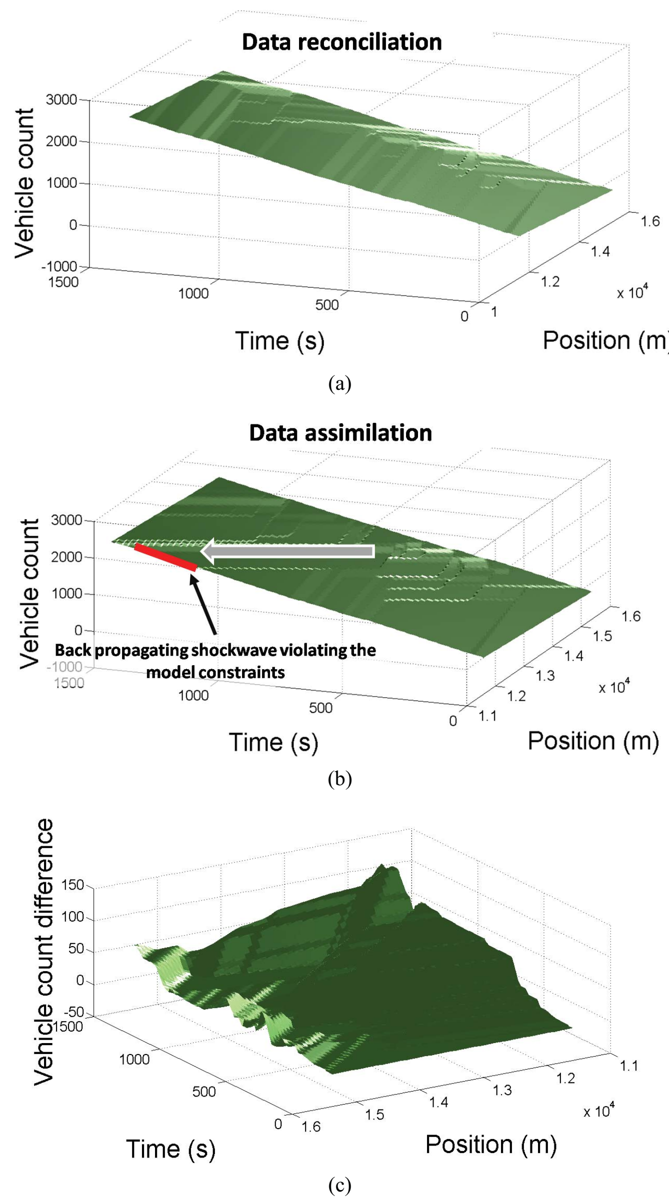

Fig. 2. Solutions to data assimilation and data reconciliation problems. We solve (20) using fixed detector data (generating upstream and downstream boundary conditions) and probe vehicle data (generating internal conditions). This experimental data was collected on February 8th, 2008. (a) Solution to the data reconciliation problem, in which the model constraints are satisfied, but the data constraints are not. (b) Solution to the data assimilation problem, in which the data constraints are satisfied, but the model constraints are not. Both problems are solved simultaneously by (20). (c) Difference (in number of vehicles) between the solution to the data reconciliation problem and the solution to the data assimilation problem.

B. Numerical Example

In this application, we consider the spatial domain defined in Section V, between the times 11:40 AM and 12:05 PM for data collected on February 8th, 2008. We use the following Hamiltonian parameters: $k_c = 0.048\ \mathrm{m}^{-1}$, $v = 24.6$ m/s, $w = -4.5$ m/s, and a maximal relative error level of $e_{\max} = 0.01$. We solve (20) for $q = 1$, using 604 variables and 17 415 linear constraints. For this specific application, the optimal value of (20) is +8.58, which ensures that the data assimilation and data reconciliation problems are well defined. As mentioned above, y_1^{optimal} and y_2^{optimal} enable us to compute the value conditions associated with the data assimilation and data reconciliation problems. We compute the solutions to (1) associated with these value conditions, and display them in Fig. 2. The solution to the data reconciliation problem at the top of Fig. 2 satisfies all the boundary and internal conditions that are prescribed on it. The model applies in the strong sense; however, the decision variable violates the data constraints (16). In contrast, the upstream and downstream boundary conditions do not apply everywhere in the solution to the data assimilation problem [Fig. 2(b)]. In the illustrated data assimilation example, the data constraints force some value conditions (boundary and internal conditions) to be set in a way that is incompatible with the model. This can be seen for instance around time $t = 1100$ s: a back propagating wave hits the upstream boundary condition at $x = 11\,000$ m, which prevents it from applying between times $t = 1100$ s and $t = 1400$ s.

VII. Traffic Flow Estimation Problems

The convex optimization framework presented in Section III can also be used to estimate traffic conditions, such as the boundary flows, the initial number of vehicles on a highway section, or the travel time required to cross the spatial domain. We now present two possible traffic estimation problems that can be solved using the proposed framework.

A. Estimation of the Initial Number of Vehicles Using Convex Programming

The initial number of vehicles Δ on the highway section can be estimated through linear programming. Indeed, Δ appears linearly in the decision variable (12), while the model (13) and data constraints (16) are linear inequalities in (12). Since the feasible set is convex by the constraints of (19), the possible values of Δ such that the model and data constraints are satisfied, are $\Delta_{\min} \le \Delta \le \Delta_{\max}$, where $\Delta_{\min}$ and $\Delta_{\max}$ are solutions to the following optimization programs:

$$\text{Min(respectively Max)} \quad \Delta \qquad \text{such that} \begin{cases} A_{\text{model}}(\psi) y \le b_{\text{model}}(\psi) \\ A_{\text{data}} y \le b_{\text{data}}. \end{cases} \tag{21}$$

We illustrate the estimation process in Fig. 3, in which we show the evolution of the interval $[\Delta_{\min}, \Delta_{\max}]$ as we increase the quantity of measurement data. In this problem, we consider the spatial domain defined in Section V, between the times 11:40 AM and 12:10 PM. We solve (20) using 60 blocks of upstream boundary conditions (8) and downstream boundary conditions (9), and a variable number of internal conditions (10).

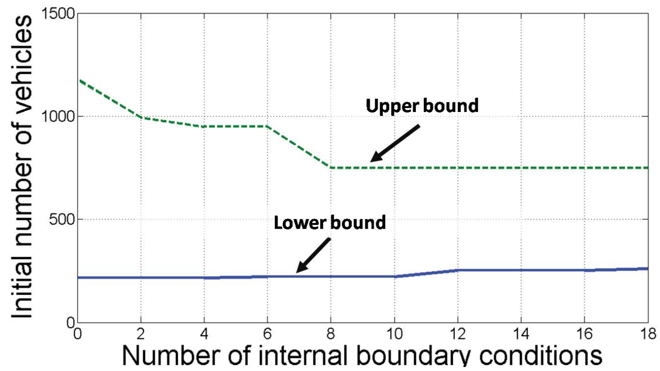

Fig. 3. Initial number of vehicles estimation using linear programming. This figure represents the evolution of the guaranteed upper and lower bounds on the initial number of vehicles Δ as new internal condition data is added into the estimation problem. The horizontal axis represents the number of probe measurement data blocks $\mu_m(\cdot,\cdot)$ as defined in (10). As predicted by Proposition 4.2, the upper bound (dashed line) on Δ decreases and the lower bound (solid line) on Δ increases when additional data is added into the estimation problem.

B. Travel Time Estimation Using Convex Programming

The same framework can also be applied for estimating other functions of the decision variable (12), such as the travel time across the highway section. Unlike the initial number of vehicles, the travel time is a nonlinear and nonconvex function of the decision variable (12), which makes the estimation problem more challenging.

In order to properly define a travel time function, we first need to assume [21] that no vehicles can pass each other, which implies in particular $r_m = 0$ for all $m \in \mathcal{M}$. In this situation, known in the transportation engineering as first-in, first-out (FIFO), the vehicle trajectories are the isolines of the state function [21]. In order to properly define the travel time function, we also have to assume that the function $\beta(\cdot,\cdot) = \min_{n\in\mathcal{N}} \beta_n(\cdot,\cdot)$ is strictly increasing. Note that by (9), imposing this last condition amounts to impose $q_{\text{out}}(\cdot) > 0$. With these two assumptions, the travel time can be defined as follows. Let t be given, and $i = \lfloor \frac{t}{\Delta t} \rfloor$. The travel time $\sigma(t)$ is defined as $\tau - t$, where $\gamma_i(t,\xi) = \beta(\tau,\chi)$. Since $\beta(\cdot,\chi)$ is strictly increasing, we can also define the travel time as

$$\sigma(y,t) = \min_{s\in\mathbb{R}_+ \text{ s. t. } \beta(s,\chi)\ge\gamma_i(t,\xi)} (s-t) \tag{22}$$

or alternatively

$$\sigma(y,t) = \max_{s\in\mathbb{R}_+ \text{ s. t. } \beta(s,\chi)\le\gamma_i(t,\xi)} (s-t). \tag{23}$$

Since $\beta_j(s,\chi)$ and $\gamma_i(t,\chi)$ are functions of the decision variable (12), the travel time function $\sigma(\cdot,\cdot)$ hereby defined is a function of the decision variable (12), though not linear. While we cannot estimate the travel time using a LP of the form (19), we can still obtain valuable information on upper and lower bounds of the travel time function using LPs, as outlined in the following proposition.

Proposition 7.1 (Upper and Lower Bounds on the Travel Time Function): Let us assume that (18) is feasible, that is, the model and data constraints are compatible. Let two times t and τ be given, and let $j = \lfloor \frac{\tau}{\Delta t} \rfloor$. We have that $\tau - t$ is a lower bound on the travel time $\sigma(y,t)$ (under the model and data constraints) if and only if the following problem is infeasible:

$$\text{Find} \quad y$$
$$\text{such that} \begin{cases} A_{\text{model}}(\psi) y \leq b_{\text{model}}(\psi) \\ A_{\text{data}} y \leq b_{\text{data}} \\ \beta_j(\tau, \chi) - \gamma_i(t, \xi) \geq 0. \end{cases} \tag{24}$$

Similarly, $\tau - t$ is an upper bound on the travel time $\sigma(y,t)$ under the model and data constraints if and only if the following problem is infeasible:

$$\text{Find} \quad y$$
$$\text{such that} \begin{cases} A_{\text{model}}(\psi) y \leq b_{\text{model}}(\psi) \\ A_{\text{data}} y \leq b_{\text{data}} \\ \beta_j(\tau, \chi) - \gamma_i(t, \xi) \leq 0. \end{cases} \tag{25}$$

Proof: We prove that $\tau - t$ is a lower bound on the travel time function if and only if (24) is infeasible. Let us thus assume that (24) is infeasible. This amounts to saying that $\beta_j(\tau,\chi) < \gamma_i(t,\xi)$ whenever the model and data constraints $A_{\text{model}}(\psi)y \leq b_{\text{model}}(\psi)$ and $A_{\text{data}} y \leq b_{\text{data}}$ are both satisfied. Hence, since $\beta(\tau,\chi) = \beta_j(\tau,\chi)$ by construction, this is equivalent to saying that $\beta(\tau,\chi) < \gamma_i(t,\xi)$ whenever the model and data constraints are both satisfied. By (23) of $\sigma(y,t)$, this is equivalent to $\sigma(y,t) > \tau - t$, whenever y satisfies the model and data constraints, which completes the proof. The proof relative to the upper bound is similar, and involves (22) of $\sigma(y,t)$. ■

Note that the feasibility programs (24) and (25) enable us to compute the largest lower bound $\sigma_{\text{d}}(y,t)$ and the smallest upper bound $\sigma_{\text{u}}(y,t)$ on the travel time by dichotomy. We illustrate the above results by computing the upper and lower bounds on the travel time function, using the experimental setup of Section V, between times 11:40 AM and 12:10 PM. For this, we check the feasibility of (24) and (25) for $\tau = j\Delta t$, and plot in Fig. 4, respectively, the lowest and highest value of $j\Delta t$ such that (25) and (24) are, respectively, infeasible. The lowest value $j_{\max}\Delta t$ for which (25) is infeasible implies that $\sigma_{\text{u}}(y,t)$ is in the interval $[(j_{\max}-1)\Delta t - t, j_{\max}\Delta t - t]$. Similarly, the highest value $j_{\min}\Delta t$ for which (24) is infeasible implies that $\sigma_d(y,t)$ is in the interval $[j_{\min}\Delta t - t, (j_{\min}+1)\Delta t - t]$. As stated in Proposition 4.2, the distance between the upper and lower bounds decreases as more data is added into the estimation problem.

Remark: The largest lower bound (or smallest upper bound) on travel time cannot be directly estimated using convex programming. Indeed, by checking the feasibility of (24) for increasing values of $\tau = n\Delta t$, we can find the integer j such that $\sigma_{\text{d}}(y,t) \in [j\Delta t - t, (j+1)\Delta t - t]$ (in this situation, (24) is infeasible for $\tau = j\Delta t$, and becomes feasible for $\tau = j\Delta t + 1$). When such a j is identified, $\sigma_d(y,t)$ is the solution to the following optimization program:

$$\text{Min} \quad \frac{z}{q_{\text{out}}(j)}$$
$$\text{such that} \begin{cases} A_{\text{model}}(\psi) y \leq b_{\text{model}}(\psi) \\ A_{\text{data}} y \leq b_{\text{data}} \\ \beta_j(\frac{z}{q_{\text{out}}(j)}, \chi) - \gamma_i(t, \xi) \leq 0. \end{cases} \tag{26}$$

The decision variable of (26) can be written as (y,z), where y is the decision variable defined by (12). The constraints $A_{\text{model}}(\psi)y \leq b_{\text{model}}(\psi)$ and $A_{\text{data}} y \leq b_{\text{data}}$ are both linear in the new decision variable (they indeed depend only upon y). The constraint $\beta_j((z/q_{\text{out}}(j)),\chi) - \gamma_i(t,\xi) \leq 0$ is also linear, since it can be written as

$$\sum_{k=0}^{j-1} q_{\text{out}}(k)\Delta t + q_{\text{out}}(j)\left(\frac{z}{q_{\text{out}}(j)} - j\Delta t\right) - \Delta - \sum_{k=0}^{i-1} q_{\text{in}}(k)\Delta t - q_{\text{in}}(i)(t - i\Delta t) \leq 0. \tag{27}$$

The objective is, however, nonconvex, since $(z,q) \to z/q$ is not convex. Problem (26) thus cannot be solved using convex programming, but may still be solved numerically using other optimization methods. ■

Remark: In experimental conditions, the vehicles may not satisfy the FIFO assumption, and their travel time may be

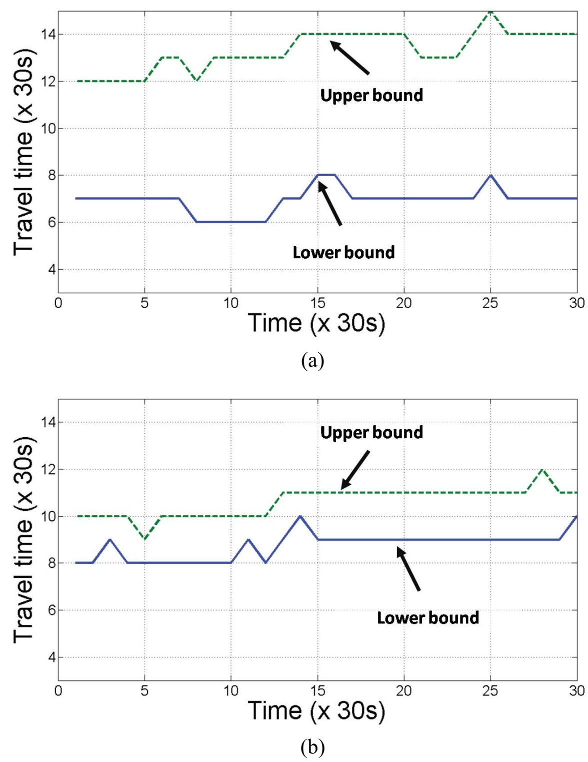

Fig. 4. Travel time estimation using linear programming. The horizontal axis represents the time t, while the vertical axis represents the travel time. The upper and lower bounds on the travel time function are represented by a dashed and solid line, respectively. (a) We consider 60 upstream and downstream boundary conditions blocks and 20 internal condition blocks. (b) We increase the number of internal condition blocks to 45. As can be seen, the corresponding bounds on the travel time function are improved since more data is added into the estimation problem, following Proposition 4.2.

out of the bounds predicted by this algorithm whenever lane shearing effects are significant. ■

VIII. Conclusion

This brief illustrated some applications of a new convex optimization-based estimation framework for systems modeled by a scalar HJ equation. Using a Hopf–Lax formula, we showed that the constraints from the model, as well as the constraints from the measurement data result in linear inequality constraints for a specific decision variable. We then posed and solved various traffic estimation problems as LPs.

Other applications of this convex optimization framework were developed, such as the detection of sensor faults in real time [11], as well as the detection of spoofing cyberattacks, or the analysis of user privacy in probe-based traffic sensing systems. Some of the above estimation programs were implemented in the Mobile Millennium traffic information system [27], [28] operated jointly by Nokia and UC Berkeley, and providing real-time traffic information to the participating public of California. Future work on the Mobile Millennium system will involve the implementation of data assimilation and data reconciliation programs for real-time inverse modeling.

Acknowledgment

Following the guidelines of [23] on reproducible research, all code and data which was used to generate the results presented in this brief have been posted on the web, and are available for download from the following URL [29]. Other information on reproducible research is also available in [25] and [26].

The authors are grateful to Prof. J.-P. Aubin for his guidance and vision, and his help to pose the HJ problem as a viability problem. Prof. P. Saint-Pierre is gratefully acknowledged for his advice on numerical computations. The numerical computations presented in this brief have been done using technologies initially developed by the company VIMADES. The authors are particularly thankful to Prof. L. El-Ghaoui for his advice and guidance on convex optimization.

References

[1] O. Alvarez, E. N. Barron, and H. Ishii, "Hopf-Lax formulas for semicontinuous data," *Indiana Univ. Math. J.*, vol. 48, no. 3, pp. 993–1036, 1999.

[2] J.-P. Aubin, A. M. Bayen, and P. Saint-Pierre, "Dirichlet problems for some Hamilton-Jacobi equations with inequality constraints," *SIAM J. Control Optimizat.*, vol. 47, no. 5, pp. 2348–2380, 2008.

[3] M. Bardi and I. Capuzzo-Dolcetta, *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, Boston, MA, 1997.

[4] M. Bardi and L. C. Evans, "On Hopf's formulas for solutions of Hamilton-Jacobi equations," *Nonlinear Anal.*, vol. 8, no. 11, pp. 1373–1381, 1984.

[5] C. Bardos, A. Y. Leroux, and J. C. Nedelec, "First order quasilinear equations with boundary conditions," *Commun. Partial Different. Equat.*, vol. 4, no. 9, pp. 1017–1034, 1979.

[6] E. N. Barron and R. Jensen, "Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians," *Comm. Partial Different. Equat.*, vol. 15, pp. 1713–1742, Sep. 1990.

[7] P. D. Christofides, *Nonlinear and Robust Control of Partial Differential Equation Systems: Methods and Applications to Transport-Reaction Processes*. Birkhäuser, Boston, MA, 2001.

[8] C. G. Claudel and A. M. Bayen, "Lax-hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part I: Theory," *IEEE Trans. Automat. Control*, vol. 55, no. 5, pp. 1142–1157, May 2010.

[9] C. G. Claudel and A. M. Bayen, "Lax-hopf based incorporation of internal boundary conditions into Hamilton-Jacobi equation. Part II: Computational methods," *IEEE Trans. Automat. Control*, vol. 55, no. 5, pp. 1158–1174, May 2010.

[10] C. G. Claudel and A. M. Bayen, "Convex formulations of data assimilation problems for a class of Hamilton-Jacobi equations," *SIAM J. Control Optimizat.*, vol. 49, no. 2, pp. 383–402, 2011.

[11] C. G. Claudel, M. Nahoum, and A. M. Bayen, "Minimal error certificates for detection of faulty sensors using convex optimization," in *Proc. 47th Annu. Allerton Conf. Commun., Control, Comput.*, Allerton, IL, Sep. 2009.

[12] M. G. Crandall, L. C. Evans, and P.-L. Lions, "Some properties of viscosity solutions of Hamilton-Jacobi equations," *Trans. Amer. Math. Soc.*, vol. 282, no. 2, pp. 487–502, 1984.

[13] M. G. Crandall and P.-L. Lions, "Viscosity solutions of Hamilton-Jacobi equations," *Trans. Amer. Math. Soc.*, vol. 277, no. 1, pp. 1–42, 1983.

[14] C. M. Crowe, "Data reconciliation-progress and challenges," *J. Process. Control*, vol. 6, no. 2, pp. 89–98, 1996.

[15] C. Daganzo, "A variational formulation of kinematic waves: Basic theory and complex boundary conditions," *Trans. Res. B*, vol. 39B, no. 2, pp. 187–196, 2005.

[16] G. Evensen, *Data Assimilation: The Ensemble Kalman Filter*. Berlin. Germany: Springer-Verlag, 2007.

[17] H. Frankowska, "Lower Semicontinuous Solutions of Hamilton-Jacobi-Bellman Equations," *SIAM J. Control Optimizat.*, vol. 31, no. 1, pp. 257–272, 1993.

[18] M. Grant and S. Boyd. (2010, Jul.). *CVX: Matlab Software for Disciplined Convex Programming, Version 1.21* [Online]. Available: http://cvxr.com/cvx

[19] J. C. Herrera, D. B. Work, R. Herring, X. J. Ban, Q. Jacobson, and A. M. Bayen, "Evaluation of traffic data obtained via GPS-enabled mobile phones: The mobile century field experiment," *Trans. Res. Part C, Emerg. Technol.*, vol. 18, no. 4, pp. 568–583, Aug. 2010.

[20] M. Krstic and A. Smyshlyaev, "Backstepping boundary control for first-order hyperbolic PDEs and application to systems with actuator and sensor delays," *Syst. Control Lett.*, vol. 57, no. 9, pp. 750–758, 2008.

[21] G. F. Newell, "A simplified theory of kinematic waves in highway traffic," *Trans. Res. B*, vol. 27B, no. 4, pp. 281–303, 1993.

[22] R. T. Rockafellar and R. J. B. Wets, *Variational analysis*, vol. 317. New York: Springer-Verlag, 2011.

[23] V. C. Stodden, "Reproducible research: Addressing the need for data and code sharing in computational science," *Comput. Sci. Eng.*, vol. 12, no. 5, pp. 8–13, 2010.

[24] R. C. Smith and M. A. Demetriou, *Research Directions in Distributed Parameter Systems*. SIAM, Philadelphia, PA, 2000.

[25] V. Stodden, "Enabling reproducible research: Licensing for scientific innovation," *Int. J. Commun. Law Policy*, vol. 13, no. 1, pp. 1–5, 2009.

[26] V. Stodden, "The legal framework for reproducible research in the sciences: Licensing and copyright," *IEEE Comput. Sci. Eng.*, vol. 11, no. 1, pp. 35–40, Jan. 2009.

[27] D. Work, S. Blandin, O. Tossavainen, B. Piccoli, and A. Bayen, "A distributed highway velocity model for traffic state reconstruction," *Appl. Res. Math. Exp.*, vol. 1, pp. 1–35, Apr. 2010.

[28] *Snapshot of Mobile Millennium Traffic in San Francisco and the Bay Area*. (2011) [Online]. Available: http://traffic.berkeley.edu/

[29] *A Matlab Toolbox for State Estimation on Scalar Hamilton-Jacobi Equations Using Linear Programming*. (2013) [Online]. Available: http://web.kaust.edu.sa/faculty/christianclaudel/code_lp.zip

[30] *Freeway Performance Measurement System*. (2002) [Online]. Available: http://pems.eecs.berkeley.edu