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Routing on Traffic Networks Incorporating Past Memory Up to Real-Time Information on the Network State

Alexander Keimer and Alexandre Bayen

Institute of Transportation Studies, University of California, Berkeley, California 94720, USA; email: keimer@berkeley.edu, bayen@berkeley.edu

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Abstract

In this review, we discuss routing algorithms for the dynamic traffic assignment (DTA) problem that assigns traffic flow in a given road network as realistically as possible. We present a new class of so-called routing operators that route traffic flow at intersections based on either real-time information about the status of the network or historical data. These routing operators thus cover the distribution of traffic flow at all possible intersections. To model traffic flow on the links, we use a well-known macroscopic ordinary delay differential equation. We prove the existence and uniqueness of the solutions of the resulting DTA for a broad class of routing operators. This new routing approach is required and justified by the increased usage of real-time information on the network provided by map services, changing the laws of routing significantly. Because these map and routing services have a huge impact on the infrastructure of cities, a more precise mathematical description of the emerging new traffic patterns and effects becomes crucial for understanding and improving road and city conditions.

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1. INTRODUCTION

In recent years, new map and routing services that provide (allegedly) almost real-time information on the status of a traffic network have increased in usage. Based on this information, these services suggest shortest routes and will continue to change the routing behavior of traffic participants on road networks all over the world (1–5). Due to these changes, well-understood and mathematically elegant models such as convex optimization problems for determining Nash-based traffic assignments (6, 7) might be less precise or appropriate, as they were not meant to incorporate changes of routes during the routing process over time.

In this article, we present the current status of mathematical modeling of the routing of traffic flow in networks macroscopically, i.e., as time- and space-dependent continuous density. We explain most of the chosen approaches dealing with the so-called dynamic traffic assignment (DTA) problem. We then introduce a broad class of routing operators and show their well-posedness on the network for specific macroscopic dynamics, represented by a system of ordinary delay differential equations. In the considered context, well-posedness means that the system of differential equations coupled with the named routing operators admits a (unique) solution. This is not straightforward, as the routing depends on the solution in real time and causes a coupling between flow allocation and state of the system. To our knowledge, such a broad and rigorous approach has not been made in the literature, in particular when the routing itself depends on the network status in real time, making it necessary to address the well-posedness by means of a fixed-point argument.

In Section 2, we present the current state of research, with an emphasis on the difference between link and node dynamics (the terminology is explained in the corresponding sections). We start with Section 2.1, explaining macroscopic traffic flow models using ordinary differential equations (ODEs) in Section 2.1.1 and partial differential equations (PDEs) in Section 2.1.2. In Sections 2.1.1.1 and 2.1.2.1, we discuss the challenges when applying the named models to networks.

After presenting the dynamic models involved in describing traffic flow on links with proper node models, in Section 2.2 we discuss different routing approaches that distribute the flow while following specific rules or laws. Section 2.2.1 briefly explains the archetypes of probabilistic routing approaches or routing operators located at intersections; Section 2.2.2 discusses the time-dependent Wardrop's principles, sometimes also called user equilibrium; and Section 2.2.3 describes optimal control approaches to obtain the best routing for a specific objective function.

Section 3 introduces a new modeling approach, routing operators, which can react to traffic situations in up to real time. After defining the needed notation and network structure in Sections 3.1 and 3.2, we introduce the considered link dynamics (an ordinary delay differential equation) in Section 3.3, present the routing operators in Section 3.4, and describe the resulting DTA problem in Section 3.5. In Section 3.6, we investigate the well-posedness of the entire time-continuous routing problem by means of a fixed-point argument, and in Section 3.7, we make some remarks about shortest-path assignments. We instantiate some well-known routing operators in Section 3.8 and discuss the applicability of our theory to these operators. The article concludes in Section 4 with suggestions for future research directions.

2. THE STATE OF THE ART

Many different approaches to the DTA problem have been considered. Usually, the problem is formulated on a network, with links and nodes representing the corresponding roads and intersections of the considered traffic network.

In the following, we distinguish between the link dynamics, meaning the dynamics that traffic flow follows when no intersections are present, and the node dynamics, which disperse incoming traffic flow according to predefined policies, or laws of routing. For the link dynamics, we consider only macroscopic traffic flow models, i.e., models that describe the flow via density and not individual traffic participants.

Depending on the link dynamics, node dynamics become more or less difficult. For instance, consider an intersection where all outgoing links are fully congested. Flow intending to enter the intersection cannot pass to the leaving links and thus spills back on the entering links. For a realistic traffic flow and DTA model, modeling of this spillback is crucial, as otherwise flow would smoothly pass through the network. Under the assumption that there is no spillback, even when travel time increases, a gridlock situation could never or would never occur, but from a modeling point of view this makes the underlying equations more complex and harder to handle.

2.1. Link and Node Dynamics

In this section, we describe different time-continuous macroscopic traffic flow models. For a general overview, we refer readers to References 8–10.

2.1.1. Ordinary differential equation models. ODE models for traffic flow have been presented in, e.g., References 11–14. For a given $T \in \mathbb{R}_{>0}$, they have the general structure

$$x'(t) = f(x(t), t),\ t \in [0, T],\ x(0) = x_0,$$

with a proper function $f : \mathbb{R} \to \mathbb{R}$ to be specified, $x(t)$ being the density at time $t \in [0, T]$, and $x_0 \in \mathbb{R}_{\geq 0}$ being the initial density. Usually, the model class is written in greater detail:

$$x'(t) = \text{inflow}[x](t) - \text{outflow}[x](t),\ t \in [0, T],\ x(0) = x_0, \qquad 1.$$

where inflow$[x]$ indicates the time-dependent inflow onto a considered link and outflow$[x]$ indicates the corresponding outflow from a given link. Both inflow and outflow might also depend on the density x of the considered link indicated in the notation. When interpreting the model with regard to an entire network, one might also prescribe additional constraints to make the model more realistic, although mathematically, these constraints can actually cause issues of well-posedness. Assume, for instance, that one adds box constraints—capacity constraints on a specific link—and additional flow enters this link. Then, when there is no other modeling taking care of the flow onto the considered link that cannot be processed, the model is not well posed, meaning that the underlying equations do not have a solution. This issue can be overcome by defining a proper node model, as discussed in Section 2.1.1.1.

In addition, even though these models have a notion of traffic density, there is no notion of travel time (as one cannot state how long a certain amount of flow takes to pass a link), so one needs to come up with objective functions that compute a type of travel time based on the given density (for how this is done for the stationary traffic assignment, see, e.g., Reference 7). However, this is confusing so far, as the model already contains time $t \in [0, T]$, so travel time should be directly defined in terms of this time.

Another approach is to place queues at the exits and entries of links, where the queuing size is modeled by ODEs. Several different models exist for this purpose; for more information, we refer readers to References 15–18 and 19–21, the latter of which consider different generalizations based on the Vickrey model (22). In addition, one can now impose constraints on links so that if a specific density is reached, the corresponding buffer increases. For short links where real flow dynamics might not develop, this model class might be fortunate. In a simplified version, following

Reference 16, the model can be written as

$$q'(t) = p(t - t_0) - \begin{cases} \min\{\overline{C}, p(t - t_0)\} & q(t) = 0 \\ \overline{C} & q(t) > 0 \end{cases} \qquad t \in [0, T],$$

$$q(0) = q_0.$$

Thereby, $q : [0, T] \to \mathbb{R}_{\geq 0}$ denotes the queue size (or density), $t_0 \in \mathbb{R}_{>0}$ the free-flow travel time, $\overline{C} \in \mathbb{R}_{>0}$ the road capacity, and $q_0 \in \mathbb{R}_{\geq 0}$ the initial queuing size or initial density. In addition, the function $p : [-t_0, T] \to \mathbb{R}_{\geq 0}$ represents the inflow, and according to Equation 1, the function

$$\text{outflow}[p, q](t) := \begin{cases} \min\{\overline{C}, p(t - t_0)\} & q(t) = 0 \\ \overline{C} & q(t) > 0 \end{cases}$$

represents the outflow. Finally, the actual travel time $\tau(t)$ can be computed as $\tau(t) = t_0 + \frac{q(t-t_0)}{\overline{C}}$.

The next step is to consider ODEs with delay. One instantiation of these equations is also used for the analysis in the present article (see Definition 4 in Section 3.3). This model class has been extensively discussed in References 23–27. The advantage is that even though it is still an ODE model, there is now an intrinsic travel time as a function of the state density, and this travel time—the delay of the ODE—increases when there is higher density on the link. One disadvantage of this model class is that only when the delay depends affine linearly on the density can a unique solution for every type of essentially bounded inflow be guaranteed, limiting to some extent its application (26, theorem 3.2]. Of course, one can then again prescribe state constraints on the dynamics to avoid links with high density; however, as pointed out above, this might affect the well-posedness of the system, as a solution might then be nonexistent after all when inflow into the system is high or too many links merge into one such that the flow can no longer be processed.

2.1.1.1. Node models. For the ODE models' modeling density and for the delayed versions, there is no need for a node model. The leaving density from one link can be directly passed to the following links according to a given routing behavior, as discussed in Section 2.2, as these models do not possess spillback. If additional capacity constraints are stated, then further adjustments must be made to create a well-defined system—e.g., introducing additional queues at the intersections or having large enough queues when one already uses queuing models for the links.

2.1.2. Partial differential equation models. The second and typically more realistic approach to modeling traffic flow on roads as a continuous density over time also takes into account the spatial component of the link representing the position on the road. This has been done in References 28 and 29, following the fundamental theory of conservation of vehicles and a fluid type of approach. The corresponding governing equation, the LWR PDE (named after Lighthill, Whitham, and Richards), reads as

$$\partial_t \rho(t, x) + \partial_x f(\rho(t, x)) = 0, \qquad (t, x) \in (0, T) \times X,$$

$$\rho(0, x) = \rho_0(x), \qquad x \in X,$$

where $X \subset \mathbb{R}$ is an interval, the boundary conditions are prescribed at ∂X, $\rho(t, x)$ is the traffic density at space-time coordinate $(t, x) \in (0, T) \times X$, and the initial datum ρ_0 prescribes the density at time $t = 0$. The function f is called the flux function and is usually chosen as $f(y) := y \cdot v(y)$,

$y \in \mathbb{R}$, with a monotone decreasing velocity function $v : \mathbb{R} \to \mathbb{R}$. A famous flux function is the Greenshields function (30) with $v(y) = 1 - y$ (when density is scaled to be between 0 and 1), but there exist many more fluxes and corresponding velocity functions dependent on the considered road setup (see 27). The choice of velocity function is usually made such that flux increases with density until a turning point when the density reaches a critical limit, and after that, the flux decreases in terms of the density until it reaches a minimum, or even zero.

Mathematically, the prescribed LWR PDE is a so-called quasi-linear scalar hyperbolic conservation law, which can develop shocks and rarefaction waves and also deal with discontinuous initial data. It is thus significantly more appropriate for modeling traffic flow when spillback is important than the ODE models described in Section 2.1.1. The existence and uniqueness of solutions for these equations are nontrivial even for the Cauchy problem (meaning without a boundary datum, $X = \mathbb{R}$) and have been studied and solved in References 31–35 by introducing entropy conditions to single out the physically relevant unique weak solution among the infinitely many possible weak solutions. The presented equation has also been studied as the limit of the well-established microscopic follow-the-leader ODEs when the number of vehicles approaches infinity (homogenization) (see, e.g., 36). It is worth mentioning that the previously defined model class can be transformed into another, so-called Hamilton–Jacobi PDE, where semiexplicit solution formulae are available based on a finite-dimensional optimization problem for every considered point in space-time (37–41).

The extension of the previous PDE to nonlocal traffic flow models has been investigated (e.g., 42–44). Here, the above-named velocity v depends not on $\rho(t, x)$ but on the averaged density ahead, i.e., on $\frac{1}{\eta}\int_x^{x+\eta} \rho(t, s)\,\mathrm{d}s,\ (t, x) \in (0, T) \times \mathbb{R}$ for a given $\eta \in \mathbb{R}_{>0}$. The advantages of this type of modeling are that it is more realistic than the previously chosen local LWR model and that even in its aggregated form—as density—it is closer to microscopic behavior due to its forward-looking parameter η. It is also mathematically interesting: As no entropy condition is required, a weak solution is by itself unique.

One disadvantage of the previously introduced LWR model class is that velocity behaves purely as a function of the density. For greater realism, one might actually want to model the velocity by its own dynamics—e.g., to model phantom shocks, which are common in high-density traffic flows. This is why the literature also considers second-order traffic flow models. We only introduce the inhomogeneous ARZ PDE (named after Aw, Rascle, and Zhang) (45, 46) for $(t, x) \in (0, T) \times X$ here:

$$\partial_t \rho(t, x) + \partial_x (\rho(t, x)u(t, x)) = 0, \tag{2.}$$

$$\partial_t (u(t, x) + h(\rho(t, x))) + u(t, x)\partial_x (u(t, x) + h(\rho(t, x))) = \frac{1}{\tau}\left(U_{\mathrm{eq}}(\rho(t, x)) - u(t, x)\right), \tag{3.}$$

$$\rho(0, x) = \rho_0(x),$$

$$u(0, x) = u_0(x),$$

where the boundary conditions are prescribed at ∂X. The traffic density at time-space coordinate $(t, x) \in (0, T) \times X$ is denoted by $\rho(t, x)$ and the velocity by $u(t, x)$. Equation 2 denotes the conservation of vehicles and is inspired by the LWR PDE, with the difference that the velocity function now is not an explicit function of the density but follows its own dynamics in Equation 3: There, the term $h : \mathbb{R} \to \mathbb{R}$ with $h' > 0$ represents the pressure term stemming from fluid dynamics, but in traffic flow, it is more reasonable to call it a hesitation function (cf. 47). $U_{\mathrm{eq}} : \mathbb{R} \to \mathbb{R}$ denotes the equilibrium velocity and $\tau \in \mathbb{R}_{>0}$ the reaction time of drivers. The considered set of equations can be posed as a system of conservation laws and is thus a quasi-linear system of hyperbolic conservation laws.

For $X = \mathbb{R}$—meaning, again, a pure Cauchy problem without a boundary datum—this model has been studied extensively for questions on the existence, uniqueness, regularity, and stability of solutions in References 48–51; for locally constrained flow in References 52 and 53; and for phase transition in Reference 54. Second-order multiclass or multicommodity models have also been studied in Reference 55 and models with creeping in Reference 56. For general conservation law models, we refer readers to Reference 57, which discusses existence and uniqueness in significantly more generality.

2.1.2.1. The node model for partial differential equations: a (modeling) challenge. In Section 2.1.2, when $X \subset \mathbb{R}$ is a finite interval, a boundary datum must be prescribed. However, this boundary datum might not be attained, as the road might already be fully congested, or higher flow might want to enter a given link than the link can take. This is why the boundary datum needs to be described in a weak sense (see 39, 40, 58–61). For conservation of flow when the boundary datum is not attained, one needs to keep track of the flow not entering the considered link. This information can be obtained by placing buffers at the intersections that make sure that flow is conserved (for supply chain modeling with buffers, see Reference 62; for traffic flow modeling with buffers and the LWR PDE, see References 63–66). When these buffers have a finite capacity, they also need to allow spillback to the incoming roads, making it possible for congestion to spread over a node. They also enable the corresponding system of conservation laws to depend in a continuous way on the input parameters.

When not using buffers, one can consider the Riemann problem at the intersection, as has been done in References 27 and 67–69 for the LWR PDEs and in Reference 70 for the ARZ model. The problem of the Riemann-solver approach is that the solution is not necessarily continuously dependent on the input datum, and additional equations must be postulated to obtain a reasonable node model (maximizing throughput; for maximizing a given objective over the nodes, see Reference 71). These dynamics as well as the first-order dynamics have extensions to multicommodity flows on networks (72).

2.2. Routing

Now that we have defined the node dynamics, the missing piece for a full consideration of the DTA dynamics is how traffic flow is actually routed at the considered intersections.

2.2.1. Routing operators. A straightforward approach to prescribing routing is to introduce flow ratios at all intersections. These flow ratios might depend on the solution itself (see, in particular, Section 3 and References 73 and 74) or might be realized by fixed turning ratios trained on existing data or stochastic modeling. Even though our analysis focuses on macroscopic modeling, many suggested approaches among route choice models for individuals can also be used for macroscopic modeling. We refer readers to Reference 75 for an overview, Reference 76 for a fuzzy route choice model, References 77–82 for general stochastic route choice models, and Reference 83 for a cognitive cost route choice model. Some of the route choice models also take into account the state of the network in real time (84, 85). For the validation of route choice models, we refer readers to References 86–88. Route choice for stationary models has also been considered in References 89–92.

Shortest-path-based routing models in real time are often used. However, in continuous dynamics, these models might not be well posed, as explained in Remark 1 in Section 3.7 for a specific class of models.

2.2.2. Time-dependent Wardrop's conditions. Wardrop's principles are famous for describing how flow might get routed in a road network (see 93, p. 345):

1. The journey times on all the routes actually used are equal to each other and less than those that would be experienced by a single vehicle on any unused route.
2. The average journey time is a minimum.

The first principle defines a rule regarding how individuals might route themselves in a network and can be interpreted as a type of Nash equilibrium (6, 94). Individuals choose their routes so that their travel time is minimal, taking into account that everyone else does the same. This requires that everyone knows about all origin destination demand. It can be motivated by the argument that drivers identify the proper routes over time as they obtain more knowledge about the state of the network and their impact on it when diverting from specific routes. By contrast, the second principle suggests a routing based on a social optimum. Clearly, the second problem can be handled by an optimal control approach in the routing, as also suggested in Section 2.2.3. However, both principles allow some interpretation when it comes to time dynamics. The reasonability of these approaches must also be discussed, as making proper routing decisions requires precise information about the status of the network. Routing based on a time-dependent Wardrop's condition is usually called dynamic user equilibrium. Mathematically, this equilibrium can be cast as a variational inequality or as a complementary condition. This idea has been investigated in detail for different user equilibria in References 18, 23, 24, and 95–100, in which variational formulations for different dynamic user equilibria are given and variational inequalities obtained. References 17, 101, and 102 take similar approaches, with solid analyses on the well-posedness of solutions and numerical studies. For general variational inequalities, we refer readers to Reference 19.

2.2.3. Optimal control. The famous M-N model (named after Merchant and Nemhauser) (11, 12) was one of the first used to model the DTA by using an optimal control framework (103). Having defined link models via ODEs, the authors presented an optimal control framework similar to that for the stationary traffic assignment (7), where the routes are optimized over the entire considered time horizon, minimizing a given objective function. Two drawbacks of this approach are evident. First, this class of link models does not explicitly possess a travel time, so one must come up with an objective that realizes this travel time in a reasonable way. Second, an optimal control problem over the entire considered time horizon is stated, although for the purpose of a real-time application, such information about when people leave might not be available. The presented formulation is path based, so the optimal paths are chosen when flows enter the network, and no rerouting will occur. A similar approach is considered in Reference 104. For cases when the link dynamics are realized via PDEs, optimal routing is considered in References 105–107 for the LWR (28, 29) traffic flow model (also with instantaneous control) and in Reference 108 for a specific class of nonlocal conservation laws.

3. ROUTING OPERATORS BASED ON THE NETWORK STATE

As we aim to propose a mathematical framework capable of handling the change of routing due to routing applications using real-time information (see Section 1), the routing on the network should be (*a*) presented in a broad way, keeping as much generality as possible and allowing the results to be applicable to very different routing scenarios; (*b*) able to distinguish among different types of flow (flow with more information, different types of vehicles, etc.); (*c*) capable of reacting to a change in the network in real time; (*d*) instantiated so that it can depend in up to real time on the solution of the entire network; (*e*) dependent on already passed information or on delayed information; (*f*) dependent on external factors, such as the closing down of specific roads at specific times; and (*g*) located at the different intersections explicitly as a functional expression. All of these properties will be satisfied by the model introduced and exploited in Section 3.

Although our aim and interest lie mainly in the routing, for a mathematical analysis, we must consider the full system consisting of the link model and node model (as described in Section 2.1) and routing (as described in Section 2.2). From the node model, we must have a travel time and an increase in travel time when the road is more congested. In addition, we require a multicommodity model capable of handling different types of flow, different types of destinations, and more. As we do not want to restrict our routing operator, we require that the node model be able to assign—at every time considered—any flow on all outgoing links and to distinguish between different commodities.

This is why we chose for the link dynamics an ordinary delay differential equation (e.g., as considered in Reference 26 and mentioned in Section 2.1.1). This equation does not model spillback, so we can use as the node model the trivial model of assigning just the incoming flow according to the routing. This missing feature is acceptable, as we focus mainly on routing. The model still has the property that travel time changes with respect to link density, so it is significantly complex for routing problems. Of course, a similar framework taking the proper node dynamics can also be introduced for more advanced link dynamics, as prescribed in Section 2.1.2. However, in this case, the node model described in Section 2.1.2.1 will not be trivial and needs to satisfy a Lipschitz continuity with respect to the input parameters, making the analysis more complex. Most of the content in this section has been discussed and explored in detail in Reference 73.

3.1. Notation and More

We will require the following function spaces.

Definition 1 (function spaces). For $p \in [1, \infty]$ and $T \in \mathbb{R}_{>0}$, we define the following Banach spaces:

$$L^p((0,T)) := \left\{ f : (0,T) \to \mathbb{R} \text{ Lebesgue measurable} : \ \|f\|_{L^p((0,T))} < \infty \right\},$$

with

$$\|f\|_{L^p((0,T))} := \left(\int_{(0,T)} |f(s)|^p \, \mathrm{d}s \right)^{\frac{1}{p}}, \ p \in [1,\infty), \qquad \|f\|_{L^\infty((0,T))} := \text{ess-sup}_{t \in (0,T)} |f(t)|$$

the space of continuous functions on the interval $[0, T]$ as

$$C([0,T]) := \left\{ f : [0,T] \to \mathbb{R} : f \text{ is continuous} \right\}$$

with the corresponding norm

$$\|f\|_{C([0,T])} := \max_{t \in [0,T]} |f(t)|, \qquad f \in C([0,T])$$

and the space of Lipschitz-continuous functions as

$$W^{1,\infty}((0,T)) := \left\{ f \in L^\infty((0,T)) : \sup_{x,y \in [0,T]} \left| \frac{f(x) - f(y)}{x - y} \right| < \infty \right\}$$

with the corresponding norm

$$\|f\|_{W^{1,\infty}((0,T))} := \|f\|_{L^\infty((0,T))} + \|f'\|_{L^\infty((0,T))},$$

where f' exists almost everywhere by Rademacher's theorem, so that the previous definition makes sense. Of course, vectorized versions of the previously defined function spaces are defined appropriately.

3.2. The Network

As we will need to describe the DTA problem with regard to the underlying network, we start by defining a network and related attributes, such as paths, destinations, and origin nodes.

Definition 2 (network and paths). We call a directed graph $G = (\mathcal{V}, \mathcal{A})$—where $\mathcal{V}$ is a finite set with $|\mathcal{V}| \in \mathbb{N}_{\geq 1}$, nodes $v \in \mathcal{V}$, and arcs $a \in \mathcal{A} \subset \mathcal{V} \times \mathcal{V}$—a network. Furthermore, we define for $v \in \mathcal{V}$ the sets of incoming arcs and outgoing arcs as

$$\mathcal{A}_{\text{in}}(v) := \{(\tilde{v}, v) \in \mathcal{A} : \tilde{v} \in \mathcal{V}\}, \qquad \mathcal{A}_{\text{out}}(v) := \{(v, \tilde{v}) \in \mathcal{A} : \tilde{v} \in \mathcal{V}\}.$$

We define the set of paths $\mathcal{P}^{v,d}$ between two nodes $(v, d) \in \mathcal{V}^2$ without cycles as

$$\mathcal{P}^{v,d} := \bigcup_{k=1}^{|\mathcal{V}|^2} \big\{\mathbf{p} \in \mathcal{A}^k : \ \forall i, j \in \{1, \ldots, k+1\}, i \neq j : v_i, v_j \in \mathcal{V} : v_i \neq v_j,$$
$$v_1 = v, \ v_{k+1} = d, \mathbf{p} = ((v_i, v_{i+1}))_{i \in \{1,\ldots,k\}}\big\}.$$

For every path $\mathbf{p} := (\mathbf{p}_1, \ldots, \mathbf{p}_k)^T \in \mathcal{P}^{v,d}$ with $k \in \mathbb{N}_{\geq 1}$, we define its length by $\text{len}(\mathbf{p}) := k$. In the network, we specify source nodes as

$$\mathcal{O} \subset \mathcal{V}$$

and destination nodes as

$$\mathcal{D} \subset \big\{d \in \mathcal{V} : \ \exists v \in \mathcal{O} : \mathcal{P}^{v,d} \neq \emptyset\big\}.$$

We define $\mathcal{OD}$ as the set of origin/source–destination pairs by

$$\mathcal{OD} := \{(v, d) \in \mathcal{O} \times \mathcal{D} : \mathcal{P}^{v,d} \neq \emptyset\}.$$

Finally, we define for $d \in \mathcal{D}$ the set $\mathcal{OP}^d$ of arcs that are part of paths starting from an arbitrary node $\tilde{v} \in \mathcal{O}$ and ending in d if $(\tilde{v}, d) \in \mathcal{OD}$,

$$\mathcal{OP}^d := \bigcup_{\ell=1}^{|\mathcal{A}|} \big\{\mathbf{p}_\ell \in \mathcal{A} : \exists \tilde{v} \in \mathcal{O} : (\tilde{v}, d) \in \mathcal{OD}, \ \mathbf{p} \in \mathcal{P}^{\tilde{v},d}, \ \ell \leq \text{len}(\mathbf{p})\big\}, \qquad 4.$$

and the outgoing arcs for a node $v \in \mathcal{V}$ from which destination $d \in \mathcal{D}$ is reachable as

$$\mathcal{A}^d_{\text{out}}(v) := \big\{a \in \mathcal{A} : \exists \tilde{v} \in \mathcal{V} \text{ with } a = (v, \tilde{v}) \ \wedge a \in \mathcal{OP}^d\big\}.$$

Assumption 1 (feasible network). The network G in Definition 2 contains at least one $\mathcal{OD}$ pair, i.e. $|\mathcal{OD}| > 0$.

3.3. The Link Model

In this section, we introduce the link dynamics that we use throughout this article and that were mentioned in Section 2.1.1. To this end, we require different commodities representing the mathematical way to express or model different types of traffic flow or different types of information-aware flow, using different routing systems or no routing suggestions at all, etc. As we want to distinguish between different commodities, we also define the multicommodity set.

Definition 3 (multicommodity set). For a given $n \in \mathbb{N}_{\geq 1}$, we define the multicommodity set as $\mathcal{C} := \{1, \ldots, n\}$.

The link dynamics read as follows.

Definition 4 (link-delay model on a single link). Denote by the superscript $(c, d) \in \mathcal{C} \times \mathcal{D}$ a tuple containing the commodity type and destination node as given in Definition 3. Assume that $\mathbf{x} : [0, T] \to \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}$ represents the flow, and let $\mathbf{u} : [0, T] \to \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}$ be the inflow. Then, for a free-flow travel time $b \in \mathbb{R}_{>0}$ and congestion factor $h \in \mathbb{R}_{\geq 0}$, we consider the following system of ordinary delay differential equations:

$$\dot{\mathbf{x}}(t) = \mathbf{u}(t) - \mathbf{g}[\mathbf{u}, x, \dot{x}](t), \qquad t \in [0, T], \tag{5}$$

$$\mathbf{x}(0) = \mathbf{0}, \tag{6}$$

$$x(t) := \sum_{c \in \mathcal{C}} \sum_{d \in \mathcal{D}} \mathbf{x}^{c,d}(t), \qquad t \in [0, T], \tag{7}$$

$$\tau[x](t) := b + hx(t), \qquad t \in [0, T], \tag{8}$$

$$\rho[x](t) := t + \tau[x](t), \qquad t \in [0, T], \tag{9}$$

$$\mathbf{g}[\mathbf{u}, x, \dot{x}](t) := \begin{cases} \mathbf{0} & \text{almost everywhere (a.e.) } t \in [0, b), \\ \frac{\mathbf{u}(\rho[x]^{-1}(t))}{1 + h\dot{x}(\rho[x]^{-1}(t))} & \text{a.e. } t \in [b, T + \tau[x](T)], \end{cases} \tag{10}$$

where $\rho[x]^{-1}$ is the inverse function of $[0, T] \ni t \mapsto \rho[x](t)$.

Some comments are in order to give a better understanding of the above-defined model. Definition 4 presents the link dynamics. We need to have the dynamics in a vectorized form, as we will need to keep track of different types of flow, including flow heading to different destinations $d \in \mathcal{D}$ and flow that follows different routing policies (e.g., flow using information about the traffic system in real time versus flow not using that information).

The dynamical process is presented in a coupled system of ordinary delay differential equations in Equation 5. Thereby, $\mathbf{u}$ denotes the inflow onto the link (vectorized with different commodities and destinations), and $\mathbf{g}$ denotes the outflow (which is detailed below and will be a function of the density of the link). Equation 6 states that the link is empty when starting. Equation 7 denotes the cumulative density of the link—i.e., the density of all flows on the link, summarizing over all different commodities and destinations. Equation 8 denotes the delay caused by the traffic density: The higher the cumulative density x is, the larger the delay becomes. b can be seen as the free flow delay or travel time when the road is empty, and h is a tuning parameter that affects the influence of the travel time or delay with regard to the cumulative density. Based on the above, Equation 9 gives the time when an inflow entering at time $t \in [0, T]$ actually exits the link, and $\mathbf{g}$

in Equation 10 shows how the outflow is determined by means of the inflow and the travel time. The denominator stands for a spreading or concentration of flow due to higher or lower density and is needed to keep the flow conserved.

Because a solution to an ordinary delay differential equation with a delay depending on the solution itself does not necessarily exist, we give the following theorem.

Theorem 1 (existence and uniqueness of the link-delay model with affine linear delay as presented in Definition 4). Let $T \in \mathbb{R}_{>0}$ and the link-delay model with affine linear delay as in Definition 4 be given, and assume that $\mathbf{u} \in L^\infty((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$ is given as well as $b \in \mathbb{R}_{>0}$ and $h \in \mathbb{R}_{\geq 0}$. Then, the system of ordinary delay differential equations in Equations 5–10 is well posed and admits a unique Carathéodory solution $\mathbf{x} \in W^{1,\infty}((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$.

Proof. The proof can be found in theorems 3.1 and 3.2 in Reference 26. It takes advantage of the delay character so that one can use an iterative approach in time to solve a series of delay equations on a small time horizon and patch them together. □

As pointed out above, the advantage of this model is that there is no spillback, so that at every intersection one can assign as much flow as needed to any of the outgoing links. This basically makes the normally required node model discussed in Section 2.1.1.1 obsolete or trivial and simplifies our analysis about routing, as we can directly continue to the definition of a routing operator.

3.4. The Routing Operator

In this section, we define the routing operator. We start with the most general definition possible in Definition 5 and later restrict the routing operator in Definition 7.

Definition 5 (general routing operator). Let $T \in \mathbb{R}_{>0}$, and for $v \in \mathcal{V}$, let $d \in \mathcal{D}$, $c \in \mathcal{C}$, and $a \in \mathcal{A}_{\text{out}}^d(v)$ be given as in Definition 2. Let $\mathbf{ext} \in L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}})$. Then, we call $\mathscr{R}_a^{c,d}$ a routing operator if and only if

$$\mathscr{R}_a^{c,d} : L^\infty\left((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) \times L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}}) \to L^\infty((0,T); [0,1]) \qquad 11.$$

such that for every $(\mathbf{x}, \mathbf{ext}) \in L^\infty((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}) \times L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}})$,

$$\sum_{\tilde{a} \in \mathcal{A}_{\text{out}}^d(v)} \mathscr{R}_{\tilde{a}}^{c,d}[\mathbf{x}, \text{ext}](t) \equiv 1, \qquad t \in [0,T] \text{ a.e.} \qquad 12.$$

As one can see from Definition 5, particularly from Equation 11, the routing operator is actually an operator that takes into account the full solution on the network over the full time horizon considered. The routing operator carries a subindex that assigns flows to all exiting links $a \in \mathcal{A}_{\text{out}}^d(v)$ from an intersection node $v \in \mathcal{V}$. In addition, it can route differently for different types of commodities $c \in \mathcal{C}$ and destinations $d \in \mathcal{D}$, a reasonable assumption when recalling that the $\mathcal{OD}$ pairs might vary significantly for different destinations $d \in \mathcal{D}$ and that different types of flow might behave or be routed differently. Finally, the routing can also be influenced by so-called externalities. As indicated by Equation 12, the routing operator must conserve flow and thus routes all incoming flows with their different destinations and commodities to the outgoing links at every time. In Equation 11, the function values of the routing operator are assumed to be between 0 and 1.

Next, we specify the routing operators, as the generality of the routing operators provided in Definition 5 is too broad to obtain any results on the existence or uniqueness of the system on the network. As it makes sense to consider routing operators, which use at time $t \in [0,T]$ at most the traffic state $\mathbf{x}$ up to time t (see Section 1), we distinguish a Lipschitz-continuous routing, which is capable of incorporating routing decisions made in up to real time, from a delay-type routing operator, where the decision can be made in real time only at finitely many points in time and is otherwise delayed, and no Lipschitz continuity or only continuity is required.

Definition 7 requires the following projection operator, which will be helpful in defining delayed routing.

Definition 6 (projection mapping). Define for $\alpha \in \mathbb{R}$ the projection mapping $\mathrm{P}_{[0,\alpha]} : \mathbb{R} \to \mathbb{R}$ on $[0, \alpha]$ as

$$\forall t \in \mathbb{R} : \qquad \mathrm{P}_{[0,\alpha]}(t) := \begin{cases} \min\{\max\{0, t\}, \alpha\} & \text{if } \alpha \in \mathbb{R}_{\geq 0}, \\ 0 & \text{else.} \end{cases}$$

Definition 7 (routing operators). Let the assumptions of Definition 5 be given. Then, we define the following under the assumption that all introduced routing operators satisfy Definition 5:

1. Lipschitz-continuous routing: We call $\mathscr{RL}_a^{c,d}$ a Lipschitz-continuous routing operator if and only if it is Lipschitz continuous in the sense that, for a $p \in (1, \infty]$, it holds that

$$\forall \mathbf{ext} \in L^\infty((0,T); \mathbb{R}^{n_{\text{ext}}})\ \forall t \in [0,T]\ \exists L \in \mathbb{R}_{\geq 0}\ \forall \mathbf{x}, \tilde{\mathbf{x}} \in C\left([0,T]; \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right) :$$

$$\left\| \mathscr{RL}_a^{c,d}[\mathbf{x}, \mathbf{ext}] - \mathscr{RL}_a^{c,d}[\tilde{\mathbf{x}}, \mathbf{ext}] \right\|_{L^p((0,t))} \leq L \|\mathbf{x} - \tilde{\mathbf{x}}\|_{C\left([0,t]; \mathbb{R}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|}\right)}.$$

2. Delay-type routing: We call $\mathscr{RD}_a^{c,d}$ a delay-type routing operator if and only if there exists for $N_a^{c,d} \in \mathbb{N}_{\geq 1}$ a time vector $\mathbf{t}_a^{c,d} := ((\mathbf{t}_a^{c,d})_1, \ldots, (\mathbf{t}_a^{c,d})_{N_a^{c,d}})^T \in (0,T]^{N_a^{c,d}}$ with $(\mathbf{t}_a^{c,d})_{N_a^{c,d}} := T$ and

$$\left(\mathbf{t}_a^{c,d}\right)_i < \left(\mathbf{t}_a^{c,d}\right)_{i+1} \qquad \forall i \in \left\{1, \ldots, N_a^{c,d} - 1\right\};$$

a delay function $\delta_a^{c,d} \in L^\infty((0,T); [0,T])$ such that for every $i \in \{1, \ldots, N_a^{c,d} - 1\}$ the regularity and estimate on the delay $\delta_a^{c,d}$

$$\begin{aligned} \delta_a^{c,d}\Big|_{\left[0, \left(\mathbf{t}_a^{c,d}\right)_1\right)} &:\equiv 0, \\ \delta_a^{c,d}\Big|_{\left[\left(\mathbf{t}_a^{c,d}\right)_i, \left(\mathbf{t}_a^{c,d}\right)_{i+1}\right)} &\in C\left(\left[\left(\mathbf{t}_a^{c,d}\right)_i, \left(\mathbf{t}_a^{c,d}\right)_{i+1}\right)\right), \\ \left\|\delta_a^{c,d}\right\|_{C\left(\left[\left(\mathbf{t}_a^{c,d}\right)_i, \left(\mathbf{t}_a^{c,d}\right)_{i+1}\right)\right)} &\leq \left(\mathbf{t}_a^{c,d}\right)_i \end{aligned} \qquad 13.$$

hold; and, finally, for a.e. $t \in [0,T]$, $\forall \mathbf{x} \in C([0,T]; \mathbb{R}_{\geq 0}^{|\mathcal{A}||\mathcal{C}||\mathcal{D}|})$, recalling Definition 6,

$$\mathscr{RD}_a^{c,d}[\mathbf{x}, \mathbf{ext}](t) = \mathscr{RD}_a^{c,d}\left[\mathbf{x} \circ \mathrm{P}_{\left[0, \delta_a^{c,d}(t)\right]}, \mathbf{ext}\right](t). \qquad 14.$$

Again, some explanations are in order. The Lipschitz-continuous operator is meant to be Lipschitz continuous in L^p, $p \in (1, \infty]$, when measuring the network state in the uniform topology. As one can see from the first item in Definition 7, the Lipschitz continuity must hold for all $t \in [0,T]$, guaranteeing that it can depend on the network state at time $t \in [0,T]$ only up to time t. As the solutions $\mathbf{x}$ on all links are at least Lipschitz continuous (as long as the inflows are essentially bounded), we know that $\mathbf{x}$ is continuous and can actually measure in the continuous topology.

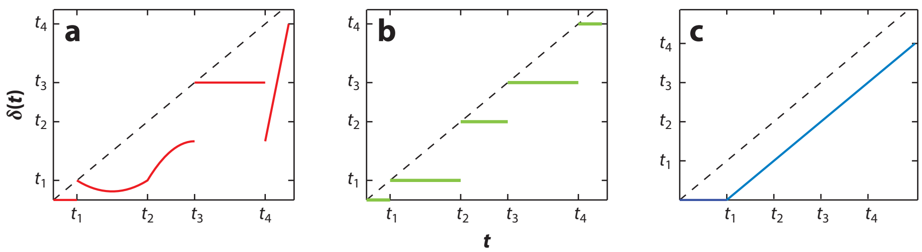

Figure 1

Example of a delay function. The diagonal dashed lines represent the identity. (*a*) A general feasible delay function δ. (*b*) The case when a piecewise vanishing delay occurs. (*c*) The case when a constant delay $\varepsilon \in \mathbb{R}_{>0}$ occurs. Figure adapted from Reference 73.

As we illustrate below, this Lipschitz-continuous dependency is not too strong a condition and is satisfied by a variety of different routing operators (see Section 3.8).

The Lipschitz-continuous dependency is needed because the real-time dependency of the routing operator gives another coupling in any of the considered ODEs on the links. The routing operator depends on the assigned outflow (which is the inflow into other links), and the inflow of these links depends again on the routing operator. This is why the problem must be addressed by means of a fixed-point argument, for which—in case one wants to obtain uniqueness of solutions—the contraction mapping principle (Banach's fixed-point theorem) is essential (109, theorem 3.A]. We refer to Theorem 3. Also note that all routing operators are coupled with each other because the entire traffic state will change according to the routing operators, and the routing operators will change their routing assignment based on the traffic state.

We also want to mention that a weaker form of routing operators is available in Reference 73 (definition 2.12, item C), where we only assume continuity. However, as this result is based on Schauder's fixed-point theorem (109, corollary 2.13], the solution cannot be expected to be unique; furthermore, we can give examples where there would be infinitely many solutions. We thus renounce the presentation of these classes of routing operators.

The delayed routing in the second item in Definition 7 does not need continuity, even Lipschitz continuity, as it acts or makes decisions based mainly on the state of the network in a delayed sense. The additional complexity in Equation 13 comes from the fact that we also want the possibility that the routing operator depends on the real-time solution at at most finitely many points. This is also illustrated in **Figures 1** and **2**. It is also worth mentioning that the presented approach

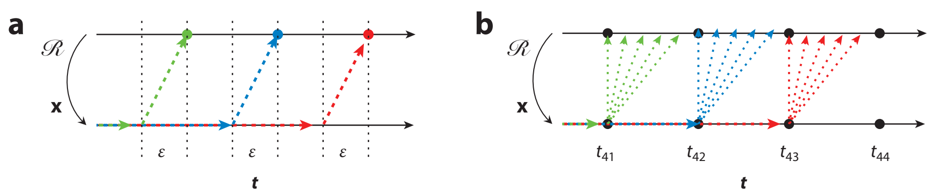

Figure 2

Example of flow information used by the routing operator, showing (*a*) a constant ε delay and (*b*) a piecewise vanishing delay. Each horizontal arrow represents the time line. The delay-type routing operator $\mathscr{R}$ maps time points (represented on the *upper horizontal arrows*) to time periods that do not occur later in time. This produces time periods (which are part of the *lower horizontal arrows*) that build the possible evaluation time periods for the flow **x**. Figure adapted from Reference 73.

does not need to know any inflow—origin–destination demand—into the network, as it will need information only at time $t \in [0, T]$ and will then react properly based on the state of the traffic.

3.5. The Dynamic Traffic Assignment Framework Incorporating Information on the Network State

In this section, we present the entire DTA problem when the routing is performed by the routing operators introduced in Section 3.4.

Definition 8 (network formulation of the link-delay model with routing operators: dynamic traffic assignment). Consider for a time horizon $T \in \mathbb{R}_{>0}$ a network $G = (\mathcal{V}, \mathcal{A})$ as in Definition 2, with origins $\mathcal{O} \subseteq \mathcal{V}$ and destinations $\mathcal{D} \subseteq \mathcal{V}$ as in Definition 2. Let the inflow into the network for $\tilde{v} \in \mathcal{O}$ be given as $\mathbf{s}_{\tilde{v}} \in L^\infty((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$. Then, we pose the dynamics subject to the link-delay model for multiple destinations with routing as proposed in Definition 5 for all $t \in [0, T]$ and $a \in \mathcal{A}$ with $b_a \in \mathbb{R}_{>0}$ and $h_a \in \mathbb{R}_{\geq 0}$ as

$$\dot{\mathbf{x}}_a(t) = \mathbf{u}_a(t) - \mathbf{g}_a[\mathbf{u}_a, x_a, \dot{x}_a](t), \qquad t \in [0,T], \quad 15.$$

$$\mathbf{x}_a(0) = \mathbf{0}, \quad 16.$$

$$x_a(t) := \sum_{c\in\mathcal{C}} \sum_{d\in\mathcal{D}} \mathbf{x}_a^{(c,d)}(t), \qquad t \in [0,T], \quad 17.$$

$$\tau_a[x_a](t) := b_a + h_a x_a(t), \qquad t \in [0,T], \quad 18.$$

$$\rho_a[x_a](t) := t + \tau_a[x_a](t), \qquad t \in [0,T], \quad 19.$$

$$\mathbf{g}_a[\mathbf{u}_a, x_a, \dot{x}_a](t) := \begin{cases} 0, & \text{a.e. } t \in [0, b_a), \\ \frac{\mathbf{u}_a(\rho_a[x_a]^{-1}(t))}{1 + h_a \dot{x}_a(\rho_a[x_a]^{-1}(t))}, & \text{a.e. } t \in [b_a, T + \tau_a[x_a](T)], \end{cases} \quad 20.$$

for a.e. $t \in [0, T]$. In addition, we define the summarized load at a junction—dependent on whether there is an external source—for $(c, d, v) \in \mathcal{C} \times \mathcal{D} \times \mathcal{V}$:

$$\mathbf{r}_v^{c,d} :\equiv \sum_{\tilde{a}\in\mathcal{A}_{\text{in}}(v)} \mathbf{g}_{\tilde{a}}^{c,d}[\mathbf{u}_{\tilde{a}}, x_{\tilde{a}}, \dot{x}_{\tilde{a}}] + \begin{cases} \mathbf{s}_v^{c,d} & (v,d) \in \mathcal{OD}, c \in \mathcal{C} \\ 0 & v \in \mathcal{V} \setminus \mathcal{O}, c \in \mathcal{C} \end{cases} \qquad \text{on } [0,T]. \quad 21.$$

Finally, recalling the routing operator $\mathscr{R}_a^{c,d}$ given in Definition 5 with $d \in \mathcal{D}$, the coupling condition for the connecting nodes is given for $(v, c, d) \in \mathcal{V} \times \mathcal{C} \times \mathcal{D}, a \in \mathcal{A}_{\text{out}}^d(v)$:

$$\mathbf{u}_a^{c,d} :\equiv \begin{cases} \mathbf{r}_v^{c,d} \cdot \mathscr{R}_a^{c,d}[\mathbf{x}, \mathbf{ext}] & \text{if } a \in \mathcal{OP}^d \\ 0 & \text{else} \end{cases} \qquad \text{on } [0,T], \quad 22.$$

with $\mathcal{OP}^d$ as in Equation 4. The presented set of equations will be called the DTA considered.

Equations 15–20 represent the link dynamics on every link $a \in \mathcal{A}$ in the network (see Section 3.3), while Equation 21 represents the cumulative inflow onto one node. Depending on whether the considered node $v \in \mathcal{V}$ is a source node, additional source $\mathbf{s}_v^{c,d}$ is either added or not added. Finally, Equation 22 represents the routing on the links exiting a node. This is why we prescribe the inflow $\mathbf{u}_a^{c,d}$ for $a \in \mathcal{A}_{\text{out}}^d(v)$ as all the flow having entered the node v and the source flow $\mathbf{s}$ (i.e., $\mathbf{r}_v^{c,d}$) and apply the routing operator $\mathscr{R}$. In case flow cannot be routed on the considered link because there is no connection (no path) to the destination, we instantiate the inflow as zero. This entire system is called the DTA subject to routing operators.

3.6. The Well-Posedness of Dynamic Traffic Assignment with Routing Operators

In this section, we present the well-posedness of the DTA system introduced in Definition 8 when performing the routing by the suggested Lipschitz-continuous or delayed routing operators. The study of the existence and uniqueness of solutions on the network is crucial because the routing operators themselves depend on the state of the network, so that another coupling between all equations emerges.

Theorem 2 (existence and uniqueness of the network model for routings with a delay property). Recall the setting described in Definition 8, where every routing operator $\mathscr{R}_a^{c,d}$ fulfills the conditions according to the second item in Definition 7, with time vectors $\mathbf{t}_a^{c,d} \in [0,T]^{N_a^{c,d}}$. Then there is a unique solution of the system presented in Definition 8, and the solution satisfies

$$\mathbf{x}_a \in W^{1,\infty}\left((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}\right) \qquad \forall a \in \mathcal{A}.$$

Proof. The proof can be found in theorem 3.1 of Reference 73. The basic idea consists of taking advantage of the delay. Using the delay in the routing, the routing choices depend on the network solution of previous times except for finitely many points in time, where the routing might actually depend on the network solution in real time. As this solution is already well known, there is no problem with any coupling of the routing operator with respect to the state of the network. For the time points where the routing might actually use real-time information, we can approach these points from previous times and take advantage of the Lipschitz continuity of the solution to identify the proper solution at the considered time point. □

Theorem 3 [existence and uniqueness of the network model for a (Lipschitz-) continuous routing operator]. Let the network with the link-delay dynamics as in Definition 8 be given and assume that $\mathscr{R}_a^{c,d}$ is a (Lipschitz-)continuous routing operator satisfying the first item in Definition 7. Then there exists a unique solution $\mathbf{x}_a,\ a \in \mathcal{A}$, of link-delay ODEs on the network, and the solution satisfies

$$\mathbf{x}_a \in W^{1,\infty}\left((0,T); \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|}\right) \qquad \forall a \in \mathcal{A}.$$

Proof. The proof can be found in theorem 3.4 of Reference 73 and is significantly more advanced than the proof of Theorem 2. Here, we take advantage of the result in Reference 26, where the solution of the link-delay ODE is constructed over a sequence of time steps, taking advantage of the delay property of the ODE. Combining this with the Lipschitz-continuous routing operators, we can define a self-mapping that is contractive in the uniform topology and obtain a unique fixed point on the entire network—i.e., a solution to the DTA subject to Lipschitz-continuous routing operators. □

3.7. Some Remarks

In this section, we present some remarks on shortest-path routing in real time and how the previously introduced non-anticipative routing operators can also incorporate forecasts.

Remark 1 (instantaneous shortest-path routing and instability). As claimed in Section 2.2.1, a shortest-path routing assignment in real time is often posed. However, consider only two possible routes and assume that travel time is identical for both routes at a specific time $t \in [0, T]$. Flow then must be distributed on both links simultaneously (although in what ratio is fully model dependent) as well as on other inflows of the links that are part of the considered paths. If the specific ratio is not met, then this directly contradicts shortest-path assignments. Also, when discretizing the composed model, the problem becomes evident: As the travel times will never be the same, at least numerically, in every time step either everything or nothing is sent on the outgoing links. This all-or-nothing assignment might change from time step to time step and is sensitive with respect to the discretization. For details and a counterexample of why a shortest-path assignment in continuous modeling is rather difficult to pose well, we refer readers to remark 4.24 in Reference 73.

Remark 2 (forecasting). Even though our approach does not allow a forecast in time because we assume a non-anticipatory behavior, one can make a statistical forecast with the proper model or a simplified forecast and can implement this as **ext**. As long as only information in delayed form (as suggested in the second item in Definition 7) is considered, nothing further needs to be prescribed in order to obtain a unique solution on the network. If information in real time is used and a forecast is run based on this information, then this forecast must depend in a Lipschitz-continuous way on the input datum, and the routing operator would be required to depend in a Lipschitz-continuous way on **ext**. We do not go into the details here.

3.8. Some Instantiations of Routing Operators

In this section, we present some routing operators for which our assumptions hold. Some of the examples are borrowed from Reference 73. To formulate the routing operators mathematically, we need a notation of path travel time, which we present in Definition 9.

Definition 9 (path travel time). Let $(v, d) \in \mathcal{V} \times \mathcal{D}$ as in Definition 2 be given. Then, we define for $t \in [0, T]$ the set $\boldsymbol{\mathcal{X}}^{v,d}(t)$ of the involved path flows from node v to destination d, where $\mathbf{x}_a \in C([0, T]; \mathbb{R}_{\geq 0}^{|\mathcal{C}||\mathcal{D}|})$, $a \in \mathcal{A}$, as

$$\boldsymbol{\mathcal{X}}^{v,d}(t) := \bigcup_{\mathbf{p} \in \mathcal{P}^{v,d}} \left\{ \left(\mathbf{x}_{\mathbf{p}_1}^T(t), \dots, \mathbf{x}_{\mathbf{p}_{\mathrm{len}(\mathbf{p})}}^T(t) \right)^T \right\}, \qquad t \in [0, T].$$

Let the travel time $\tau_a[x_a]$ on every arc $a \in \mathcal{A}$ and $\ell := |\mathcal{P}^{v,d}|$ be given. After redefining its components for better readability to $\tilde{\mathbf{p}}^1, \dots, \tilde{\mathbf{p}}^\ell$, we define the vector of possible travel times $\boldsymbol{\tau}^{v,d}$ from node v to destination d as

$$\begin{aligned}
\boldsymbol{\tau}^{v,d}\left[\boldsymbol{\mathcal{X}}^{v,d}\right](t) &:= \left(\sum_{i=1}^{\mathrm{len}(\tilde{\mathbf{p}}^1)} \tau_{\tilde{\mathbf{p}}_i^1}\left[x_{\tilde{\mathbf{p}}_i^1}\right](t), \dots, \sum_{i=1}^{\mathrm{len}(\tilde{\mathbf{p}}^\ell)} \tau_{\tilde{\mathbf{p}}_i^\ell}\left[x_{\tilde{\mathbf{p}}_i^\ell}\right](t) \right), \qquad t \in [0, T],\\
&= \left(\sum_{i=1}^{\mathrm{len}(\tilde{\mathbf{p}}^j)} \tau_{\tilde{\mathbf{p}}_i^j}\left[x_{\tilde{\mathbf{p}}_i^j}\right](t) \right)_{j \in \{1,\dots,\ell\}}, \qquad t \in [0, T].
\end{aligned}$$

Routing 1 (routing with a logit function). Given $v \in \mathcal{V}$ and $d \in \mathcal{D}$, we define weighted path distribution routing as follows:

$$\text{for } a \in \mathcal{A}^d_{\text{out}}(v), \quad \mathscr{R}^{c,d}_a[\mathbf{x}, \mathbf{ext}](t) := \sum_{\substack{\mathbf{p} \in \mathcal{P}^{v,d} \\ \mathbf{p}_1 = a}} \frac{e^{-(\boldsymbol{\tau}^{v,d}[\boldsymbol{\mathcal{X}}^{v,d}](t))_{\mathbf{p}}}}{\sum_{\tilde{\mathbf{p}} \in \mathcal{P}^{v,d}} e^{-(\boldsymbol{\tau}^{v,d}[\boldsymbol{\mathcal{X}}^{v,d}](t))_{\tilde{\mathbf{p}}}}}, \qquad t \in [0, T].$$

This routing operator, sometimes called a nested logit model, assigns on all exiting links at node $v \in \mathcal{V}$ a percentage of the incoming flow to the outgoing links $a \in \mathcal{A}^d_{\text{out}}(v)$ based on the travel time needed to reach the destination. The exponential terms lead to the facts that most flow is sent onto the shortest route and that a flow (even a very small flow) is assigned on all possible routes. The routing operator satisfies the first item in Definition 7, so that we obtain by Theorem 3 the existence and uniqueness of the solution on the network. Clearly, the operator can be enriched by tuning parameters.

In case of delay, a shortest-path assignment can be considered (contrary to Remark 1) and yields a unique solution on the network.

Routing 2 (shortest path with delay). Given $v \in \mathcal{V}$ and $d \in \mathcal{D}$, we define shortest-path routing for $t \in [0, T]$ and $a \in \mathcal{A}_{\text{out}}(v)^d$ with delay $\varepsilon \in \mathbb{R}_{>0}$ as

$$\mathscr{R}^{c,d}_{\tilde{a}}[\mathbf{x}, \mathbf{ext}](t) := \frac{1}{\left| \left\{ \mathbf{p}_1 : \mathbf{p} \in \arg\min_{\mathbf{p} \in \mathcal{P}^{v,d}} \boldsymbol{\tau}^{v,d}\left[\boldsymbol{\mathcal{X}}^{v,d} \circ \mathrm{P}_{[0,t-\varepsilon]}\right](t) \right\} \right|}. \tag{23.}$$

This operator assigns flow only on leaving nodes $a \in \mathcal{A}^d_{\text{out}}(v)$ with minimal travel time, and if there is more than one shortest route, it assigns flow equally. Due to the delay character, this operator satisfies the second item in Definition 7, so that we obtain by Theorem 2 the existence and uniqueness of the solution on the network. We emphasize that a real-time shortest-path delay might not be well posed, as detailed in Remark 1.

Routing 3 (variational inequalities as routing operators). With Definition 9, we can formulate the variational inequalities for routing as investigated in Section 2.2.2 as time-dependent Wardrop's conditions (93), recalling that for every origin destination pair $(v, d) \in \mathcal{V} \times \mathcal{D}$, every commodity $c \in \mathcal{C}$, and every time $t \in [0, T]$, the used paths' travel times must be smaller than or equal to the travel time of the minimal paths, or—in formulae—when the cumulative assignment $x^*_a, a \in \mathcal{A}$ is an optimal assignment, $\forall t \in [0, T]$ and $\forall (v, d) \in \mathcal{V} \times \mathcal{D}$,

$$\sum_{\mathbf{p} \in \mathcal{P}^{v,d}} \sum_{a \in \{\mathbf{p}_1, \ldots, \mathbf{p}_{\text{len}(\mathbf{p})}\}} \tau_a\left[x^*_a\right](t)\left(x_a(t) - x^*_a(t)\right) \geqq 0 \tag{24.}$$

for every x_a cumulative admissible flow (also cf. chapter 7.3 of Reference 13). Altogether, variational inequalities can also be interpreted as routing operators in the sense of Definition 7; however, our results are not applicable, as there is no Lipschitz continuity or delay property holding, and even a continuous dependency of the obtained routing operator is questionable.

For a more comprehensive list of routing operators fitting in the proposed framework, we refer readers to section 4 of Reference 73. We want to emphasize once more that the introduced class of routing operators is very broad, so many routing operators used in the literature fit in the proposed framework.

4. CONCLUSION AND FUTURE RESEARCH

In this article, we have, for an ordinary delay differential equation on the links, presented a rigorous framework for up to real-time routing based on the state of the traffic in the network and how it connects to existing research in the literature. We have shown the existence and uniqueness of solutions in the network and have considered so-called routing operators that are rather general and broadly applicable. Future research should address the following related problems:

1. Replacing the link dynamics instantiated here by ordinary delay differential equations with more advanced traffic flow models (as, e.g., in the LWR model, the ARZ model, or the recent nonlocal models presented in Section 2.1.2): As mentioned above, the additional complexity will come from the need to define the proper node models given in Section 2.1.2.1. For simple PDE models, a first approach has been made in Reference 74.
2. Numerical testing: Because the structure of the routing operators potentially depends on the status of the network at any time, every routing operator must know this status at all times, requiring a clever distribution of this information on the computational side. Additionally, because every destination must be realized as a commodity, the density on each link is at least of the dimensionality of the number of destinations (multiplied by the number of different commodities) at every time step. As there might be many links in a given network where specific destinations cannot be reached, it might be worthwhile to analyze the network first for its connectivity.
3. Testing of specific routing operators: Once the specific structure for the routing operator has been chosen, there are many parameters that can still be chosen in the routing. Therefore, a study on real data that optimizes the routing parameters accordingly would be insightful.
4. Stability analysis: For the Lipschitz-continuous routing, a stability analysis should be carried out. Slightly changing the Lipschitz routing in the proper topology and all demand, one can conclude that the solution in the full network is also close in the proper topology. Clearly, for a specific link, this question of the mentioned stability can easily be answered positively.

DISCLOSURE STATEMENT

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