River Flow Control using the Hayami Model

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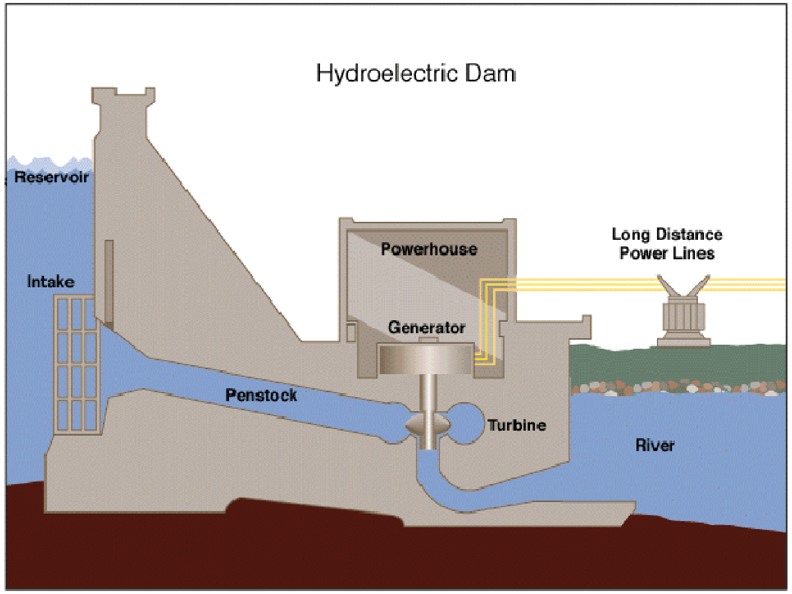
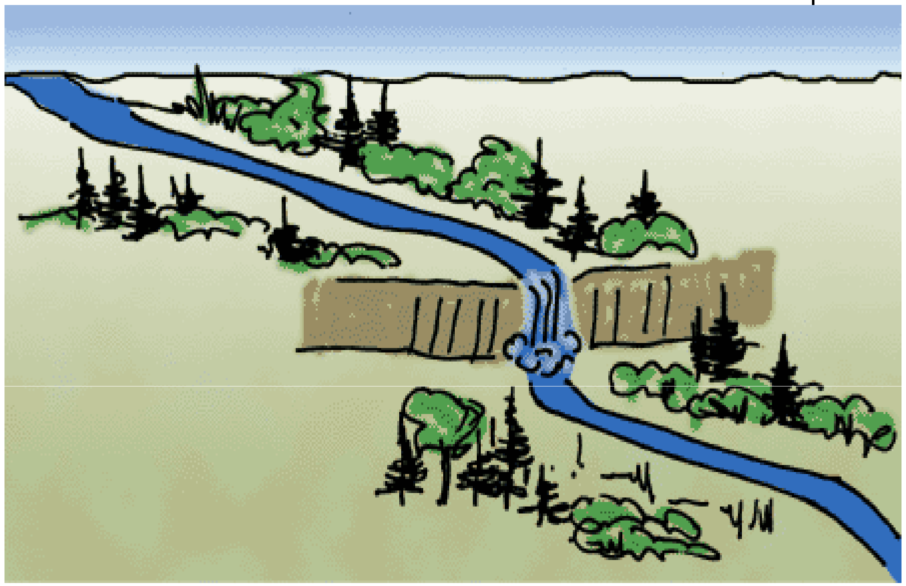
ME 236 – Professor Bayen

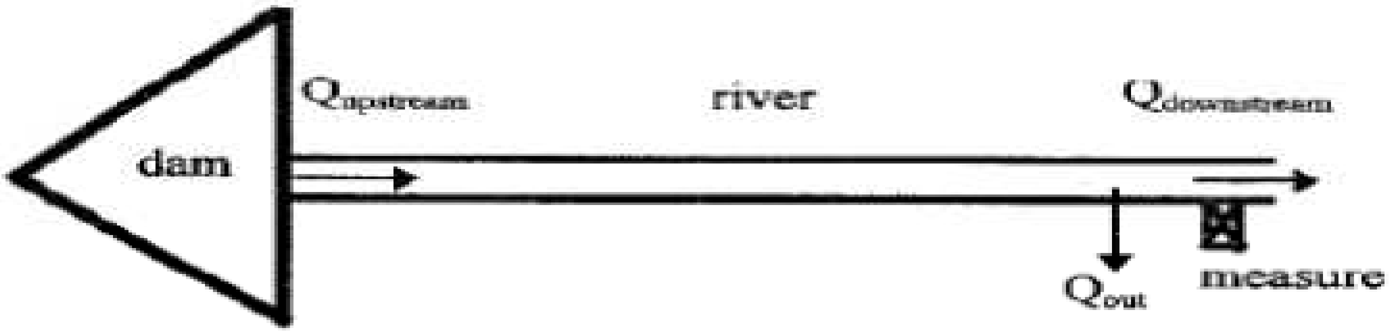
Project Purpose

- Irrigation systems waste too much water, especially during low water periods, when transported from the river to the crop fields.
- Identify a control strategy to convey the right amount of water
  - Open loop control
    - Differential flatness
  - Closed loop control
    - State-Space Model

Formulation of the Control Problem







Formulation of the Control Problem

$$\partial_t Q + C\,\partial_x Q = D\,\partial_{x,x} Q$$

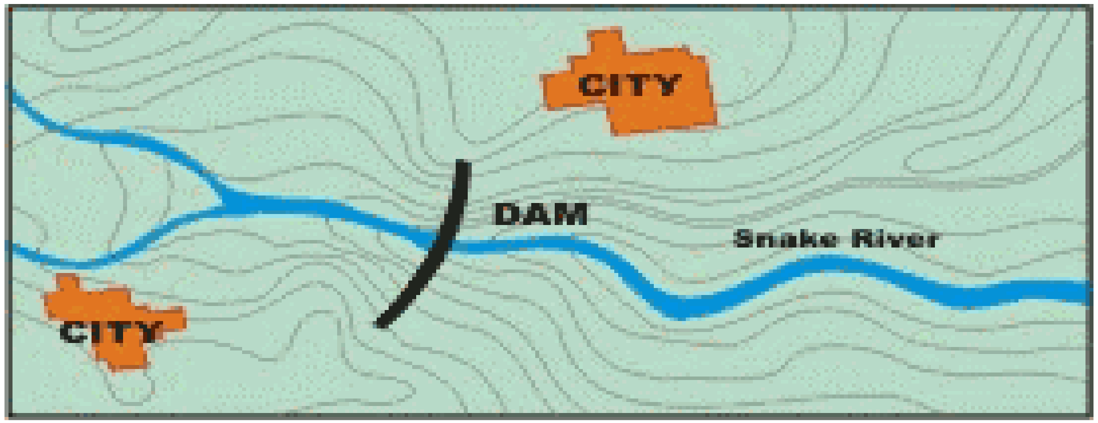
$$\partial_x Q(0, t) = u(t)$$

$$\partial_x Q(L, t) = 0$$

$$Q(x, 0) = Q_0(x)$$

$$y(t) = Q(L, t)$$

Given a flow at the reach of the river, the control input is to be determined to produce the desired output.



Formulation of the Control Problem-Open Loop

- Using Differential flatness

$$Q = \sum_{k=0}^{\infty} a_k(t) \frac{(x-L)^k}{k!}$$

$$\sum_{k=0}^{\infty} (x-L)^k [a'_k + Ca_{k+1} - Da_{k+2}] = 0$$

$$a_{k+2} = \frac{C}{D} a_{k+1} + \frac{1}{D} a'_k \qquad k \geq 0$$

$$a_k = \frac{C}{D} a_{k-1} + \frac{1}{D} a'_{k-2} \qquad k \geq 2$$

Formulation of the Control Problem-Open Loop

- Verification of the results using Flatness-based Control of a Nonlinear Parabolic Equation Modeling of a Tubular Reactor.

$$\partial_t C + \nu \partial_x C = \partial_{x,x} C$$
$$\partial_x C(0, t) = 0$$
$$C(-1, t) = u(t)$$
$$C(x, 0) = \psi(x)$$

$$C(x, t) = \sum_{k=0}^{\infty} a_k(t) \frac{(x)^k}{k!}$$

$$a_k = a'_{k-2} + \nu a_{k-1}$$

$$\partial_T Q + C \partial_X Q = D \partial_{X,X} Q$$
$$\partial_X Q(0, T) = u(T)$$
$$\partial_X Q(L, T) = 0$$
$$Q(X, 0) = Q_0(X)$$

$$Q = \sum_{k=0}^{\infty} A_k(t) \frac{(X - L)^k}{k!}$$

$$A_k = \frac{C}{D} A_{k-1} + \frac{1}{D} A'_{k-2}$$

Formulation of the Control Problem-Open Loop

Apply the transformation

$$\begin{pmatrix} x \\ t \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{L} & 0 & 0 \\ 0 & \frac{D}{L^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} X \\ T \\ Q \end{pmatrix}$$

with $\nu = \frac{C\,L}{D}$

To get: $a_k = A_k L^k$

$$a^{(n)}{}_k = \frac{L^k}{\left(\frac{D}{L^2}\right)^n} A^{(n)}{}_k$$

Substitute in:

$$a_{k+2} = a'_k + \nu a_{k+1}$$

To get:

$$L^{K+2} A_{k+2} = \frac{L^k}{\frac{D}{L^2}} A'_k + \frac{C\,L}{D} L^{k+1} A_{k+1}$$

$$A_{k+2} = \frac{C}{D} A_{k+1} + \frac{1}{D} A'_k$$

Same Result

Convergence of the series:

$$Q = \sum_{k=0}^{\infty} a_k(t) \frac{(x-L)^k}{k!}$$

The formal series converges if $y : R \longrightarrow R$ is a Gevery function of class $\alpha \leq 2$;

i.e. a C^∞ function which satisfies : $\sup_{t \in R} \left| y^{(l)}(t) \right| \leq \frac{m l!^\alpha}{\gamma^l}$ $l \geq 0$ and $\alpha \leq 2$

Proof:

1. Use the triangular inequality to show that $\sup_{t \in R} \left| y^{(l)}(t) \right| \leq \frac{m l^\alpha}{\gamma^l}$ implies $\sup_{t \in R} \left| a_k^{(l)}(t) \right| \leq \frac{m M^k (l+k)!^\alpha}{\gamma^l k!^{\alpha-1}}$

2. Apply the Cauchy – Hadamard Formula to compute the Radius of Convergence.

Result

The Radius of Convergance is $R \geq \frac{1}{M}$ where M is the largest solution of $\frac{1}{D}\frac{1}{\gamma M^2} + \frac{C}{D}\frac{1}{2M} = 1$

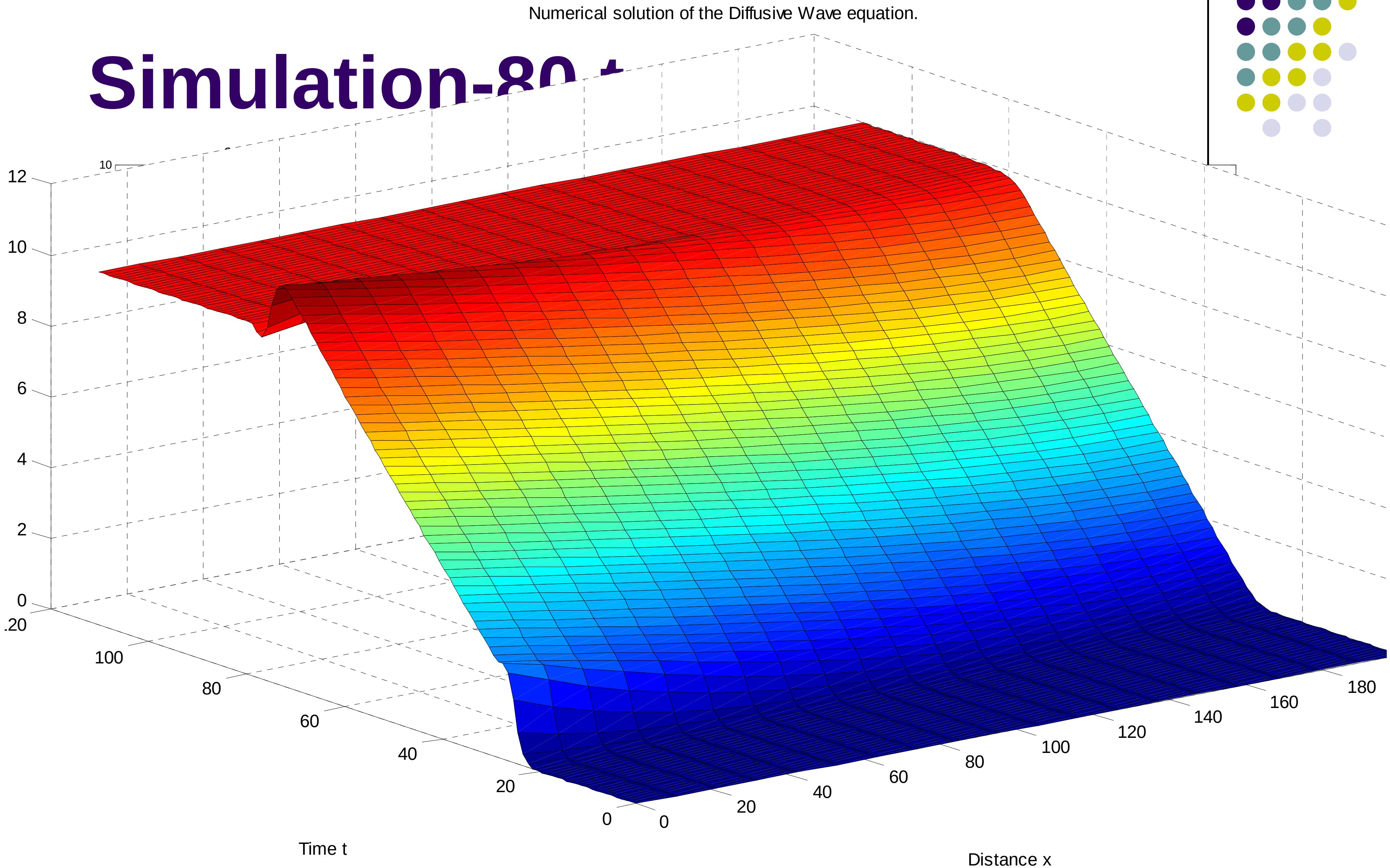
or $R \geq \dfrac{4}{\frac{C}{D} + \sqrt{\frac{16}{\gamma D} + \left(\frac{C}{D}\right)^2}}$

Since the river has length of L, we require an L radius of convergence. Hence,

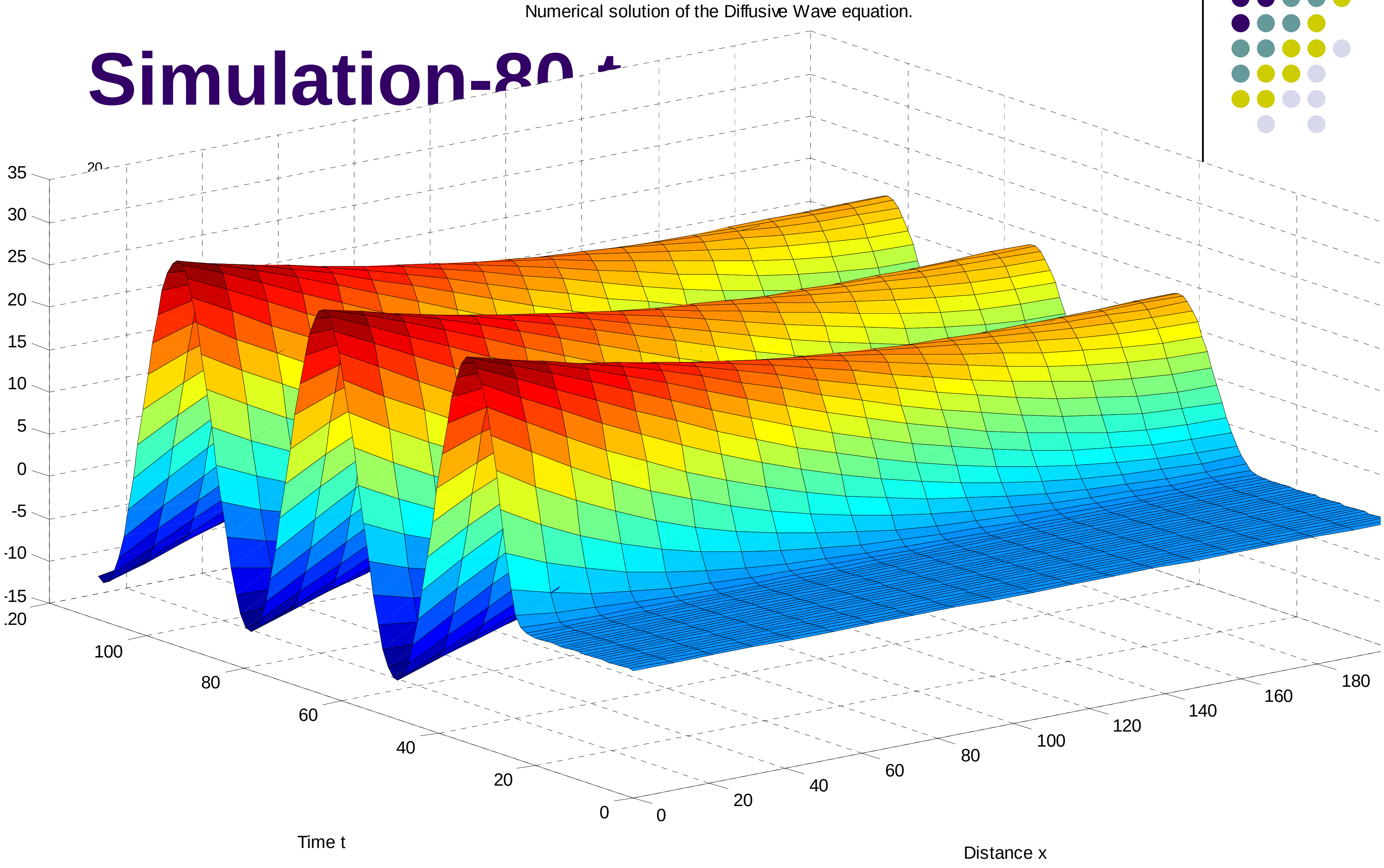
$2 \geq \frac{2L^2}{D\gamma} + \frac{CL}{D}$ $\alpha \leq 2$

Infinite if $\alpha < 2$

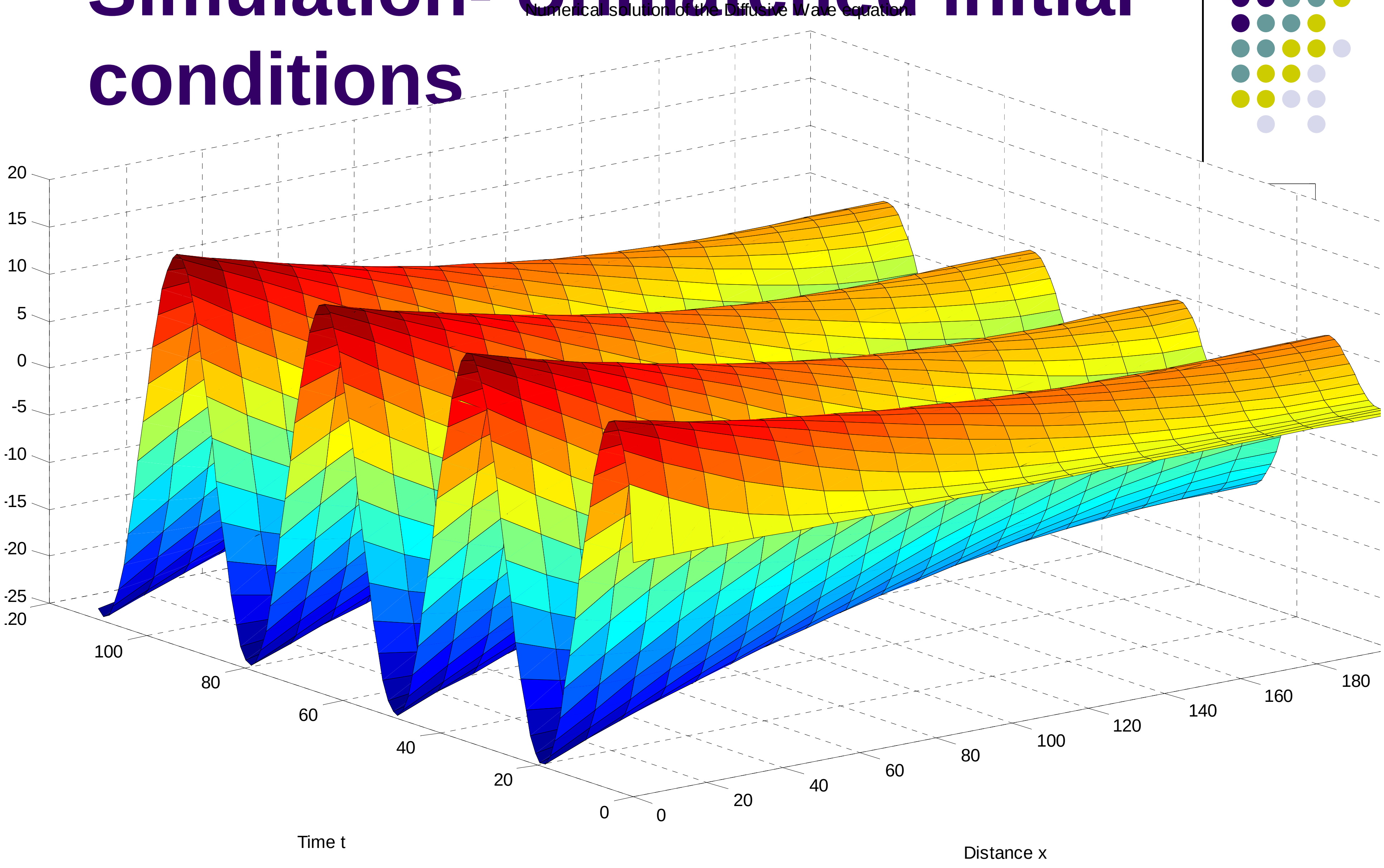
Simulation-80 t



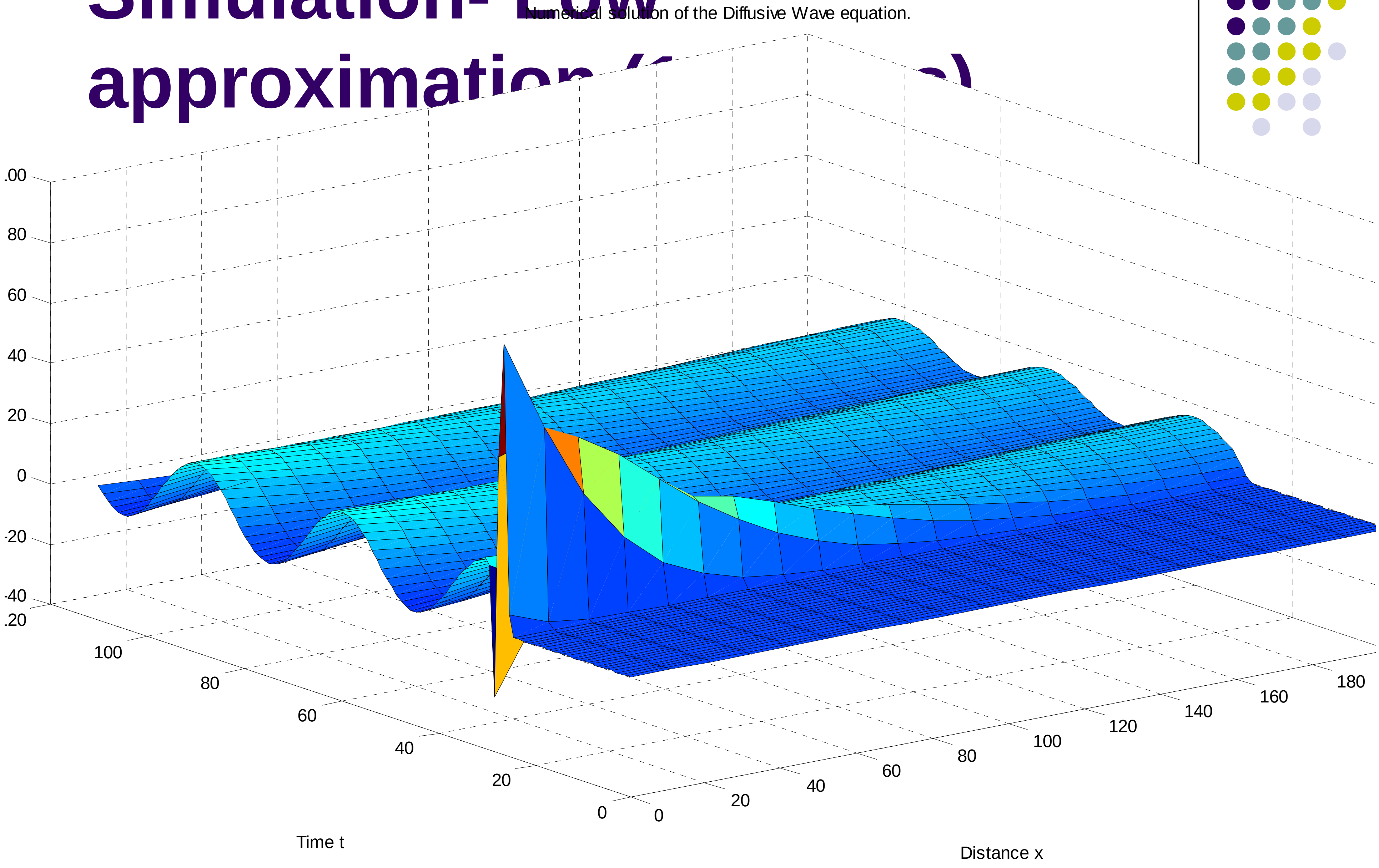
Simulation-80 t



Simulation- Unmatched initial conditions



Simulation- Low approximation



Closed Loop Control– X. Litrico

$$y(s) = F_{Hayami}(s)u(s)$$

$$F_{Hayami}(s) = e^{\frac{\Theta_e - \sqrt{\Theta_e^2 + 4E_e s}}{2E_e} X}$$

$$F_{2r}(s) = \frac{e^{-\tau s}}{1 + Ss + Ps^2}$$

$$\begin{cases} \dot{\delta x}(t) = A(\lambda)\delta x(t) + B(\lambda)\delta u(t) \\ \delta y(t) = C(\lambda)\delta x(t - \tau(\lambda)) \end{cases}$$

with $A(\lambda) = \begin{pmatrix} -\frac{S(\lambda)}{P(\lambda)} & -\frac{1}{P(\lambda)} \\ 1 & 0 \end{pmatrix}$, $B(\lambda) = \begin{pmatrix} \frac{1}{P(\lambda)} \\ 0 \end{pmatrix}$,

$C(\lambda) = \begin{pmatrix} 0 & 1 \end{pmatrix}$ and $\lambda = Q_e$.

Close loop Control – State Space Model using Differential Flatness

Think of the recursion $a_{k+2} = \frac{C}{D} a_{k+1} + \frac{1}{D} a'_k$ as a set of nonlinear odes :

$$a'[0] = D*a[2] - C*a[1]$$

$$a'[1] = D*a[3] - C*a[2]$$

$$a'[2] = D*a[4] - C*a[3]$$

Due to boundary conditions, $a'[1] = a[1] = 0. \to a[3] = \frac{C}{D} a[2]$. Morever,

all odd number series can be expressed interms of even

ones.

For example,

$$a'[3] = D*a[5] - C*a[4]$$

$$\frac{C}{D} * (a'[2]) = D*a[5] - C*a[4]$$

$$\frac{C}{D} * (D*a[4] - C*a[3]) = D*a[5] - C*a[4]$$

$$a[5] = \frac{\left(\frac{C}{D} * \left(D*a[4] - C*\frac{C}{D} a[2]\right) + C*a[4]\right)}{D} = -\frac{C^3 a[2]}{D^3} + \frac{2 C a[4]}{D}$$

Closed Loop Control – 2 Dimensional State Space

$$a'[0] = D * a[2]$$

$$a'[2] = \frac{-C^2}{D} * a[2] + D * a[4]$$

$$u(t) = \sum_{k=0}^{3} a_{k+1}(t)\frac{(-L)^k}{k!} = a[4]\frac{(-L)^3}{3!} + a[1]\frac{(-L)^0}{0!} + a[2]\frac{(-L)^1}{1!} + a[3]\frac{(-L)^2}{2!}$$

$$a[4] = \frac{3!}{(-L)^3}u(t) - \frac{3!}{(-L)^3}\left(a[1]\frac{(-L)^0}{0!} + a[2]\frac{(-L)^1}{1!} + \frac{C}{D}a[2]\frac{(-L)^2}{2!}\right)$$

$$a[4] = \frac{-3!}{L^3}u(t) - \frac{3!}{L^2}a[2] - \frac{3!}{(-L)^3}\frac{C}{D}a[2]\frac{(-L)^2}{2!}$$

$$a'[2] = \frac{-C^2}{D} * a[2] + D * \left(\frac{-3!}{L^3}u(t) - \frac{3!}{L^2}a[2] - \frac{3!}{(-L)^3}\frac{C}{D}a[2]\frac{(-L)^2}{2!}\right)$$

Closed Loop Control – 2 Dimensional State Space

$$a'[0] = D * a[2]$$

$$a'[2] = \left(\frac{-C^2}{D} - \frac{6D}{L^2} + \frac{3C}{L}\right) * a[2] - 6\frac{D}{L^3} u$$

$$\begin{pmatrix} a[0] \\ a[2] \end{pmatrix}^{(1)} = \begin{pmatrix} 0 & D \\ 0 & \left(\frac{-C^2}{D} - \frac{6D}{L^2} + \frac{3C}{L}\right) \end{pmatrix} \begin{pmatrix} a[0] \\ a[2] \end{pmatrix} + \begin{pmatrix} 0 \\ -6\frac{D}{L^3} \end{pmatrix} u$$

$$y = a[0] = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} a[0] \\ a[2] \end{pmatrix}$$

Closed Loop Control – N Dimensional State Space

Determining the A,B, and C matrices for an N-state space boils down to determing a transformation between the odd and even series. i.e.

$$\begin{pmatrix} a[1] \\ a[3] \\ \vdots \\ a[2N-1] \end{pmatrix} = T \begin{pmatrix} a[0] \\ a[1] \\ \vdots \\ a[2N-2] \end{pmatrix}$$

Closed Loop Control – N Dimensional State Space

In order to find N series, we need N-1 explicit iterations and sum minor manipulations.

The iteration is given by:

$$A_{i+1} = [I_{N\times N} - e_{i+1}^T e_{i+1}]A_i + e_{i+1}^T e_i A_i^2$$

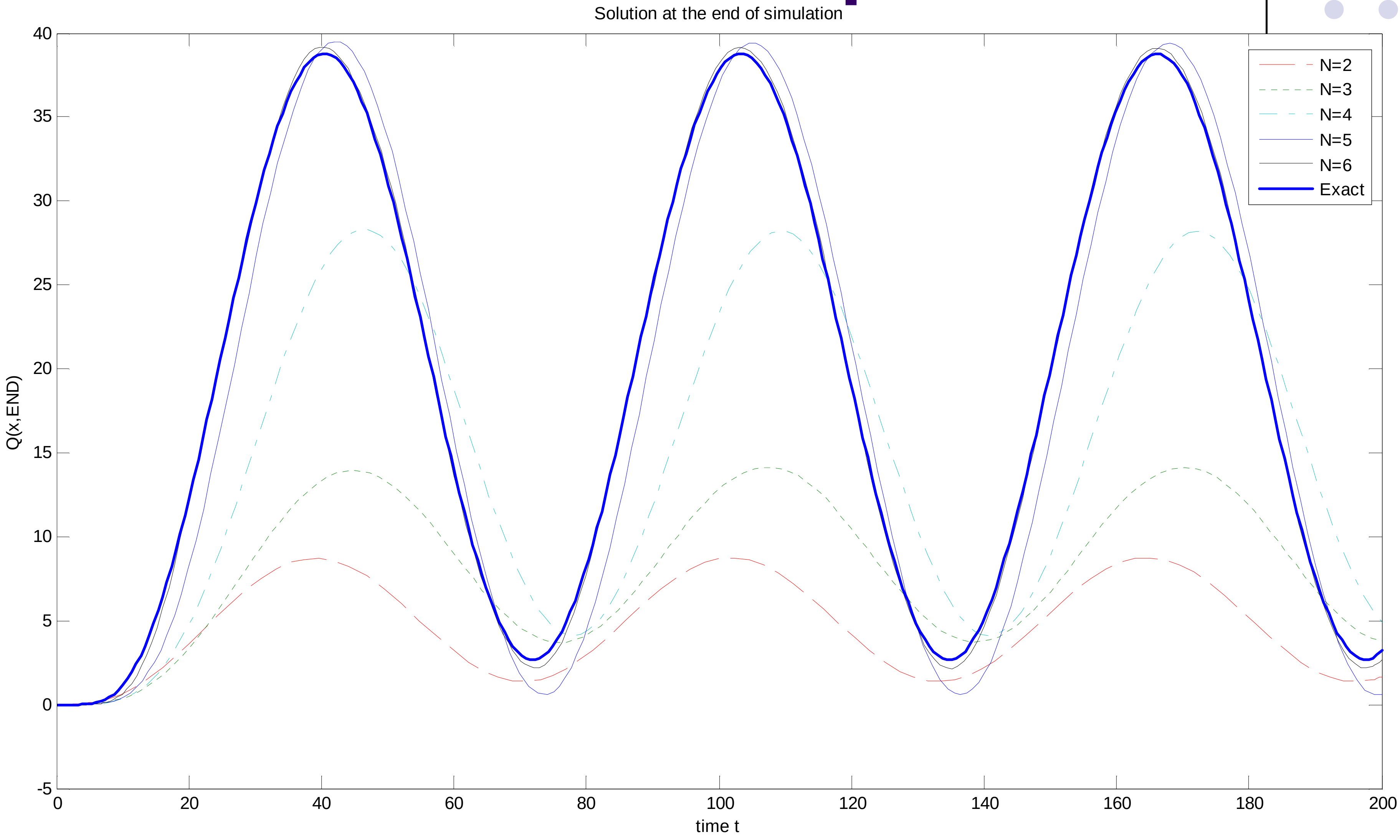
$$where \quad A_1 = \begin{bmatrix} 1 & 1 & 0 & \dots & 0 \\ 0 & 1 & 1 & & 0 \\ \vdots & & 1 & \ddots & \vdots \\ & & & \ddots & 1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$e_i = \begin{bmatrix} 0 & \cdots & e(i)=1 & 0 & \cdots & 0 \end{bmatrix}$$

Closed Loop Control – N Dimensional State Space

```
>> [A B C]=buildABC(2)

A =

[                                        0,                                      DD]
[                                        0, -1/L^2*(-3*DD*L*C+6*DD^2+C^2*L^2)/DD]


B =

         0
 -6*DD/L^3


C =

     1     0
```

Closed Loop Control – N Dimensional State Space



Comparison of Flatness based State Space with other methods

Flatness Bases State Space	Others (X. Litrico)
One approximation Used	More than one approximation
Observable, Controllable, Full State linearizable	There are no guarantees for any of the properties
Has information capable of constructing the whole flow, in space and time.	Just contains information of the reach flow
More accurate, easier to construct, can be used for other partial differential equations with negative coefficients	Limited to the diffusive wave equation with positive coefficients, needs hand calculations. i.e. can’t be automated

Conclusions

- The open-loop control has to be truncated at a certain integer, due to this fact and in view of disturbances and model errors, a closed loop control is needed
- State space model using flatness can allow observers design, inspection of state, feedback stabilization
- The discovered series for the transformation matrix allows the generation of an N symbolic state space equations.
- It is possible to extend the solution for the diffusive wave equation to others, like that of the tubular reactor.