Regrets in Routing Networks: Measuring the Impact of Routing Apps in Traffic

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The impact of the recent increase in routing apps usage on road traffic remains uncertain to this day. The article introduces, for the first time, a criterion to evaluate a distance between an observed state of traffic and the user equilibrium of the traffic assignment: *the average marginal regret*. The average marginal regret provides a quantitative measure of the impact of routing apps on traffic using only link flows, link travel times, and travel demand. In non-atomic routing games (or static traffic assignment models), the average marginal regret is a measure of selfish drivers' behaviors. Unlike the *price of anarchy*, the average marginal regret in the routing game can be computed in polynomial time without any knowledge of user equilibria and socially optimal states of traffic. First, this article demonstrates on a small example that the average marginal regret is more appropriate to define proximity between an observed state of traffic and an user equilibrium state of traffic than comparing flows, travel times, or total cost. Then, experiments on two different models of app usage and three networks (including the whole L.A. network with more than 50,000 nodes) demonstrate that the average marginal regret decreases with an increase of app usage. Sensitivity analysis of the equilibrium state with respect to the app usage ratio proves that the average marginal regret monotonically decreases to 0 with an increase of app usage. Finally, using a toy example, the article concludes that app usage could become the new Braess paradox.

CCS Concepts: • **Theory of computation** → **Algorithmic game theory**; *Convex optimization*; • **Applied computing** → *Transportation*;

Additional Key Words and Phrases: Routing games, network flows, regret, navigonational GPS routing applications

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1 INTRODUCTION

1.1 Context

The Importance of Reducing Traffic. Road traffic costs the U.S. billions in GDP every year [14]. Reducing road congestion is a way to reduce fuel consumption and emissions, to increase economic productivity, and to improve the daily life of motorists. During the past decade, new mobility services have grown with the rise of the mobile internet (*mobility as a Service*, *GPS-routing apps*, *carpooling*). These new mobility services have not translated in reduced road congestion, however.

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On the contrary, some recent traffic services have been blamed for contributing to alleged new congestion patterns [22, 30, 36, 37] due to the increase in "through traffic."

New Traffic Patterns Generated by Routing Apps. The past decade has witnessed the explosion of cell phone use [10], in particular in the context of mobility. Today, INRIX, HERE, Google, Apple, Waze, and TomTom are used by a large number of motorists [8]. These routing apps have created new traffic patterns called *cut-through* traffic [37]. As the number of app users increases, arterial roads are subjected to higher flows of vehicles due to cut-through traffic [9]. Solving this problem will require new public policy approaches, such as regulation of congestion and routing to reduce the negative externalities of cut-through traffic [26].

Modeling App Usage. In some approaches, traffic engineers have used game theory to model traffic macroscopically [28, 32]. In these models, every vehicle is assumed to follow an optimal travel time path. Since routing apps propose a fastest path [39], their impact needs to be explained with new traffic models that take into account the newly available information. Such new models investigate this problem by introducing a distinction between non-app users and app-users [9, 18, 37].

Addressing Drivers' Selfishness. The core of the problem is that mapping apps provide fastest paths to app users by design [39]. It is known that providing the best "selfish" path to users competing for the same commodity can lead to outcomes that are not socially optimal [25, 28]. The *price of anarchy* [32] can be used to measure how bad selfish routing is [33]. However, there is no known "function" that allows one to measure how "selfish" an observed traffic assignment is. Using game theory [1] and the framework of routing games [27], this article introduces a new concept to address this question: *the average marginal regret.*

1.2 Contributions

The key contributions of the article consist of the following:

- ***The average marginal regret*** for the static traffic assignment problem. From a given static traffic assignment model, this article introduces the average marginal regret. The average marginal regret is the expected time-saving a driver has in the network if they change their path to an optimal one. It enables to understand how close an observed traffic assignment is from a Nash equilibrium. This article shows that the average marginal regret can be computed with only the knowledge of the traffic demand and the link flows and link travel times of the network. To extend traffic assignment frameworks from traffic planning to real-time control, the ability to define metrics from only observable data is imperative. Note that this is not the case for the *price of anarchy* [33], where socially optimal patterns and Nash equilibrium patterns need to be known in order to compute it.
- Simulation-based assessment of the impact of navigational apps on traffic using the average marginal regret. Through simulations, we illustrate that an increase of app usage leads to a decrease in the average marginal regret of the traffic state. The traffic state converges to a Nash equilibrium when app usage increases. Due to having full information about the status of the network, drivers can actually make their choices based on Wardrop's first principle (compare Remark 2.6). Before the emergence of apps, drivers could only learn from iterations of commuting. Therefore, they would much likely never explore the "full capacity of the network," i.e., would never reach Wardrop's equilibrium.
- Theoretical convergence of the traffic state toward a Nash equilibrium when app usage increases. By modelling app usage with the restricted path choice model [9] and splitting vehicles between app users and non-app users uniformly across the network, we show that

the average marginal regret monotonically decreases to 0 when the ratio of app users increases to 100%. The traffic state converges to a Nash equilibrium when app usage increases.

1.3 Organization of the Article

The remainder of this article is organized as follows: First, the average marginal regret for the static traffic assignment is introduced in Section 2. Then, in Section 3, it is shown—using models described in [9, 37]—that an increase in app usage leads to a decrease of the average marginal regret.

2 STATIC TRAFFIC ASSIGNMENT AND THE AVERAGE MARGINAL REGRET

In this section, the classical framework of the *static traffic assignment* [28] is introduced. Then, a benchmark example is presented explaining how one can define a measure of how far a flow allocation is from a Nash equilibrium. Finally, the average marginal regret is introduced.

2.1 The Static Traffic Assignment

The static traffic assignment [28] is traditionally defined on a network $\mathcal{G}$. For each path $\mathbf{p}$ of the network, the path flow h_{p} of the path $\mathbf{p}$ is the flow of vehicles using this path $\mathbf{p}$. Assuming that the path flow allocation is static, the link flow on an edge e—the flow of vehicles using the edge e of $\mathcal{G}$—is the sum of the path flows of all paths using the edge e. Then, for a given flow demand vector $\boldsymbol{d}$—which assigns for each origin o and destination d in the network, a flow demand d_{od}—we say that a path flow allocation $\mathbf{h} = (h_{\mathrm{p}})_p$ is feasible if for any origin destination pair (o, d), the demand between o and d is equal to the sum of the flows on the paths between o and d. We assume that the travel time t_{p} of each path $\mathbf{p}$ is the sum of the travel times t_e of every edge e used by the path $\mathbf{p}$. We assume that the travel time t_e of the edge e is only a function of the link flow on the edge f_e and of the characteristics of the link e (such as length, speed limit, etc.). A flow allocation is called a *user equilibrium* if all paths used between an origin o and a destination d have the same travel time for every o, d pair (Wardrop's first condition [38]). We precise this in the following:

2.1.1 *Framework [28].*

Definition 2.1 (Network and Paths). Given a finite strongly connected directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the set of edges (links), we call, for each origin $o \in \mathcal{V}$ and destination $d \in \mathcal{V}$, $\mathcal{P}_{od}$ the set of feasible paths without cycles from o to d.

Definition 2.2 (Path Flows). For each path $\mathbf{p} \in \mathcal{P} = \bigcup_{o,d \in \mathcal{V}} \mathcal{P}_{od}$, we define:

- The flow h_{p} using path $\mathbf{p}$ (path flows). We note the path flow vector $\mathbf{h} = (h_{\mathrm{p}})_{\mathrm{p} \in \mathcal{P}}$.
- The indicator $\delta_{\mathrm{p}} = (\delta_{\mathrm{p}}(e))_{e \in \mathcal{E}}$, where $\delta_{\mathrm{p}}(e) = \begin{cases} 1 \text{ if } e \in \mathrm{p}, \\ 0 \text{ else.} \end{cases}$
  This vector is called indicator of links included in the path $\mathbf{p}$. We denote the incidence matrix $\Delta = (\delta_{\mathrm{p}})_{\mathrm{p} \in \mathcal{P}}$.

Definition 2.3 (Link Flows). For each link $e \in \mathcal{E}$, we define f_e as the flow using link e (link flow). We note the link flow vector $\mathbf{f} = (f_e)_{e \in \mathcal{E}}$.

Remark 2.1 (Static Equilibrium Model). We assume static equilibrium conditions, i.e., $\forall \mathbf{p} \in \mathcal{P},\ h_{\mathrm{p}}$ is constant over time and we have $\mathbf{f} = \Delta \mathbf{h}$ (with Δ the incidence matrix in Definition 2.2).

Definition 2.4 (Feasible Assignment). Given a demand $\boldsymbol{d} \in \mathbb{R}_+^{|\mathcal{V}| \times |\mathcal{V}|}$, we define:

- $\mathcal{H}_{\boldsymbol{d}} = \{\mathbf{h} \in \mathbb{R}_+^{|\mathcal{P}|} \mid \forall (o, d) \in \mathcal{V}^2, \sum_{\mathrm{p} \in \mathcal{P}_{od}} h_{\mathrm{p}} = \mathrm{d}_{od}\}$, the set of feasible path flow allocations.
- $\mathcal{F}_{\boldsymbol{d}} = \Delta\, \mathcal{H}_{\boldsymbol{d}} = \{\mathbf{f} \mid \exists\, \mathbf{h} \in \mathcal{H}_{\boldsymbol{d}},\ \text{s.t. } \mathbf{f} = \Delta \mathbf{h}\}$, the set of feasible link flow allocations.

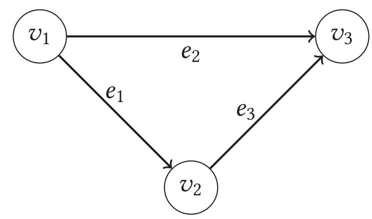


Fig. 1. Network to illustrate the framework. We have $\mathcal{V} = \{v_1, v_2, v_3\}$. $\mathcal{E} = \{e_1, e_2, e_3\}$ where $e_1 = (v_1, v_2)$, $e_2 = (v_1, v_3)$, and $e_3 = (v_2, v_3)$. $\mathcal{P}_{v_1,v_3} = \{(e_1, e_3), (e_2)\}$. If we name $\mathbf{p} = (e_1, e_3) \in \mathcal{P}_{v_1,v_3}$, then $\delta_{\mathbf{p}} = (1, 0, 1)$. $\mathcal{P} = \{(e_1), (e_2), (e_3), (e_1, e_3)\}$. If we have 100 travelers with $(o, d) = (v_1, v_3)$, then we would have $\mathcal{F}_{\boldsymbol{d}} = \{(x, 100 - x, x),\ x \in [0, 100]\}$.

Figure 1 illustrates this framework on a benchmark network.

Definition 2.5 (Travel Time Function). For each edge $e \in \mathcal{E}$, we define the edge (link) travel time function that gives the edge (link) travel time given the edge (link) flow: $t_e : f_e \rightarrow t_e(f_e)$. We assume t_e to be continuous. We then define the travel time vector $\mathbf{t}(\mathbf{f}) = (t_e(f_e))_{e \in \mathcal{E}}$.

For each path $\mathbf{p} \in \mathcal{P}$, we define the travel time function of the path p as: $t_{\mathbf{p}} : \mathbf{f} \rightarrow t_{\mathbf{p}}(\mathbf{f}) = \sum_{e \in \mathcal{E}} t_e(f_e) \cdot \delta_{\mathbf{p}}(e)$.

We sometimes refer to the travel time of a path as the cost of the path.

Remark 2.2 (Travel Time Function). During the remainder of this article, travel time functions are assumed to be increasing as functions of the corresponding link flows. More vehicles on a link increase the travel time of this specific link. The article assumes that drivers want to minimize their travel time. However, the travel time can be interpreted as any type of cost (i.e., gas emission [19], . . .), and it will not change the following results.

Definition 2.6 ([27] Non Atomic Routing Game). The set $(\mathcal{G}, \boldsymbol{d}, \mathbf{t})$ defines the non-atomic routing game [27].

Definition 2.7 ([28] User Equilibrium). For the traffic demand $\boldsymbol{d}$, a flow allocation $\mathbf{f} = \Delta\mathbf{h} \in \mathcal{F}_{\boldsymbol{d}}$ is a user equilibrium if and only if:

$$\forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od},\ h_{\mathbf{p}} \cdot \left(t_{\mathbf{p}}(\mathbf{f}) - \min_{\tilde{p} \in \mathcal{P}_{od}} t_{\tilde{p}}(\mathbf{f}) \right) = 0. \tag{1}$$

Theorem 2.8 ([27] Routing Game). *The user equilibrium is the Nash equilibrium of the non-atomic routing game $(\mathcal{G}, \boldsymbol{d}, \mathbf{t})$ [23, 27, 31, 34].*

Remark 2.3 (Routing Game). Non-atomic routing games are games with an uncountable (or continuous) number of players (referred to as potential games [23, 34]). Players are divided in types corresponding to every origin destination pair. Some notions defined in a classic non-cooperative game with a finite number of players [1, test] (like regret, ϵ-Nash equilibrium, average regret, or average-ϵ-Nash equilibrium) can be extended to potential games (uncountable number of players) using measure theoretical tools to deal with the continuous number of players.

Definition 2.9 ([11] Marginal Regret). We define the (marginal) regret of a driver as the difference between their travel times and the optimal travel times between their origins to their destinations. Because in the non-atomic routing game the number of drivers is uncountable, we define the regret for the type of drivers on a path $\mathbf{p}$ (called then regret of path $\mathbf{p}$) as:

$$r_p(\mathbf{f}) = t_{\mathbf{p}}(\mathbf{f}) - \min_{\tilde{p} \in \mathcal{P}_{od}} t_{\tilde{p}}(\mathbf{f}).$$

Definition 2.10 (ϵ-User Equilibrium). For the traffic demand $\boldsymbol{d}$, a flow allocation $\mathbf{f} = \Delta\mathbf{h} \in \mathcal{F}_{\boldsymbol{d}}$ is an ϵ-user equilibrium for $\epsilon \in \mathbb{R}_{>0}$ if and only if:

$$\forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od},\ h_{\mathbf{p}} > 0 \Rightarrow t_{\mathbf{p}}(\mathbf{f}) - \min_{\tilde{p} \in \mathcal{P}_{od}} t_{\tilde{p}}(\mathbf{f}) < \epsilon. \tag{2}$$

Remark 2.4 (ϵ-User Equilibrium). A strategy profile of a non-cooperative game is an ϵ-Nash equilibrium if no one can increase their payoff by ϵ by changing their strategy unilaterally [1]. In an ϵ-Nash equilibrium no one has a (marginal) regret above ϵ. This notion is an extension from the discrete game framework to non-atomic routing game framework using measure theoretical tools. To our knowledge, it is the first time that an ϵ-Nash equilibrium is defined for potential games.

As an ϵ-user equilibrium considers the worst case for one path, we introduce the notion of average ϵ-user equilibrium:

Definition 2.11 ([2] Average ϵ-User Equilibrium). Let $\epsilon \in \mathbb{R}_{>0}$. For the traffic demand $\boldsymbol{d}$, a flow allocation $\mathbf{f} = \Delta\mathbf{h} \in \mathcal{F}_{\boldsymbol{d}}$ is an average ϵ-user equilibrium if and only if:

$$\frac{1}{\|\boldsymbol{d}\|_1} \sum_{o,d \in \mathcal{V}} \sum_{p \in \mathcal{P}_{od}} h_{\mathbf{p}} \cdot \left(t_{\mathbf{p}}(\mathbf{f}) - \min_{\tilde{p} \in \mathcal{P}_{od}} t_{\tilde{p}}(\mathbf{f}) \right) < \epsilon. \tag{3}$$

Remark 2.5 (Average ϵ-User Equilibrium). In an average ϵ-user equilibrium, an average "player" (infinitesimal fraction of traffic) can expect to save ϵ by changing unilaterally her/his strategy (path). As highlighted in Reference [2], this notion is similar to the definition of (ϵ, δ)-Nash equilibria in Reference [15].

Definition 2.12 (Social Optimum). For the traffic demand $\boldsymbol{d}$, a flow allocation $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$ is a social optimum if and only if:

$$\forall \mathbf{f}' \in \mathcal{F}_{\boldsymbol{d}}, \quad \mathbf{t}(\mathbf{f})^\top \mathbf{f} \leq \mathbf{t}(\mathbf{f}')^\top \mathbf{f}'. \tag{4}$$

Remark 2.6 ([38] Wardrop's Conditions).

- At a ***user equilibrium,*** the cost on all routes—with the same origin and destination—used in the road network are equal, and less than the ones that would be experienced by a single player on any unused route in the network (Wardrop's first principle).
- At a ***social optimum,*** the average cost is a minimum (Wardrop's second principle).
- A ***user equilibrium*** is not necessarily a ***social optimum*** [7, 33].

2.1.2 *Variational Inequality and Minimization Problem Formulation.*

For a non-atomic routing game, Nash equilibria are known to be easy to compute, as they can be expressed as the solution to a convex optimization problem, using a convex potential function (Rosenthal function) [31].

Definition 2.13 ([28] Variational Inequality). We define the variational inequality problem as finding an $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$, such that:

$$\forall \mathbf{f}' \in \mathcal{F}_{\boldsymbol{d}},\ \mathbf{t}(\mathbf{f})^\top (\mathbf{f}' - \mathbf{f}) \geq 0. \tag{5}$$

Definition 2.14 ([31] Minimization Problem). We define the following optimization problem with the classical Rosenthal potential:

$$\min_{\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}} \sum_{e \in \mathcal{E}} \int_0^{f_e} t_e(x)\, \mathrm{d}x. \tag{6}$$

PROPERTY 2.1 ([31] INTERPRETATION AND EQUIVALENCES). *If for all links $e \in \mathcal{E}$, the link travel time functions $t_e(f_e)$ are strictly increasing functions of the link flow f_e, then the solution of the minimization problem (6), the variational inequality (5), and the user equilibrium (1) are the same: (i.e., (1) $\Longleftrightarrow$ (5) $\Longleftrightarrow$ (6)).*

2.2 Introducing the Average Marginal Regret

This section is focused on introducing a function to evaluate how far a traffic state is from a user equilibrium. We present the characteristics desired for this function. We first discuss several quantities one might consider using and explain their advantages and disadvantages. Then, we introduce a new function that best responds to this goal: **the average marginal regret**.

2.2.1 Evaluating the Distance between a Traffic State and a User Equilibrium. An observed traffic state—defined as a feasible flow allocation $\mathbf{f}$ (Definition 2.4)—could be neither a user equilibrium nor a social optimum. It can be interesting to know how far from a user equilibrium this traffic state is. For example, it would constitute a way to know whether drivers efficiently chose the route that minimizes their own travel time given the flow allocation $\mathbf{f}$. This is particularly relevant for Section 3 to experimentally assess the impact of routing apps on traffic. It can also be used to understand under which conditions the use of apps will have an impact on a given traffic assignment (given that app usage decreases regrets of drivers). To achieve this goal, we define a function $\bar{\mathcal{R}}$ that takes as an input the traffic state $\mathbf{f}$ and returns a positive real value that quantifies how far this traffic state $\mathbf{f}$ is from a user equilibrium.

Formally, given a network $\mathcal{G}$, a traffic demand $\boldsymbol{d}$, and a travel time vector function $\mathbf{t}$, $\bar{\mathcal{R}}$ should satisfy the following properties:

i. $\bar{\mathcal{R}} : \mathcal{F}_{\boldsymbol{d}} \mapsto \mathbb{R}_+$, is a function of a feasible flow allocation $\mathbf{f}$ and returns a non-negative real value.
ii. $\bar{\mathcal{R}}(\mathbf{f}) = 0 \iff \mathbf{f}$ is a user equilibrium. The function should characterize all user equilibria.
iii. If $\bar{\mathcal{R}}(\mathbf{f}) < \epsilon$, then $\mathbf{f}$ is at an average-ϵ-user equilibrium for $\epsilon \in \mathbb{R}_{>0}$ given.
iv. For every $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$, $\bar{\mathcal{R}}(\mathbf{f})$ is tractable, i.e., $\bar{\mathcal{R}}(\mathbf{f})$ can be computed in polynomial time with respect to $|\mathcal{E}|$ and $|V|$.
v. $\bar{\mathcal{R}}$ is a continuous function of the flow allocation $\mathbf{f}$.

These properties should satisfy for every network $\mathcal{G}$, traffic demand $\boldsymbol{d}$ and travel time vector function $\mathbf{t}$.

Remark 2.7 (Distance). As we want $\bar{\mathcal{R}}$ to be only a function of the observed state of traffic $\mathbf{f}$, it cannot be defined as a mathematical distance. Section 2.2.2 presents why defining $\bar{\mathcal{R}}(\mathbf{f}) = \min_{\mathbf{f}^{\text{ue}} \in S_{\text{ue}}} d(\mathbf{f}, \mathbf{f}^{\text{ue}})$—where S_{ue} is the set of user equilibrium flow allocation and d is a metric like $d(x, y) = \|x - y\|_2$—would not satisfied properties we want for $\bar{\mathcal{R}}$.

2.2.2 A Benchmark Example to Give Some Context and Refute Possible Nash Gap Quantifier Candidate Functions. In this section, we present different functions that can be considered to quantify the gap between an observed state of traffic and a user equilibrium. We present a benchmark network to show that the average marginal regret is more appropriate to define $\bar{\mathcal{R}}$ than the other candidates of functions.

The Case Considered. Let us consider the network in Figure 2. The network consists of two nodes (O and D) and two paths (p_1 and p_2). The cost of each path depends on the flow on this path (f_1 and f_2). The functions are given in Figure 2: $t_1(f_1) = 1 + f_1$ and $t_2(f_2) = M + 2 + f_2$ with $M \geq 0$ given. We consider a demand of $d_{OD} = 1$ between O and D. The user equilibrium flow can be directly determined from the Wardrop first principle: $f_1^{\text{ue}} = 1$ and $f_2^{\text{ue}} = 0$. We note $\mathbf{f}^{\text{ue}}$ the vector $\mathbf{f}^{\text{ue}} = (f_1^{\text{ue}}, f_2^{\text{ue}})$. Now, imagine that we observe a traffic flow $\hat{\mathbf{f}}$ where $\hat{f}_1 = 1 - \alpha$ and $f_2^{\text{ue}} = \alpha$ with $0 \leq \alpha \leq 1$ given. We seek to understand what kind of functions could serve as good candidates (in the sense of properties (i), (ii), (iii), (iv), and (v) defined in Section 2.2.1) to measure how far the observed flow $\hat{\mathbf{f}}$ is from a Nash equilibrium (here $\mathbf{f}^{\text{ue}}$).

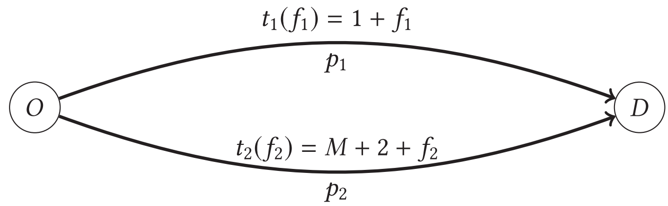


Fig. 2. Benchmark network example to illustrate the average marginal regret.

Inadequacy of a Flow-based Function. A first intuition approach might consist of comparing the link flows between the observed state $\hat{\mathbf{f}}$ and the Nash equilibrium flow $\mathbf{f}^{ue}$: $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \|\mathbf{f}^{ue} - \hat{\mathbf{f}}\|_2$. In the case considered, we have that $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \sqrt{2}\alpha$. The function $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ satisfies properties (i), (ii), and (iv) for the case we consider. But the function $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ does not satisfy the property (iii). Indeed, $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ does not depend on M (or on the differences on the cost of both paths). Therefore, we consider that a flow-based function is not relevant to quantify a gap to Nash.

Inadequacy of a Cost-based Function. Another possible approach would be to consider the travel time function $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \|\mathbf{t}(\mathbf{f}^{ue}) - \mathbf{t}(\hat{\mathbf{f}})\|_2$. In the case considered, the travel time function is just a translation of the flow vector, so we still have $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \sqrt{2}\alpha$. For the same reasons than the flow-based function, we consider the cost-based function inadequate.

Inadequacy of a Hybrid-based Function. A third approach is to consider norm on $\mathbf{f} \cdot \mathbf{t}(\mathbf{f})$: $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = |\mathbf{f}^{ue} \cdot \mathbf{t}(\mathbf{f}^{ue}) - \hat{\mathbf{f}} \cdot \mathbf{t}(\hat{\mathbf{f}})|$ (like in Reference [29]). But, as seen in Definition 2.12, $\mathbf{f} \cdot \mathbf{t}(\mathbf{f})$ is the total travel time on the network (the sum of the travel times of all cumulated flows). So, any function on $\mathbf{f} \cdot \mathbf{t}(\mathbf{f})$ will be related to the social optimality of the solution. This will not satisfy the fact that we want to measure a gap between a traffic state $\mathbf{f}$ and a user equilibrium. In the case considered, $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \alpha|M - 1 + 2\alpha|$. If $|M - 1 + 2\alpha| = 0$, $\hat{\mathbf{f}}$ is not a Nash equilibrium, but $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = 0$. So, $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ does not satisfy the property (ii).

Inadequacy of a Price of Anarchy-like Function. Considering an idea similar to the price of anarchy [33], one can define $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \frac{\hat{\mathbf{f}} \cdot \mathbf{t}(\hat{\mathbf{f}})}{\mathbf{f}^{ue} \cdot \mathbf{t}(\mathbf{f}^{ue})}$, where $\mathbf{f}^{ue}$ is a user equilibrium. This is similar to the hybrid-based function: $\frac{\hat{\mathbf{f}} \cdot \mathbf{t}(\hat{\mathbf{f}})}{\mathbf{f}^{ue} \cdot \mathbf{t}(\mathbf{f}^{ue})} = 1 + \frac{1}{\mathbf{f}^{ue} \cdot \mathbf{t}(\mathbf{f}^{ue})} \cdot (\hat{\mathbf{f}} \cdot \mathbf{t}(\hat{\mathbf{f}}) - \mathbf{f}^{ue} \cdot \mathbf{t}(\mathbf{f}^{ue}))$. Therefore, this type of function is equivalent to a hybrid-based function and thus is inadequate for our purpose.

The Worst "Marginal Regret" of Vehicles Function. Another type of approach is to define $\bar{\mathcal{R}}$ using the game theoretical framework. Using marginal regret, we do not need to know every user equilibrium flow allocation $\mathbf{f}^{ue}$ to find out whether a state of traffic is a user equilibrium. A user equilibrium is defined as a state of traffic where nobody can achieve a better travel time by being the only one changing their route (Wardrop's first condition 2.6): i.e., no one has marginal regret. One idea is to define $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ as the worst marginal regret of drivers in the observed traffic assignment. This is equivalent to define $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ as the smallest ϵ such that $\hat{\mathbf{f}}$ is a ϵ-Nash equilibrium:

$$\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \max_{o,d} \max_{\substack{p \in \mathcal{P}_{od} \\ h_p > 0}} \left(t_p(\hat{\mathbf{f}}) - \min_{\tilde{p} \in \mathcal{P}_{od}} \mathbf{t}_{\tilde{p}}(\hat{\mathbf{f}}) \right).$$

As the marginal regret is defined for a path flow allocation h_p and not a link flow allocation $\mathbf{f}$, property (i) is not satisfied. Assuming h_p is known, properties (i) and (ii) are satisfied. As $\frac{\|\cdot\|_1}{n} \leq \|\cdot\|_\infty$, property (iii) is also satisfied. Satisfying property (iv) is more complicated, as the number of paths of a network is generally an exponential function of the number of nodes $|V|$ and the number of links $|\mathcal{E}|$.

Property (v) is not satisfied. For the considered case, we have that $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \begin{cases} M + 2\alpha & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0 \end{cases}$. We see that $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ is not continuous in $\hat{\mathbf{f}}$. If $\alpha = 0$, then $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = 0$. However, if $\alpha = 0^+$, then $\bar{\mathcal{R}}(\hat{\mathbf{f}}) = M$.

The Average "Marginal Regret" of Vehicles Function. Based on the definition of the average-ϵ-Nash equilibrium, we define $\bar{\mathcal{R}}(\hat{\mathbf{f}})$ as the smallest ϵ such that $\hat{\mathbf{f}}$ is at average-ϵ-Nash equilibrium:

$$\bar{\mathcal{R}}(\hat{\mathbf{f}}) = \frac{1}{\|\boldsymbol{d}\|_1} \cdot \sum_{(o,d)\in\mathcal{V}^2} \sum_{\mathrm{p}\in\mathcal{P}_{od}} h_{\mathrm{p}} \cdot (t_{\mathrm{p}}(\hat{\mathbf{f}}) - \pi_{od}(\hat{\mathbf{f}})),$$

where $\pi_{od}(\hat{\mathbf{f}}) = \min_{\tilde{p}\in\mathcal{P}_{od}} t_{\mathrm{p}}(\hat{\mathbf{f}})$. We will use this function as $\bar{\mathcal{R}}$ in the remainder of the article.
Section 2.2.3 shows that the function satisfies properties (i), (ii), (iii), (iv), and (v).

2.2.3 The Average Marginal Regret. In this section, we formulate **the average marginal regret**, which quantifies how much time an average driver can expect to save by changing their path to the optimal one. Then, some properties of the average marginal regret are presented. In particular, the average marginal regret satisfies the properties in Section 2.2.1.

Definition 2.15 (Best Path, Optimal Flow Pattern). Given a flow allocation $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$, which provides the cost vector $\mathbf{t}(\mathbf{f})$, we define:

- An optimal path between o and d, as $\mathbf{p}^*_{od}(\mathbf{f}) \in \operatorname{argmin}_{\mathrm{p}\in\mathcal{P}_{od}} \delta_{\mathrm{p}}^\top \mathbf{t}(\mathbf{f}) = \mathcal{P}^*_{od}(\mathbf{f})$.
- An all-or-nothing allocation $\mathbf{y}(\mathbf{f})$ based on the travel times at the flow $\mathbf{f}$, as $\mathbf{y}(\mathbf{f}) = \sum_{o,d\in\mathcal{V}} \mathrm{d}_{od} \cdot \delta_{\mathrm{p}^*_{od}(\mathbf{f})}$ for $\delta_{\mathrm{p}^*_{od}(\mathbf{f})} \in \mathcal{P}^*_{od}(\mathbf{f})$. In this definition, the full od demand d_{od} is allocated to an optimal path (between o and d) computed with the current flow allocation $\mathbf{f}$.

Remark 2.8 (Existence and Non-uniqueness). For all $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$, $\mathbf{p}^*_{od}(\mathbf{f})$ and $\mathbf{y}(\mathbf{f})$ exist but might not be unique.

We define the average marginal regret as the inner product of the travel time vector and the actual flow allocation minus the all-or-nothing flow allocation normalized with the total demand.

Definition 2.16 (Average Marginal Regret). We define the average marginal regret of the flow pattern $\mathbf{f} \in \mathcal{F}_{\boldsymbol{d}}$ as follows:

$$\bar{\mathcal{R}}(\mathbf{f}) := \frac{1}{\|\boldsymbol{d}\|_1} \mathbf{t}(\mathbf{f})^\top (\mathbf{f} - \mathbf{y}(\mathbf{f})), \tag{7}$$

where $\|\boldsymbol{d}\|_1 = \sum_{o,d\in\mathcal{V}} \mathrm{d}_{od}$ and $\mathbf{y}(\mathbf{f})$ is an all-or-nothing solution as in Definition 2.15.

Remark 2.9 (Measuring the Average Marginal Regret). Because the average marginal regret is a function of only the link flow, the link travel time, and the traffic demand, it can be accessed with loop detectors and demand survey. Knowing the path flows is not required to measure the average marginal regret. The function $\bar{\mathcal{R}}$ satisfies the property (iv) in Section 2.2.1.

After defining the average marginal regret, we introduce its properties.

Definition 2.17 (Shortest Travel Time). Any optimal path $\mathbf{p}^*_{od}(\mathbf{f})$ (see Definition 2.15) is a shortest path (with respect to cost) between o and d given the cost on each link $\mathbf{t}(\mathbf{f})$: $\mathbf{t}(\mathbf{f})^\top \delta_{\mathrm{p}^*_{od}(\mathbf{f})} = \min_{\mathrm{p}\in\mathcal{P}_{od}} \mathbf{t}(\mathbf{f})$. For every $o, d \in \mathcal{V}$, we define:

$$\pi_{od}(\mathbf{f}) := \mathbf{t}(\mathbf{f})^\top \delta_{\mathrm{p}^*_{od}(\mathbf{f})}.$$

Remark 2.10 (Interpretation of the Average Marginal Regret). Since $t_{\mathrm{p}}(\mathbf{f}) - \pi_{od}(\mathbf{f})$ represents the time a driver on path $\mathbf{p} \in \mathcal{P}_{od}$ could save by choosing the best path for their trip, $\bar{\mathcal{R}}$ can be interpreted as the average time a driver could save by changing unilaterally their path.

Using $\mathbf{f} = \Delta\mathbf{h}$, we have:

$$\mathbf{t}(\mathbf{f})^\top \mathbf{f} = \sum_{(o,d)\in\mathcal{V}^2} \sum_{\mathrm{p}\in\mathcal{P}_{od}} h_{\mathrm{p}} \cdot t_{\mathrm{p}}(\mathbf{f}),$$

$$\mathbf{t}(\mathbf{f})^\top \mathbf{y}(\mathbf{f}) = \sum_{(o,d)\in\mathcal{V}^2} \mathbf{t}(\mathbf{f})^\top \delta_{\mathrm{p}^*_{od}(\mathbf{f})} \cdot \mathrm{d}_{od} = \sum_{(o,d)\in\mathcal{V}^2} \pi_{od}(\mathbf{f}) \cdot \mathrm{d}_{od},$$

$$\bar{\mathcal{R}}(\mathbf{f}) = \frac{1}{\|\boldsymbol{d}\|_1} \cdot \sum_{(o,d)\in\mathcal{V}^2} \left(\left(\sum_{\mathrm{p}\in\mathcal{P}_{od}} h_{\mathrm{p}} \cdot t_{\mathrm{p}}(\mathbf{f}) \right) - \mathrm{d}_{od} \cdot \pi_{od}(\mathbf{f}) \right),$$

$$\sum_{\mathrm{p}\in\mathcal{P}_{od}} h_{\mathrm{p}} = \mathrm{d}_{od} \Rightarrow \quad \bar{\mathcal{R}}(\mathbf{f}) = \frac{1}{\|\boldsymbol{d}\|_1} \cdot \sum_{(o,d)\in\mathcal{V}^2} \sum_{\mathrm{p}\in\mathcal{P}_{od}} h_{\mathrm{p}} \cdot \left(t_{\mathrm{p}}(\mathbf{f}) - \pi_{od}(\mathbf{f}) \right).$$

Note that this shows that $\bar{\mathcal{R}}$ is defined even if $\mathbf{y}(\mathbf{f})$ is not unique.

Property 2.2 [The Average Marginal Regret is a Positive Real Value and Characterizes All User Equilibria]. *As $\mathbf{p}^*_{od}(\mathbf{f})$ is the fastest path between o and d, we have*

$$\frac{1}{\|\boldsymbol{d}\|_1} \cdot \mathbf{t}(\mathbf{f})^\top (\mathbf{f} - \mathbf{y}(\mathbf{f})) = \max_{\mathbf{x}\in\mathcal{F}_{\boldsymbol{d}}} \frac{1}{\|\boldsymbol{d}\|_1} \cdot \mathbf{t}(\mathbf{f})^\top (\mathbf{f} - \mathbf{x}) \geq 0. \tag{8}$$

Thus:

$$\bar{\mathcal{R}}(\mathbf{f}) = 0 \iff \forall \mathbf{f}' \in \mathcal{F}_{\boldsymbol{d}},\ \mathbf{t}(\mathbf{f})^\top (\mathbf{f}' - \mathbf{f}) \geq 0. \tag{9}$$

Equation (8) implies that $\forall \mathbf{f} \in \mathcal{F}_{\boldsymbol{d}},\ \bar{\mathcal{R}}(\mathbf{f}) \geq 0$: $\bar{\mathcal{R}} : \mathcal{F}_{\boldsymbol{d}} \mapsto \mathbb{R}_+$, is a function of a feasible flow allocation and returns a positive real value (property (i) in Section 2.2.1).

Equation (9)—using the variational inequality definition of user equilibrium (Definition 2.13)—provides that $\bar{\mathcal{R}}(\mathbf{f}) = 0 \iff \mathbf{f}$ is a user equilibrium (property (ii) in Section 2.2.1).

Remark 2.11. The variational inequality tells us that none of the players can have a better outcome by choosing a different path in isolation. The average marginal regret identifies which travel time any player could expect to save by rerouting.

Property 2.3 [The Average Marginal Regret as a Measure of Driver Efficiency]. *Given $\epsilon \in \mathbb{R}_{>0}$, from Remark 2.10, it is straightforward that $\forall \mathbf{f} \in \mathcal{F}_{\boldsymbol{d}},\ \bar{\mathcal{R}}(\mathbf{f}) \leq \epsilon \iff \mathbf{f}$ is an average-ϵ-Nash equilibrium (Definition 2.11). This is property (iii) in Section 2.2.1.*

So, the average marginal regret is a good way to characterize how close to a user equilibrium the state of traffic is.

A player is defined as efficient if they take one best route between their origin and destination as their path. Then $\bar{\mathcal{R}}$ can be interpreted as a measure of the efficiency of the drivers.

The closer $\bar{\mathcal{R}}$ is to 0, the less inclined players are to change their paths. If $\bar{\mathcal{R}} = 0$, we are at a user equilibrium.

Property 2.4 [Continuity]. *The average marginal regret $\bar{\mathcal{R}}(\mathbf{f})$ is continuous with respect to $\mathbf{f}$. Property (v) in Section 2.2.1 is satisfied.*

Proof. We have $\bar{\mathcal{R}}(\mathbf{f}) = \frac{1}{\|\boldsymbol{d}\|_1}\mathbf{t}(\mathbf{f})^\top(\mathbf{f} - \mathbf{y}(\mathbf{f})) = \frac{1}{\|\boldsymbol{d}\|_1}(\mathbf{t}(\mathbf{f})^\top\mathbf{f} - \min_{\tilde{\mathbf{f}}\in\mathcal{F}_{\boldsymbol{d}}} \mathbf{t}(\mathbf{f})^\top\tilde{\mathbf{f}})$. Because $\mathbf{t}(\mathbf{f})$ is continuous with respect to $\mathbf{f}$ (Definition 2.5), it suffices to show that $\min_{\tilde{\mathbf{f}}\in\mathcal{F}_{\boldsymbol{d}}} \mathbf{t}(\mathbf{f})^\top\tilde{\mathbf{f}}$ is continuous with respect to $\mathbf{f}$. This is a linear program (LP) ($\mathcal{F}_{\boldsymbol{d}}$ defined in Definition 2.3 is a polytope). The optimal objective value of an LP is continuous with respect to perturbation on the objective function [6]. □

Remark 2.12 (Computational Time). An optimal path $\mathbf{p}_{od}^*$ can be found in $O(|\mathcal{E}| \cdot \log(|\mathcal{V}|))$ with Dijkstra's algorithm. $\bar{\mathcal{R}}$ can be found in $O(|\mathcal{E}| \cdot |\mathcal{V}| \log(|\mathcal{V}|))$ with a sequential application of $|\mathcal{V}|$ Dijkstra's algorithms. Therefore, the average marginal regret satisfies property (iv).

The average marginal regret $\bar{\mathcal{R}}$ can be computed with "local data" only, i.e., link cost, link flow, and demand. Definition 2.16 shows that we only need to compute the inner product of the travel time vector and the difference between the flow allocation and the all-or-nothing flow allocation given the current travel time vector. To compute the all-or-nothing flow allocation, only the demand and the current travel time vector are needed.

ALGORITHM 1: Frank Wolfe's algorithm: the average marginal regret as a criterion of convergence

Data: $(\mathcal{V}, \mathcal{E})$, $\epsilon \in \mathbb{R}_{>0}$, $\boldsymbol{d} \in \mathbb{R}_+^{|\mathcal{V}| \times |\mathcal{V}|}$, $\mathbf{t} \in \mathbb{R}_+^{|\mathcal{E}|} \to \mathbb{R}_+^{|\mathcal{E}|}$
Set k = 1;
Take any $\mathbf{f}^k \in \mathcal{F}_{\boldsymbol{d}}$;
while $\bar{\mathcal{R}}(\mathbf{f}^k) > \epsilon$ **do**
  $k = k + 1$;
  $\mathbf{f}^k = \mathbf{f}^{k-1} + \frac{1}{k} \cdot (\mathbf{y}(\mathbf{f}^{k-1}) - \mathbf{f}^{k-1})$;
end
Result: $\mathbf{f}^k$

Remark 2.13 (Frank Wolfe's Algorithm [2, 16]). The user equilibrium can be seen as a routing game [27]. It has been shown that a no-regret learning algorithm in selfish routing converges to a user equilibrium of the system [2, 3, 20].

For solving the minimization problem (6), we can use Frank Wolfe's algorithm [16], a projected gradient descent algorithm. This algorithm minimizes $\bar{\mathcal{R}}$ over each iteration of the algorithm. It is equivalent to a no-regret learning algorithm [2]:

At the termination of this algorithm, we have $\bar{\mathcal{R}}(\mathbf{f}) \leq \epsilon$. Here, ϵ is an input parameter of the algorithm that represents the accuracy threshold on the value of $\mathbf{f}$.

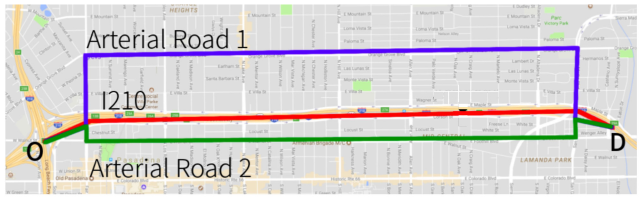
3 ADDRESSING THE IMPACT OF NAVIGATIONAL APPS ON TRAFFIC USING THE AVERAGE MARGINAL REGRET

The aim of this section is to apply the previous considerations to the impact of navigational apps on traffic. The average marginal regret is used to achieve this goal. First, we present traffic models that take into account app usage [9, 37], used as a baseline to apply our framework. Then, we show on simulations that the average marginal regret decreases when app usage increases. Finally, sensitivity analysis of the equilibrium state of the restricted path choice model with respect to the app usage ratio proves that the average marginal regret monotonically decreases to 0 with an increase of app usage.

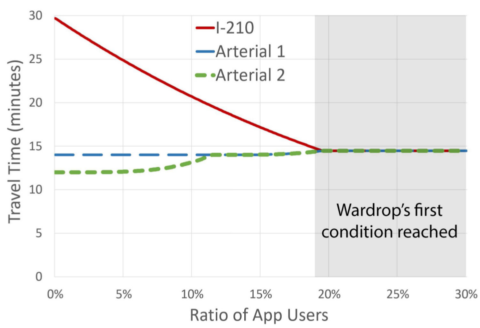
3.1 Modeling App Usage

We use two models from previous works to capture app usage [9, 37]. Simulations of these models are used to compute the evolution of the average marginal regret with the increase of app usage in the next section.

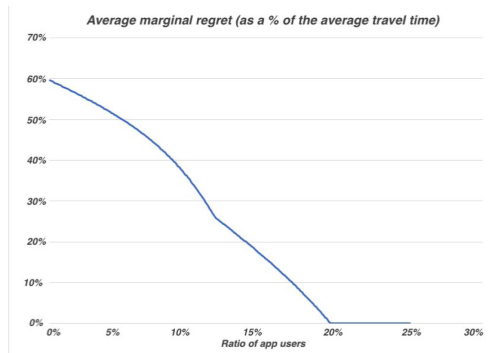
Both models separate drivers into populations: the app users and non-app-users. It is assumed that both types of drivers want to minimize their own travel time. In both models, app users are assumed to know the travel time of every path of the network. The demand is assumed to be static, and so the travel time of paths. The difference between the two models is the way to model non-app users. In the first one, the *cognitive cost model* [37], non-app users pay a "cognitive cost" to



(a) The benchmark network



(b) Path travel time



(c) The average marginal regret

Fig. 3. Benchmark network (above) and path travel times (on the left) and the average marginal regret (on the right) as a function of the percentage of app usage. On a benchmark network, the traffic converges to a user equilibrium state when app usage increases.

access arterial roads. In the second one, the *restricted path choice model* [9], non-app users stay on paths given by road signs.

3.2 App Usage Leads to a Decrease of the Average Marginal Regret

In this section, we compute the user equilibrium of the restricted path choice model for different ratios of app users/non-app users on two networks. First, on a benchmark network with three paths, we show that the average marginal regret converges to 0 as app usage increases. Then, on the full L.A. network, we show the same phenomenon.

The Bureau of Public Roads (BPR) Function [4]. For each link, we define a link capacity $c_e \in \mathbb{R}_{>0}$, which is roughly the total flow a link can have before being congested [4]. Then, we define the link travel function as $t_e(f_e) = \frac{d_e}{v_e}(1 + 0.15(\frac{f_e}{c_e})^4)$ with $d_e \in \mathbb{R}_{>0}$ and $v_e \in \mathbb{R}_{>0}$ the length and the free-flow speed of the link. $\frac{d_e}{v_e}$ is the free-flow travel time of the link e. The function t_e is referred to as the BPR function in transportation literature [4, 28].

Open-source Code to Solve Static Traffic Assignment Used for the Work. Transportation networks with road capacity, free-flow travel times, and travel demand are available in open-source form on the website noted in Reference [35]. A static traffic assignment solver can be found on the website noted in Reference [21]. Open Street Map and OSMNX [5] can be used to model the considered road network.

3.2.1 App Usage on a Benchmark Network. The first computation (shown in Figure 3) studies the impact of the number of app users on the traffic state. The highway capacity is 6,000 veh./h. The arterial road capacity is fixed at 2,000 veh./h. The *od* demand is set to 20,000 veh./h (more than

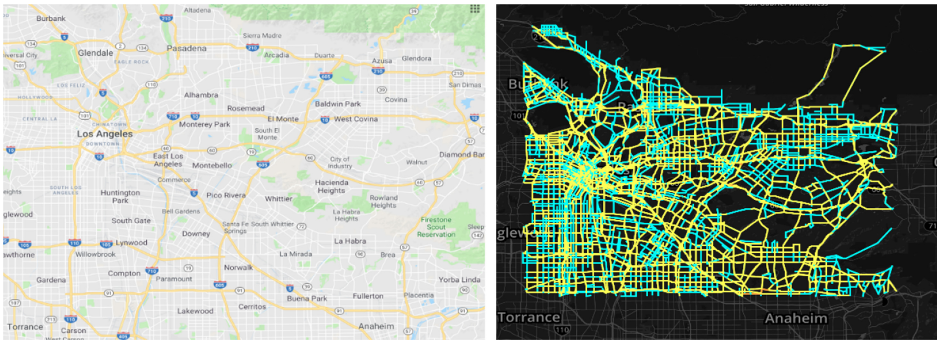


Fig. 4. L.A. network considered for Section 3.2.2. On the left: a map of the L.A. basin; on the right: the graph we use to model the L.A. basin.

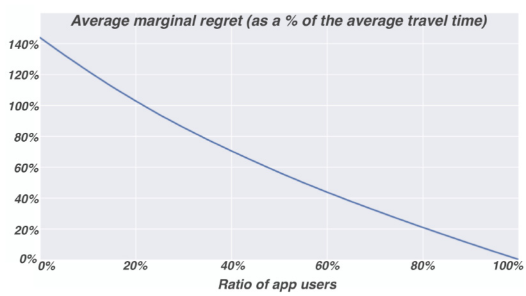


Fig. 5. Average marginal regret as a function of the percentage of app usage on the L.A. network. The average marginal regret decreases monotonically to 0 when app usage increases.

the capacity to observe congestion). We split the demand between the two populations of drivers: app users $\boldsymbol{d}_a$ and non-app users $\boldsymbol{d}_{na}$. We have $\boldsymbol{d}_a + \boldsymbol{d}_{na} = \boldsymbol{d}$ and $\boldsymbol{d}_a = \alpha\, \boldsymbol{d}$ with $\alpha \in [0, 1]$. We call α the ratio (or percentage) of app users. Using the restricted path choice model [9] (and Section 3.3), non-app users stay on the highway regardless of the traffic conditions.

With 0% routed users, the entire flow stays on the highway. As the ratio of app users increases, app users start using Arterial Road 2 (AR2), because it is faster than the congested highway. This transfer relieves the freeway but increases congestion on AR2. When the travel time on AR2 becomes as high as the travel time on Arterial Road 1 (AR1), app users start taking AR1 as well. Travel times stop evolving when app usage reaches 18%, which corresponds to a travel time equalization phenomenon: in these conditions, no app user can reroute to decrease their travel time and Wardrop's first condition is reached.

The average marginal regret decreases with the increase of app usage. It reaches 0 when app usage reaches 18%; at this point, the user equilibrium condition is reached.

3.2.2 App Usage on the L.A. Network. The second simulation is performed on the L.A. network (Figure 4). Simulations use the cognitive cost static model [37] with app user percentages ranging from 0% to 100%, with a 1% increment. For each of these simulations, traffic demand data is collected from the American Community Survey, composed of 96,077 *od* pairs. The network is built from Open Street Map. Traffic demand is set consistently at rush-hour levels to find the effects of app usage when networks are congested.

We see that the average marginal regret decreases monotonically with the increase of navigational app usage (Figure 5). The fact that the decrease is monotonic is important here. This shows

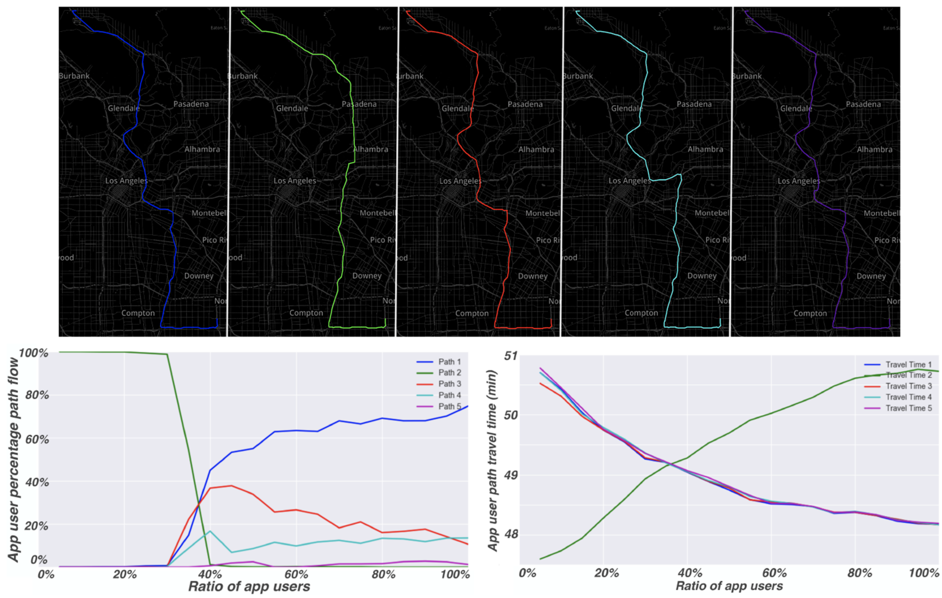


Fig. 6. Impact of the increase of app usage on path choice and path travel time for a specific (o, d pair) pair with the increase of app usage. Above: the five main paths are used by app users. The blue path is the main path used by non-app users. Below on the right: the travel time of the five paths as a function of app usage. Below on the left: the percentage of flow of app users on the five paths as a function of app usage. When there are no app users, every vehicle uses the highway. The green side road is a shortcut for app users. When there is more than 35% of app users, the green path is not a shortcut anymore. This path gets congested because of other motorists that use this path for their trips. App users always use paths that have the smallest travel time.

that, whatever the percentage of app users is, the traffic will be closer to a user equilibrium when app usage increases.

Remark 3.1. For every simulation run and on every type of network, the average marginal regret monotonically decreases with the increase of navigational app usage. This is not the case with the price of anarchy, which depends on the network configuration.

We show the evolution of path flow for a particular o, d pair. This (o, d pair) has been chosen to be one of the (o, d pairs) with the highest demand. This particular (o, d pair) starts slightly southeast of Compton and ends just north of Burbank. Figure 6 shows the top five paths taken for this o, d pair. Almost all of these paths take the SR 2 through Glendale. One takes the I-210 through Pasadena (the green one).

In the 0% to 35% app usage range, almost all app users take the green path, which is the fastest. But then, with 35% app usage, app users begin to take other paths, particularly the blue (Path 1) and red (Path 3) ones. 35% app usage is exactly when the travel time of path green, blue, and red equalize. App users always follow the fastest paths. Then, after 35% app usage, the travel time of the other paths fall below that of the green path and all app users leave the green path for other paths.

Remark 3.2 (Travel Time Evolution). It is important to see that here the travel time of these paths depends on other o, d pairs. Even after 35% app usage, when no rerouting occurs, the path travel time still varies, Mainly because drivers from other o, d pairs still change their path while the ratio of app usage increases.

3.3 Theoretical Convergence of the Restricted Path Choice Model to Nash with the Increase of App Usage

Let us model traffic assignment with app usage with the restricted path choice model [9]. App users possess perfect knowledge of the path set $\mathcal{P}_{od}$ between every origin $o \in V$ and destination $d \in V$ (see Definition 2.1). Non-app users route themselves on a non-empty subset $\mathcal{P}_{od}^{na}$ of the possible path $\mathcal{P}_{od}$ ($\mathcal{P}_{od}^{na} \subset \mathcal{P}_{od}$) between every origin $o \in V$ and destination $d \in V$. The app users' path flow vector is denoted by $\mathbf{h}^a$ and the non-app users' path flow vector by $\mathbf{h}^{na}$ (see Definition 2.2). We note $\mathbf{h} = \mathbf{h}^a + \mathbf{h}^{na}$. Let $\alpha \in [0, 1]$ be the ratio (of percentage) of app users (as in Section 3.2.1). If $\alpha = 1$, the restricted path choice model is equivalent to the static traffic assignment defined in Section 2.1 and Reference [28]. In this case, every vehicle possesses perfect information and user equilibrium is reached.

Definition 3.1 ([9] User Equilibrium of the Restricted Path Choice Model). Wardrop's first condition for the restricted path choice model can be expressed as:

$$
\begin{aligned}
h_{\mathbf{p}}^a \cdot \left(t_{\mathbf{p}}(\Delta \mathbf{h}) - \pi_{od}^a\right) &= 0, && \forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od}, && (10)\\
h_{\mathbf{p}}^{na} \cdot \left(t_{\mathbf{p}}(\Delta \mathbf{h}) - \pi_{od}^{na}\right) &= 0, && \forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od}^{na}, && (11)\\
h_{\mathbf{p}}^a &\geq 0, && \forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od}, && (12)\\
h_{\mathbf{p}}^{na} &\geq 0, && \forall o, d \in \mathcal{V},\ \forall \mathbf{p} \in \mathcal{P}_{od}^{na}, && (13)\\
\pi_{od} &\geq 0, && \forall o, d \in \mathcal{V}, && (14)\\
\pi_{od}^{na} &\geq 0, && \forall o, d \in \mathcal{V}, && (15)\\
\sum_{p \in \mathcal{P}_{od}^{na}} h_{\mathbf{p}}^{na} &= (1 - \alpha)\mathrm{d}_{od}, && \forall o, d \in \mathcal{V}, && (16)\\
\sum_{p \in \mathcal{P}_{od}} h_{\mathbf{p}}^a &= \alpha \mathrm{d}_{od}, && \forall o, d \in \mathcal{V}. && (17)
\end{aligned}
$$

With the assumption of strictly increasing travel time functions (as in Property 2.1), only a unique flow allocation satisfies the above Wardrop's condition (Reference [9, Property 4.2] and Reference [6]). Following Remark 2.1, Definition 2.2, and Definition 2.4, we denote it $\mathbf{f}_\alpha^\star = \Delta(\mathbf{h}^{a,\star} + \mathbf{h}^{na,\star})$. For this flow allocation, app users are routed on the shortest path inside $\mathcal{P}_{od}$ and non-app users routed themselves on the shortest path inside $\mathcal{P}_{od}^{na}$. Therefore, app users do not have "regrets" while non-app users "regret" to not know $\mathcal{P}_{od}$. We can express the average marginal regret associated with $\mathbf{f}_\alpha^\star$ (as in Remark 2.10):

$$\bar{\mathcal{R}}(\mathbf{f}_\alpha^\star) = \sum_{o, d \in \mathcal{V}} \sum_{p \in \mathcal{P}_{od}} h_{\mathbf{p}} \cdot \left(t_{\mathbf{p}}(\mathbf{f}_\alpha^\star) - \min_{\tilde{p} \in \mathcal{P}_{od}} t_{\tilde{p}}(\mathbf{f}_\alpha^\star)\right),$$

$$\textit{Equation } (10), \textit{Equation } (11) \Rightarrow \bar{\mathcal{R}}(\mathbf{f}_\alpha^\star) = \sum_{o, d \in \mathcal{V}} \sum_{p \in \mathcal{P}_{od}} h_{\mathbf{p}}^a \cdot (\pi_{od}(\mathbf{f}_\alpha^\star) - \pi_{od}(\mathbf{f}_\alpha^\star)) + h_{\mathbf{p}}^{na} \cdot \left(\pi_{od}^{na}(\mathbf{f}_\alpha^\star) - \pi_{od}(\mathbf{f}_\alpha^\star)\right),$$

$$\pi_{od}(\mathbf{f}_\alpha^\star) = \pi_{od}(\mathbf{f}_\alpha^\star) \Rightarrow \bar{\mathcal{R}}(\mathbf{f}_\alpha^\star) = \sum_{o, d \in \mathcal{V}} \sum_{p \in \mathcal{P}_{od}} h_{\mathbf{p}}^{na} \cdot \left(\pi_{od}^{na}(\mathbf{f}_\alpha^\star) - \pi_{od}(\mathbf{f}_\alpha^\star)\right).$$

Then, Equation (16) gives:

$$\bar{\mathcal{R}}(\mathbf{f}_\alpha^\star) = (1 - \alpha) \sum_{o, d \in \mathcal{V}} \mathrm{d}_{od} \cdot \left(\pi_{od}^{na}(\mathbf{f}_\alpha^\star) - \pi_{od}(\mathbf{f}_\alpha^\star)\right). \tag{18}$$

Similarly to satisfying property (v) of the average marginal regret (continuity with respect to the link flow allocation, see Section 2.2.1), we are interested in the continuity of $\bar{\mathcal{R}}\,(\mathbf{f}^\star_\alpha)$ with respect to α.

Theorem 3.2 [Continuity of the Average Marginal Regret With the Ratio of App Users]. *The average marginal regret $\bar{\mathcal{R}}\,(\mathbf{f}^\star_\alpha)$ is continuous as a function of α.*

Proof. This follows from the continuity of the average marginal regret (see Property 2.4) and of the continuity of $\mathbf{f}^\star_\alpha$ with respect to α. The continuity of $\mathbf{f}^\star_\alpha$ with respect to α is due to the convexity of the restricted path choice model (Definition 3.1, see Reference [9, Property 4.2] for the convexity, and Reference [17, Theorem 1] for a more detailed proof). □

Theorem 3.3 [Monotonicity and Convergence to Nash]. *For α_1, α_2 such that $0 \le \alpha_1 \le \alpha_2 \le 1$:*

$$\bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha_2}) \le \bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha_1}) \text{ and } \lim_{\alpha_2 \to 1} \bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha_2}) = 0.$$

Proof. First, given that $\bar{\mathcal{R}}\,(\mathbf{f}^\star_\alpha)$ is continuous with α (Theorem 3.2), $\bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha=1}) = 0$ (Equation (18)) gives that $\lim_{\alpha \to 1} \bar{\mathcal{R}}\,(\mathbf{f}^\star_\alpha) = 0$.

Then, we can use the sensitivity analysis of the travel cost $\pi_{od}(\mathbf{f}^\star_\alpha)$ and $\pi^{na}_{od}(\mathbf{f}^\star_\alpha)$ with respect to α (as in References [13, 24]). By using Equation (18), we have:

$$\begin{aligned}
\bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha_1}) - \bar{\mathcal{R}}\,(\mathbf{f}^\star_{\alpha_2}) = (1-\alpha_1) & \sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot \left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_1})\right) \\
& - (1-\alpha_2) \sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot \left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2}) - \pi_{od}(\mathbf{f}^\star_{\alpha_2})\right), \\
= (\alpha_2 - \alpha_1) & \sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot \left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_1})\right) \\
& + (1-\alpha_2) \sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot \left(\left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2})\right) - \left(\pi_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_2})\right)\right).
\end{aligned}$$

Because $\alpha_1 \le \alpha_2$ and $\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) \ge \pi_{od}(\mathbf{f}^\star_{\alpha_1})$, then $(\alpha_2 - \alpha_1) \sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot (\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_1})) \ge 0$.

Using Dafermos sensitivity analysis of travel cost with respect to the demand [13, Theorem 4.2], we will show that $\sum_{o,d \in \mathcal{V}} \mathrm{d}_{od} \cdot ((\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2})) - (\pi_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_2}))) \ge 0$. Since $(1 - \alpha_2) \ge 0$, it will complete the proof.

Changing the problem (in Definition 3.1) into a stationary traffic assignment problem by vectorizing it, we denote $\tilde{\pi}_{o,d} = (\pi_{od}(\mathbf{f}^\star_{\alpha_1}),\ \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}))$, $\tilde{\mathrm{d}}_{o,d} = (\alpha_1\ \mathrm{d}_{od},\ (1-\alpha_1)\ \mathrm{d}_{od})$, and $\tilde{\pi}^\star_{o,d} = (\pi_{od}(\mathbf{f}^\star_{\alpha_2}),\ \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2}))$, $\tilde{\mathrm{d}}^\star_{o,d} = (\alpha_2\ \mathrm{d}_{od},\ (1-\alpha_2)\ \mathrm{d}_{od})$. This notation is inspired by Dafermos [13]. Then, the Dafermos sensitivity analysis of the travel cost with respect to the demand [13, Theorem 4.2] gives:

$$\sum_{o,d \in \mathcal{V}} (\tilde{\pi}^\star_{o,d} - \tilde{\pi}_{o,d})^\top (\tilde{\mathrm{d}}^\star_{o,d} - \tilde{\mathrm{d}}_{o,d}) \ge 0.$$

Going back to previous notations:

$$\sum_{o,d \in \mathcal{V}} (\pi_{od}(\mathbf{f}^\star_{\alpha_2}) - \pi_{od}(\mathbf{f}^\star_{\alpha_1}))^\top ((\alpha_2 - \alpha_1)\ \mathrm{d}_{o,d}) - \left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2}) - \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1})\right)^\top ((\alpha_2 - \alpha_1)\ \mathrm{d}_{o,d}) \ge 0,$$

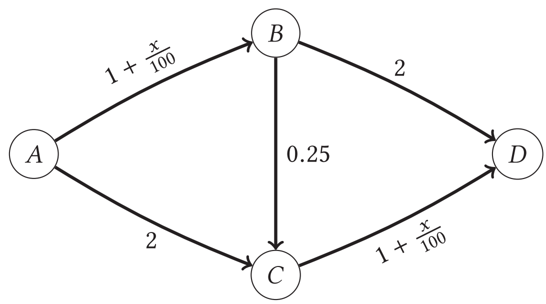


Fig. 7. Braess network considered. Cost on every link are given as a function of the link flow.

$$(\alpha_2 - \alpha_1) \sum_{o,d \in \mathcal{V}} \mathrm{d}_{o,d} \cdot \left(\left((\pi_{od}(\mathbf{f}^\star_{\alpha_2}) - \pi_{od}(\mathbf{f}^\star_{\alpha_1})) - \left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2}) - \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) \right) \right) \right) \geq 0,$$

$$\sum_{o,d \in \mathcal{V}} \mathrm{d}_{o,d} \cdot \left(\left(\left(\pi^{na}_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi^{na}_{od}(\mathbf{f}^\star_{\alpha_2}) \right) - \left(\left(\pi_{od}(\mathbf{f}^\star_{\alpha_1}) - \pi_{od}(\mathbf{f}^\star_{\alpha_2}) \right) \right) \right) \right) \geq 0.$$

This shows the claim $\bar{\mathcal{R}}(\mathbf{f}^\star_{\alpha_1}) \geq \bar{\mathcal{R}}(\mathbf{f}^\star_{\alpha_2})$. □

The average marginal regret decreases monotonically to zero when the ratio of app users increases uniformly using the restricted path choice model.

3.4 When App Usage is Worse for Everybody

This section shows on a toy example that the increase of app usage could lead to a situation worse for everybody (even for the app users). Indeed, Section 3.2 shows that the increase of app usage leads the traffic to converge to a Nash equilibrium. It is known that Nash equilibria traffic assignment can be worse than socially optimal traffic assignment [25, 28, 33]. In traffic, this is known as the Braess paradox [7]. We claim that routing apps can reproduce the Braess paradox!

To this end, let us consider a network with four nodes (A, B, C, D) and three paths (ABD, ACD, and $ABCD$). The demand is set to 100 vehicles that want to go from A to D. Link cost functions are defined in Figure 7. This is equivalent to the experiment of Reference [12].

App users are routed on the shortest path between $ABCD$, ABD, and ACD. Non-app users are assumed to not know the path BC. Therefore, they are routed on the shortest path between ABD and ACD.

We observe the evolution of the travel time of app users and non-app users as a function of the percentage of app users (Figure 8).

In this particular case, even if the usage of apps decreases the average marginal regret, the travel time of every vehicle increases with an increase of app usage. This toy example shows that the use of apps might not be beneficial for society. Indeed, the article shows that app usage leads the state of traffic to converge to a Nash equilibrium (using the average marginal regret as quantifier), which usually is not socially optimal.

Remark 3.3 ([36] The Average Marginal Regret for Dynamic Traffic Assignment). The average marginal regret can also be defined and evaluated for the dynamic traffic assignment problem. Simulations using Aimsun show that the average marginal regret also decreases when app usage increases in a dynamic traffic environment [36]. However, more research is required to prove properties like monotonically decrease of the marginal regret when the percentage of app users increases in a time-dynamical environment.

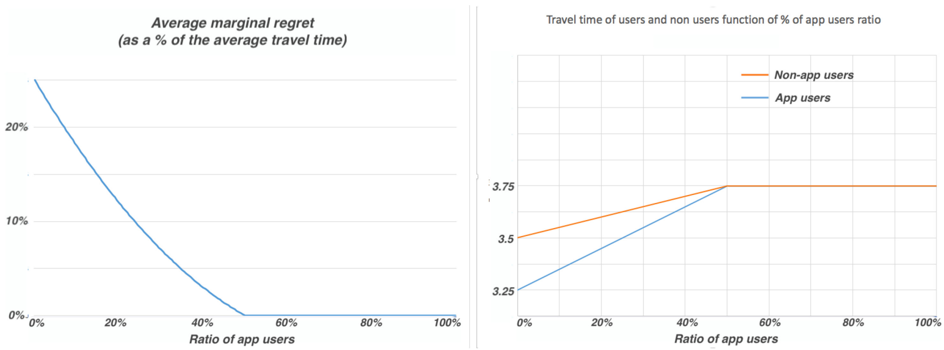


Fig. 8. Impact of the increase of app usage on the Braess network: everybody gets a worse travel time when app usage increases. On the left: the average marginal regret as a function of app usage. The average marginal regret of the drivers decreases monotonically when app usage increases. On the right: the travel time of app users (blue) and non-app users (orange) as a function of app usage. The travel time of every traveller (non-app users and app users) increases when app usage increases.

4 CONCLUSION

In conclusion, this article introduces the average marginal regret, which is a measure of how selfish drivers behave. The average marginal regret is defined in order to characterize user equilibria. This article shows that the average marginal regret is the best candidate for quantifying how far an observed state of traffic is from a Nash equilibrium. Simulations show that app usage increase leads to a decrease of the average marginal regret. This article proves that the average marginal regret decreases monotonically to 0 when app usage increases using the restricted path choice model. This proves that the traffic state converges to a Nash equilibrium when app usage increases. As a Nash equilibrium is not necessarily socially optimal, app usage can have a bad impact on drivers' travel times. A toy example is presented, where app usage translates in a worse travel time for everyone.

Future work will define the concept of average-ϵ-Nash equilibrium from a measure theory point of view and try to extend the current work to the dynamic traffic assignment framework [18].

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