Real-time Estimation of Flow States in Open Channels via Lagrangian Sensing

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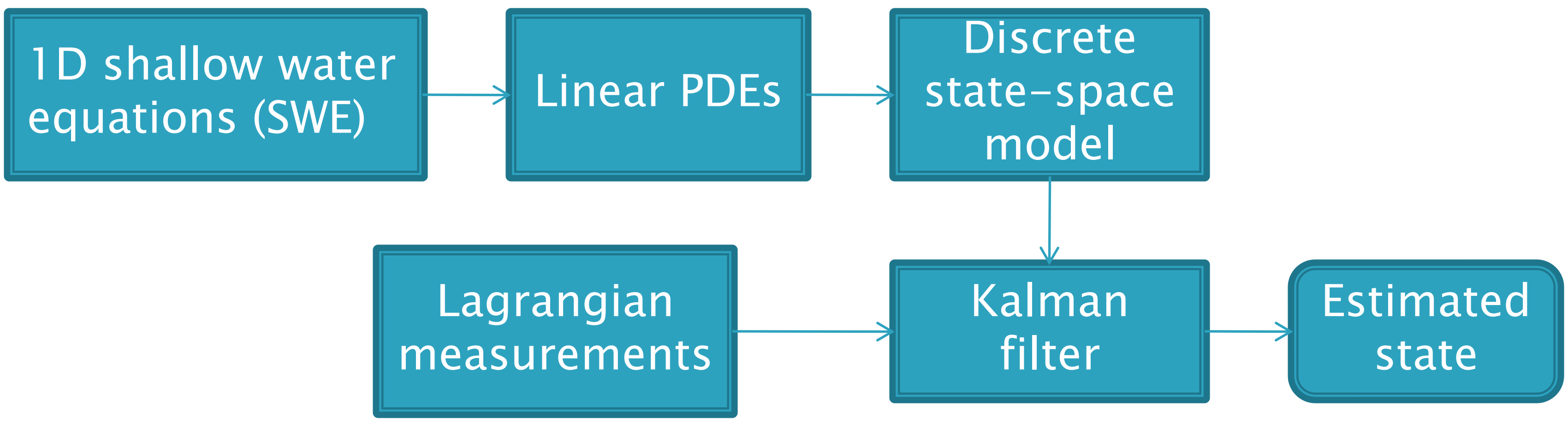
ME 236 – Spring 2009

Outline

- Problem statement
- Mathematical model
  - 1D Saint-Venant equations
  - Linearization & Discretization
  - State-space model
- Estimation set-up
- Implementation and Numerical results
- Future work

Problem Statement

- Real-time estimation of flow states (average velocity and average stage)
  - Based on a 1D model
  - Measurements obtained from Lagrangian sensors (drifters)



Governing equations

1D Saint-Venant equations:

$$T\frac{\partial H}{\partial t} + \frac{\partial (THV)}{\partial x} = 0$$
$$\frac{\partial V}{\partial t} + V\frac{\partial V}{\partial x} + g\frac{\partial H}{\partial x} + g(S_f - S_b) = 0$$

Friction slope:

$$S_f = \frac{m^2 V|V|}{(H)^{4/3}}$$

Linearization of the PDEs

$$V(x,t) = \bar{V}(x) + v(x,t)$$
$$H(x,t) = \bar{H}(x) + h(x,t)$$

Backwater curves (steady state):

$$\frac{d\bar{V}(x)}{dx} = -\frac{\bar{V}(x)}{\bar{H}(x)}\frac{d\bar{H}(x)}{dx} - \frac{\bar{V}(x)}{T(x)}\frac{dT(x)}{dx}$$
$$\frac{d\bar{H}(x)}{dx} = \frac{S_b - \bar{S}_f}{1 - \bar{F}(x)^2}$$

Carrying out the following approximation:

$$f(V,H) \approx f(\bar{V},\bar{H}) + (f_V)\,|_{(\bar{V},\bar{H})} v(x,t) + (f_H)\,|_{(\bar{V},\bar{H})} h(x,t)$$

Linearization of the PDEs

Linearized PDEs:

$$h_t + \bar{H}(x)v_x + \bar{V}(x)h_x + \alpha(x)v + \beta(x)h = 0$$
$$v_t + \bar{V}(x)v_x + gh_x + \gamma(x)v + \eta(x)h = 0$$

where

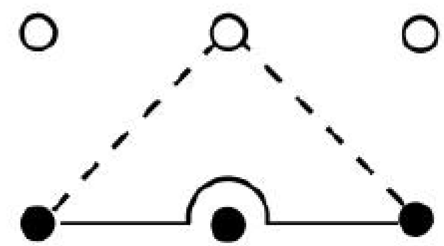
$$\alpha(x) = \frac{d\bar{H}}{dx} + \frac{\bar{H}}{T}\frac{d\bar{T}}{dx}$$
$$\beta(x) = -\frac{\bar{V}}{\bar{H}}\frac{d\bar{H}}{dx} - \frac{\bar{V}(x)}{T(x)}\frac{dT(x)}{dx}$$
$$\gamma(x) = 2gm^2\frac{|\bar{V}|}{\bar{H}^{\frac{4}{3}}} - \frac{\bar{V}}{\bar{H}}\frac{d\bar{H}}{dx} - \frac{\bar{V}(x)}{T(x)}\frac{dT(x)}{dx}$$
$$\eta(x) = -\frac{4}{3}gm^2\frac{\bar{V}|\bar{V}|}{\bar{H}^{\frac{7}{3}}}$$

Discretization

Lax Diffusive scheme:

$$\frac{\partial f}{\partial t} = \frac{f_i^{k+1} - \frac{1}{2}(f_{i+1}^k + f_{i-1}^k)}{\Delta t}$$

$$\frac{\partial f}{\partial x} = \frac{(f_{i+1}^k - f_{i-1}^k)}{2\Delta x}$$



$$\begin{aligned} h_i^{k+1} = & \frac{1}{2}(h_{i+1}^k + h_{i-1}^k) \\ & - \frac{\Delta t}{4\Delta x}(\bar{H}_{i+1} + \bar{H}_{i-1})(v_{i+1}^k - v_{i-1}^k) \\ & - \frac{\Delta t}{4\Delta x}(\bar{V}_{i+1} + \bar{V}_{i-1})(h_{i+1}^k - h_{i-1}^k) \\ & - \frac{\Delta t}{2}(\alpha_{i+1} v_{i+1}^k + \alpha_{i-1} v_{i-1}^k) \\ & - \frac{\Delta t}{2}(\beta_{i+1} h_{i+1}^k + \beta_{i-1} h_{i-1}^k) \end{aligned}$$

Discretization

$$
\begin{aligned}
v_i^{k+1} = \frac{1}{2}(v_{i+1}^k + v_{i-1}^k) \\
- \frac{\Delta t}{4\Delta x}(\bar{V}_{i+1} + \bar{V}_{i-1})(v_{i+1}^k - v_{i-1}^k) \\
- \frac{g\Delta t}{2\Delta x}(h_{i+1}^k - h_{i-1}^k) \\
- \frac{\Delta t}{2}(\gamma_{i+1} v_{i+1}^k + \gamma_{i-1} v_{i-1}^k) \\
- \frac{\Delta t}{2}(\eta_{i+1} h_{i+1}^k + \eta_{i-1} h_{i-1}^k)
\end{aligned}
$$

CFL condition:

$$
\frac{\Delta t}{\Delta x}|V + C| \leq 1
$$

State-space model

The state-space model:

$$z(k+1) = Az(k) + Bu(k)$$

where

$$z(k) = (v_2^k, \cdots, v_I^k, h_2^k, \cdots, h_I^k)^T$$

$$u(k) = (v_1^k, h_1^k)^T$$

Esimation set-up

Stochastic model:

$$z_{k+1} = Az_k + Bu_k + w_k$$

$$y_k = C_k z_k + e_k$$

where the process and measurement noises are zero-mean Gaussian noise and,

$$E[w_k w_l^T] = Q_k \delta_{kl}$$

$$E[e_k e_l^T] = R_k \delta_{kl}$$

$$z_0 = \mathcal{N}(\bar{z}_0, P_0)$$

Kalman Filter

Notation:

$$\hat{z}_k = E[z_k | y_0, \cdots, y_k]$$
$$\hat{z}_k^- = E[z_k | y_0, \cdots, y_{k-1}]$$
$$P_k^- = \Sigma_{k|k-1}$$
$$P_k = \Sigma_{k|k}$$

Time update:

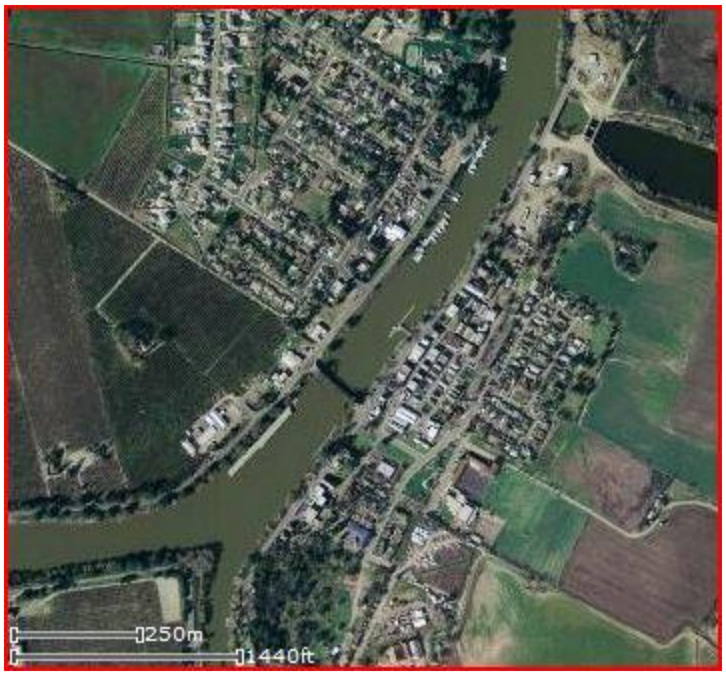
$$\hat{z}_k^- = A\hat{z}_{k-1} + Bu_k$$
$$P_k^- = AP_{k-1}A^T + Q$$

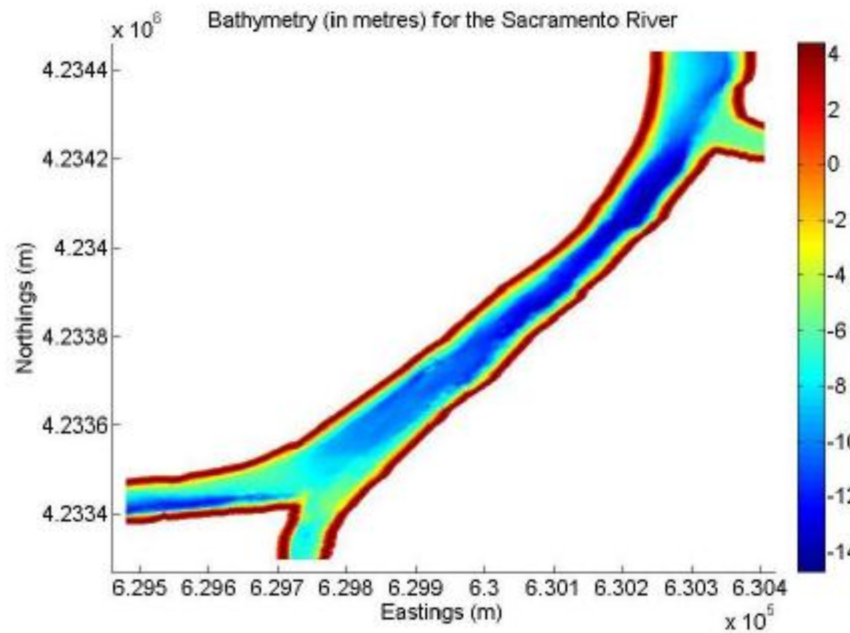
Measurement update:

$$K_k = P_k^- C_k^T (C_k P_k^- C_k^T + R)^{-1}$$
$$\hat{z}_k = \hat{z}_k^- + K_k(y_k - C_k \hat{z}_k^-)$$
$$P_k = (I - K_k C_k) P_k^-$$

Implementation & Numerical Results

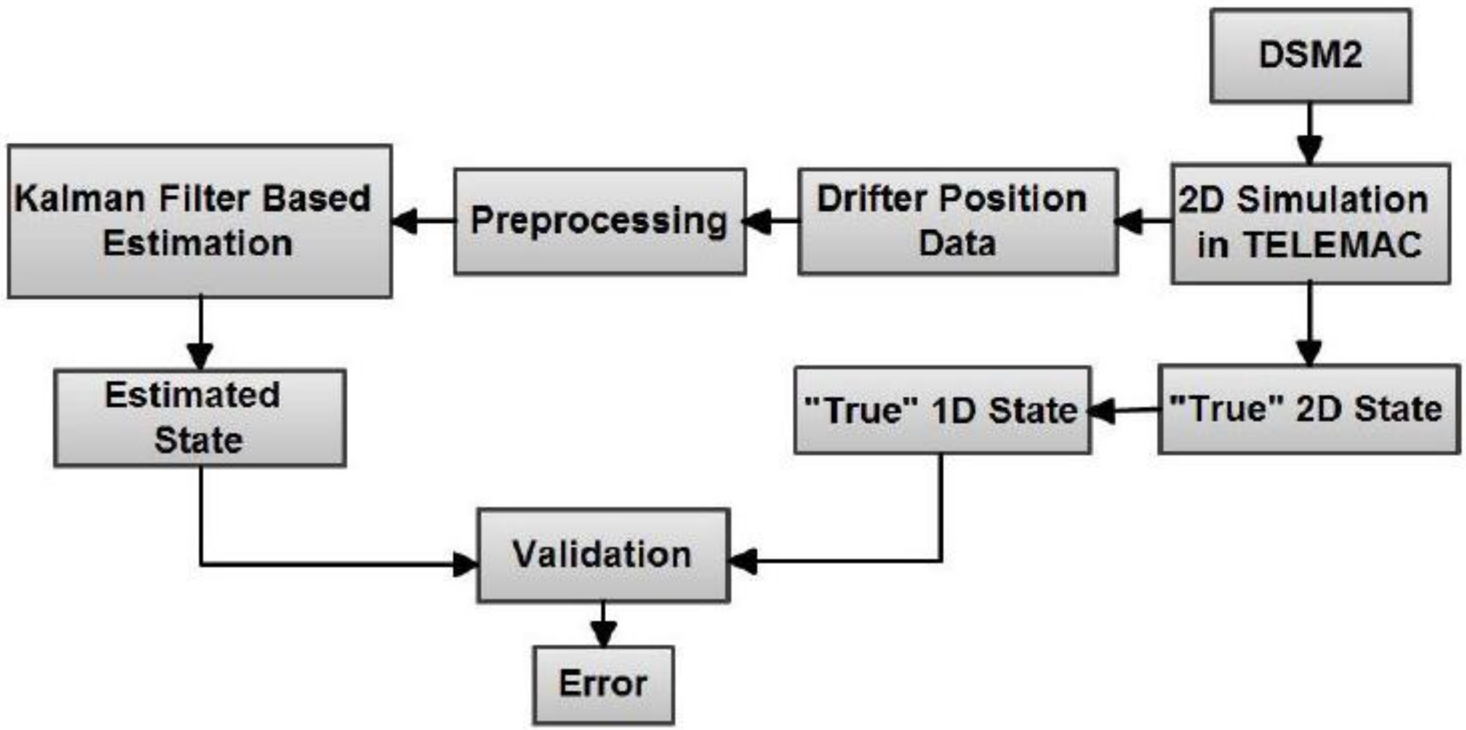
Experiment area: Sacramento River





(Courtesy of Qingfang Wu)

Implementation & Numerical Results



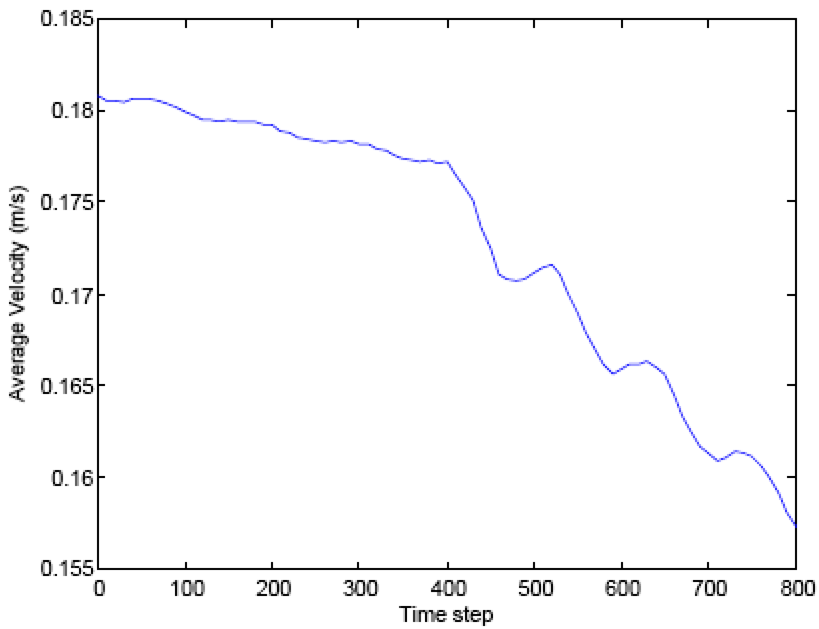
Implementation & Numerical Results

- Start at 3:40PM March 16 2007
- A single drifter deployed
- Number of cells: 30
- Spatial step size = 30 m
- Temporal step size= 3 sec
- Experiment duration: 38 min

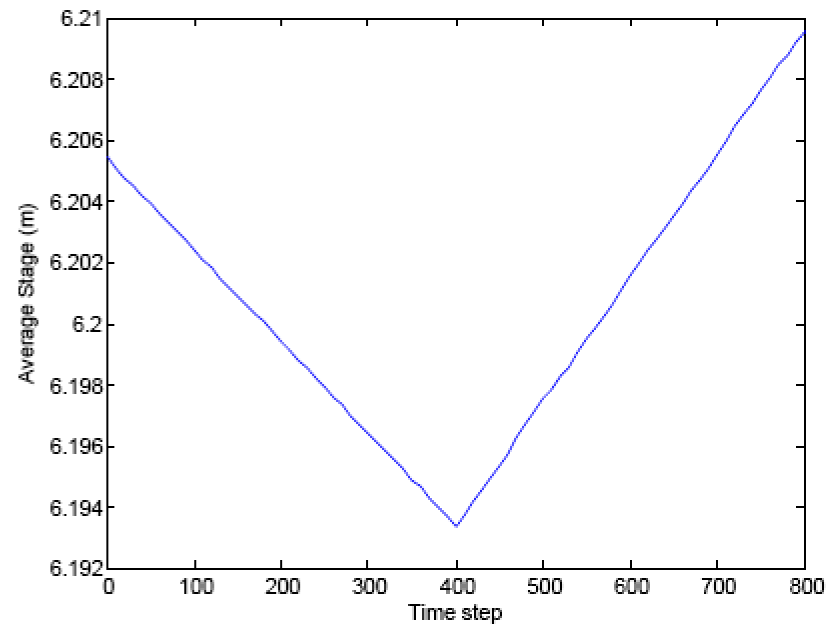
Implementation & Numerical Results

Boundary conditions:

Average Velocity (m/s)

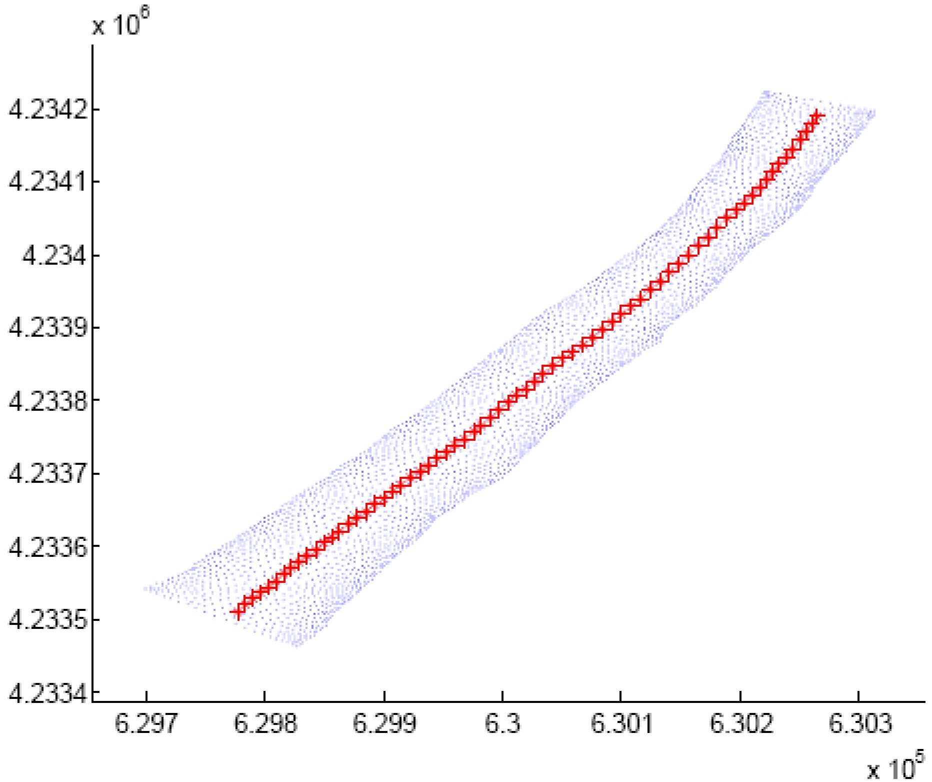


Average Stage (m)



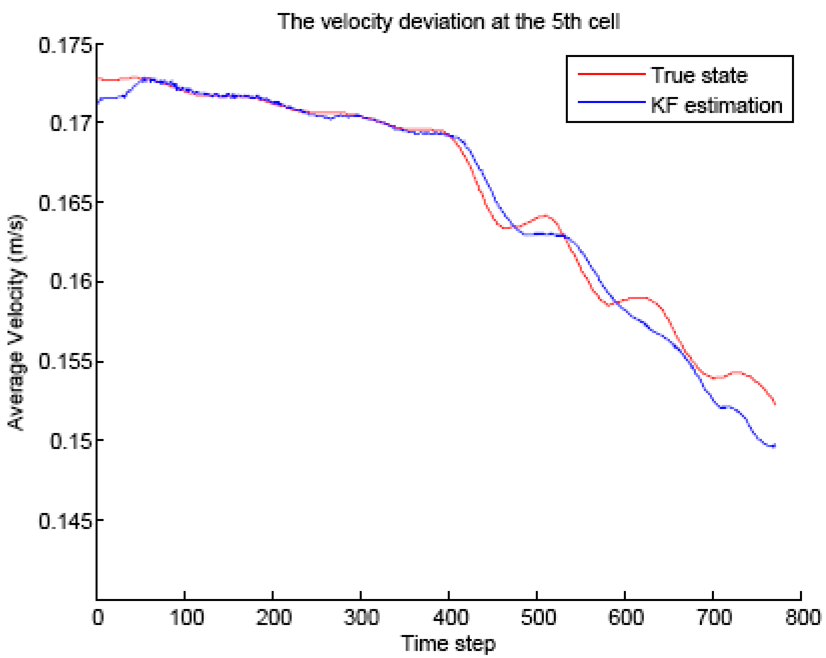
Implementation & Numerical Results

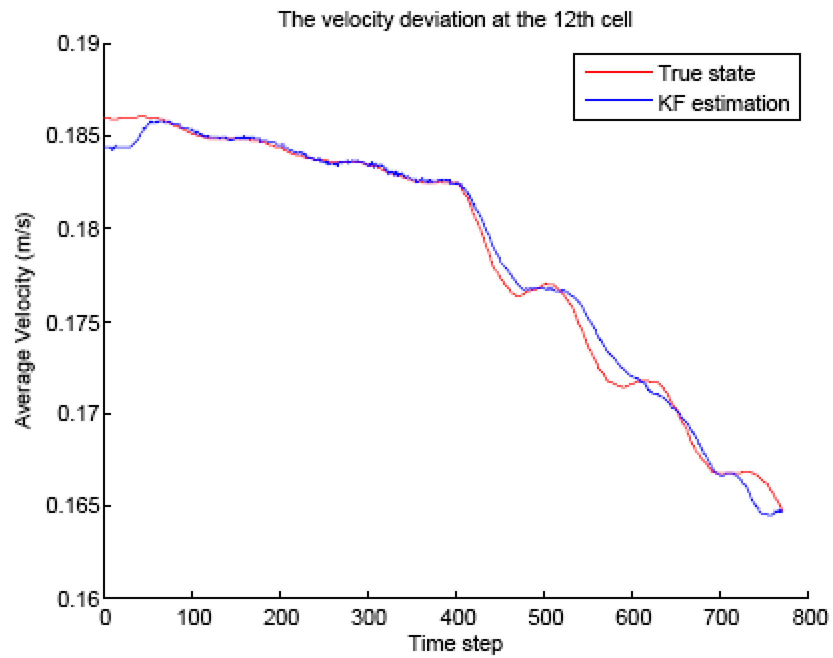
Drifter positions recorded every 3 seconds:



Implementation & Numerical Results

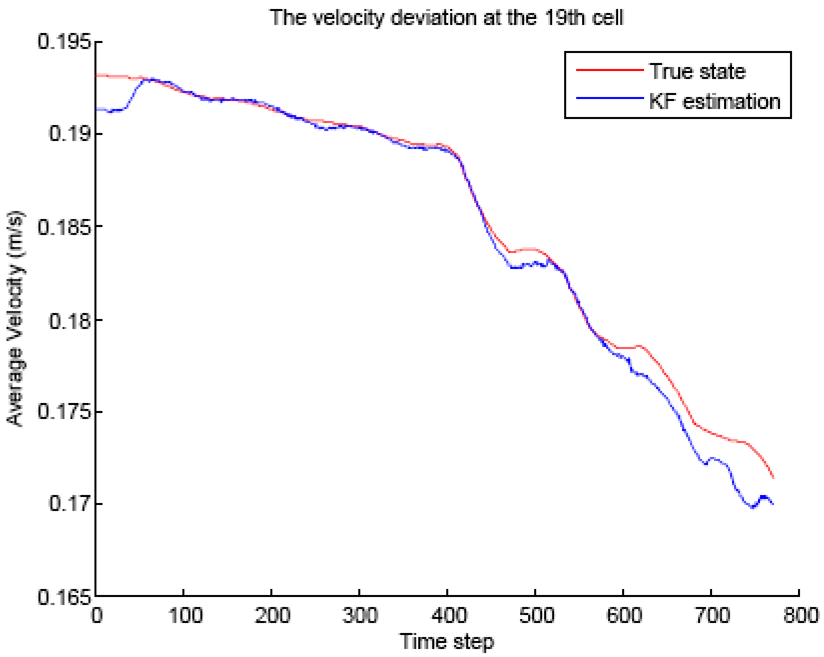
The estimated and the true average velocity at the 5th and 12th cells

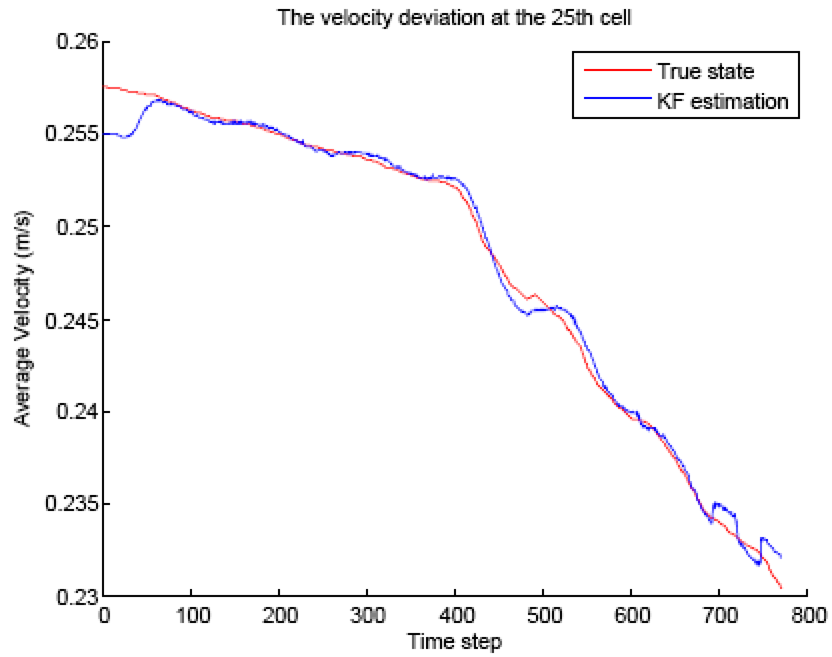




Implementation & Numerical Results

The estimated and the true average velocity at the 19th and 25th cells



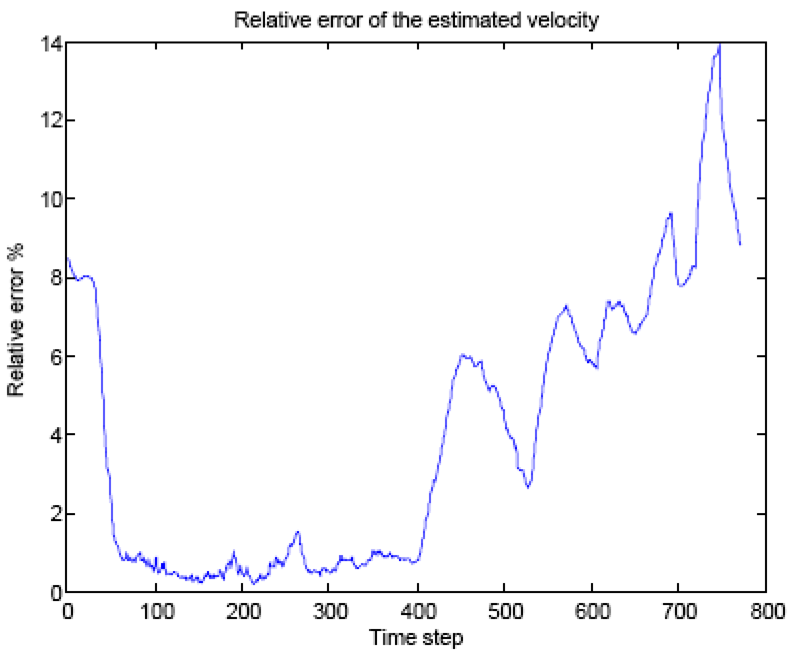


Implementation & Numerical Results

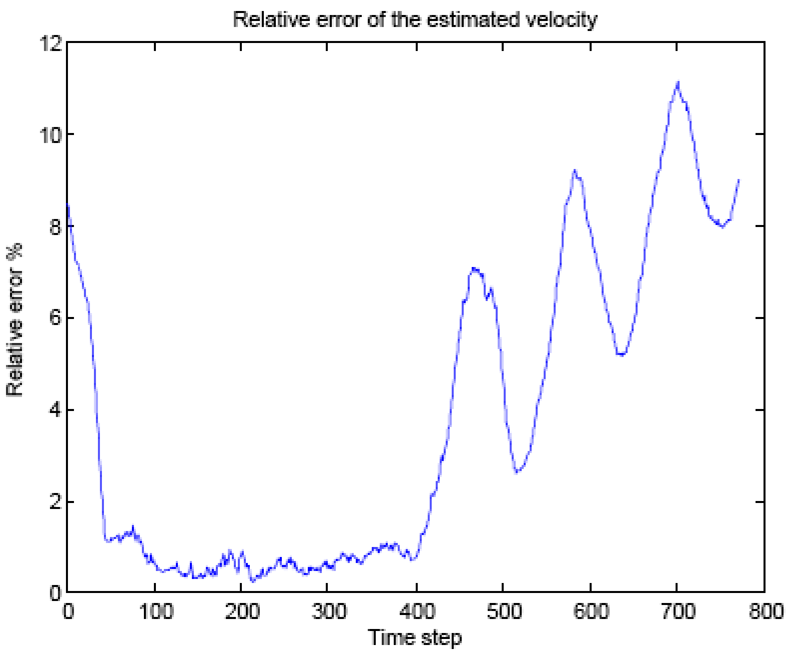
Relative error:

$$\text{error}(k) = \sqrt{\frac{\sum_{i=1}^{N_{cell}} (u_i^k - \hat{u}_i^k)^2}{\sum_{i=1}^{N_{cell}} (u_i^k)^2}}$$

A single drifter



A static sensor at the 6th cell



Future Works

- Implementing the method in a real experiment in real time
- Application to complex interconnected networks of channels
- Estimation of other quantities of interest (salinity, etc)
- Decentralized communication architecture
- Active drifters