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Probabilistic formulation of estimation problems for a class of Hamilton-Jacobi equations

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***Abstract*— This article presents a method for deriving the probability distribution of the solution to a Hamilton-Jacobi partial differential equation for which the value conditions are random. The derivations lead to analytical or semi-analytical expressions of the probability distribution function at any point in the domain in which the solution is defined. The characterization of the distribution of the solution at any point is a first step towards the estimation of the parameters defining the random value conditions.**

This work has important applications for estimation in flow networks in which value conditions are noisy. In particular, we illustrate our derivations on a road segment with random capacity reductions.

I. INTRODUCTION

The computation of numerical solutions to the *Hamilton-Jacobi* (HJ) *partial differential equation* (PDE) subject to boundary conditions, initial conditions or sometimes terminal conditions is a topic which has generated significant interest in the control and numerical analysis community [19], [17], [14]. More recently, researchers have studied how to impose internal value conditions to the HJ-PDE [13], [6], [7].

For numerous applications, initial, boundary and internal value conditions shall be regarded as random processes, rather than deterministic functions. Stochastic formulations of the HJ-PDE have been studied, in particular in the financial mathematics community [18], [11]. This research uses diffusion theory and in particular Itô's formula to show existence and uniqueness of the solution under certain conditions [11]. The research focuses on specific classes of stochastic HJ-PDE, such as backward differential equations, which is a different problem than what we are interested in solving in the present article.

The contributions of this article are as follows. We derive the probability distribution of the solution of a class of HJ-PDEs subject to random value conditions. These derivations allow for the statistical estimation of parameters characterizing the distribution of value conditions based on (noisy) measurements of the solution.

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An application of interest is the design of reliable real-time traffic monitoring systems [4], [20]. Arterial traffic is inherently probabilistic and it is natural to consider the internal, boundary and value conditions as random processes and estimate the probability distribution of the macroscopic state variables (flow, density and velocity) at any location x and time t.

The article is organized as follows. In Section II, we introduce the mathematical background related to solving a HJ-PDE subject to initial and boundary conditions. In Section III, we derive the probability distribution of the solution to the HJ-PDE when the boundary conditions are random processes. We indicate how these derivations can be used to estimate value conditions parameters statistically. We illustrate the importance of the approach for applications through an example in traffic flow networks in Section IV. We analyze the effects of random capacity reductions on the dynamics of congestion.

II. DETERMINISTIC SOLUTION

We first introduce the solution to the following deterministic HJ–PDE on the domain $(t,x) \in [0,t_{\max}] \times [\xi,\chi]$, sometimes known as the *Moskowitz* HJ-PDE [16], [10].

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0 \tag{1}$$

A. Mathematical background

The *Hamiltonian*, ψ, is assumed to be concave on its domain of definition $D_\psi = [0,\rho_{\max}]$ and to satisfy $\psi(0) = \psi(\rho_{\max}) = 0$. We call $q_{\max}$ the maximum value of ψ on D_ψ and define $v^\flat = \psi'(0)$ and $v^\sharp = -\psi'(\rho_{\max})$.

The solution to (1) with initial and boundary condition has been well studied in the literature [9]. The mathematical properties of the solution of (1) require specific treatments to introduce internal boundary conditions, which we solve within a specific control framework based on Lax-Hopf's formula and viability theory [3], [6]. We define the convex transform φ^* of the Hamiltonian as follows.

Definition 1 (Convex transform): Let ψ be a concave function defined in D_ψ, its convex transform φ^* takes

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finite values on $D_{\varphi^*} = [-\nu^\flat, \nu^\sharp]$:

$$\varphi^*(u) = \begin{cases} \sup_{p \in D_\psi} [pu + \psi(p)] & \text{if } u \in [-\nu^\flat, \nu^\sharp] \\ +\infty & \text{otherwise} \end{cases} \quad (2)$$

Let $\mathbf{c}$ be a lower semi-continuous function defined in a subset of $[0, t_{\max}] \times [\xi, \chi]$. It represents a *value condition*, *i.e.* a value that we want to impose on the solution of (1). The viability epi-solution [2], [6], [7] $\mathbf{M_c}$ associated with $\mathbf{c}$ is explicitly given by a Lax-Hopf formula and is the unique generalized solution of (1) in the *Barron-Jensen/Frankowska* (B-J/F) sense [2]. The formula implies an inf-morphism property [2], [6], [7], which is a key property used in this article.

Proposition 1 (Inf-morphism): Let $\mathbf{c}$ be the minimum of a finite number of functions $\mathbf{c}_i$, $i \in I$. The Lax-Hopf formula implies that:

$$\forall (t,x) \in [0, t_{\max}] \times [\xi, \chi]\ \mathbf{M_c}(t,x) = \inf_{i \in I} \mathbf{M}_{\mathbf{c}_i}(t,x)$$

The inf-morphism property is a practical tool to integrate new value conditions and separate a complex problem involving multiple value conditions into a set of more tractable subproblems [6], [7].

B. State estimation with affine initial, boundary and internal value conditions

The solution associated with an affine initial, boundary or internal value condition has an analytical expression [7]. We introduce the following notation and definitions, used in the analytical derivations of the solution, referring to [7] for the proof of their existence.

Definition 2 (Upper and lower critical densities [7]): We define the upper (resp. lower) critical density $\overline{\rho}_c$ (resp. $\underline{\rho}_c$) as the maximum (resp. minimum) $\rho \in [0, \rho_{\max}]$ such that $\psi(\rho) = q_{\max}$.

Definition 3 (Densities associated with q [12]): For $q \in [0, q_{\max}]$ we define $\overline{\rho}(q)$ (resp. $\underline{\rho}(q)$) as the unique solution of $\psi(\rho) = q$ for $\rho \in [\overline{\rho}_c, \rho_{\max}]$ (resp. for $\rho \in [0, \underline{\rho}_c]$).

Following [5], we define the sub- and super-derivative (∂_- and ∂_+) as follows:

$$v \in \partial_- f(x_0) \Leftrightarrow \forall x \in D_f,\ f(x) \geq f(x_0) + v(x - x_0)$$
$$v \in \partial_+ f(x_0) \Leftrightarrow \forall x \in D_f,\ f(x) \leq f(x_0) + v(x - x_0)$$

Definition 4: For $\rho \in [0, \rho_{\max}]$, we define $u_0^+(\rho)$ (resp. $u_0^-(\rho)$) as an element of $-\partial_+\psi(\rho) \cap \mathbb{R}^+$ (resp. $-\partial_+\psi(\rho) \cap \mathbb{R}^-$). Note that $u_0^+(\rho)$ (resp. $u_0^-(\rho)$) is not uniquely defined if ψ is not differentiable in ρ. It was shown [7] that any choice of $u_0^+(\rho)$ (resp. $u_0^-(\rho)$) in $-\partial_+\psi(\rho)$ provides the expression of the solution of the HJ-PDE.

Definition 5 (Capture times [7]): The capture times $\overline{T}_0$ and $\underline{T}_0$ are defined as follows:

$$\overline{T}_0(\rho, x) = \begin{cases} \frac{\chi - x}{u_0^+(\rho)} & \text{if } u_0^+(\rho) \neq 0 \\ +\infty & \text{otherwise} \end{cases} \quad \forall (\rho, x) \in [\overline{\rho}_c, \rho_{\max}] \times [\xi, \chi],$$

$$\underline{T}_0(\rho, x) = \begin{cases} \frac{\xi - x}{u_0^-(\rho)} & \text{if } u_0^-(\rho) \neq 0 \\ +\infty & \text{otherwise} \end{cases} \quad \forall (\rho, x) \in [0, \underline{\rho}_c] \times [\xi, \chi],$$

Proposition 2 (Explicit component solutions): The analytical solutions of the viability episolution associated with affine value conditions are as follows:

- The solution associated with an upstream condition γ_j, defined on $[\bar{\gamma}_j, \bar{\gamma}_{j+1}] \times \{\xi\}$ by $\gamma_j(t,x) = d_j + (t - \bar{\gamma}_j)\psi(\rho_j)$, $\rho_j \in [0, \overline{\rho}_c]$ takes finite values for $(t,x) \in [0, t_{\max}] \times [\xi, \chi]$ such that $t \geq \bar{\gamma}_j + \frac{x - \xi}{\nu^\flat}$. On this domain, the solution is defined as follows:

$$\mathbf{M}_{\gamma_j}(t,x) = \begin{cases} (i)\ (t - \overline{\gamma}_j)\psi(\rho_j) + \rho_j(\xi - x) + d_j \text{ if } \underline{T}_0(\rho_j, x) \in [t - \overline{\gamma}_{j+1}, t - \overline{\gamma}_j] \\ (ii)\ d_j + (t - \overline{\gamma}_j)\varphi^*\left(\frac{\xi - x}{t - \overline{\gamma}_j}\right) \text{ if } \underline{T}_0(\rho_j, x) \geq t - \overline{\gamma}_j \\ (iii)\ (\overline{\gamma}_{j+1} - \overline{\gamma}_j)\psi(\rho_j) + d_j + (t - \overline{\gamma}_{j+1})\varphi^*\left(\frac{\xi - x}{t - \overline{\gamma}_{j+1}}\right) \\ \qquad \text{if } \underline{T}_0(\rho_j, x) \leq t - \overline{\gamma}_{j+1} \end{cases} \quad (3)$$

- The solution associated with a downstream condition β_k defined on $[\bar{\beta}_k, \bar{\beta}_{k+1}] \times \{\chi\}$ by $\beta_k(t,x) = f_k + (t - \bar{\beta}_k)\psi(\rho_k)$, $\rho_k \in [\underline{\rho}_c, \rho_{\max}]$ takes finite values for $(t,x) \in [0, t_{\max}] \times [\xi, \chi]$ such that $t \geq \bar{\beta}_k + \frac{\chi - x}{\nu^\sharp}$. On this domain, the solution is defined as follows:

$$\mathbf{M}_{\beta_k}(t,x) = \begin{cases} (i)\ (t - \bar{\beta}_k)\psi(\rho_k) - \rho_k(\chi - x) + f_k \text{ if } \overline{T}_0(\rho_k, x) \in [t - \bar{\beta}_{k+1}, t - \bar{\beta}_k] \\ (ii)\ f_k + (t - \bar{\beta}_k)\varphi^*\left(\frac{\chi - x}{t - \bar{\beta}_k}\right) \text{ if } \overline{T}_0(\rho_k, x) \geq t - \bar{\beta}_k \\ (iii)\ (\bar{\beta}_{k+1} - \bar{\beta}_k)\psi(\rho_k) + f_k + (t - \bar{\beta}_{k+1})\varphi^*\left(\frac{\chi - x}{t - \bar{\beta}_{k+1}}\right) \\ \qquad \text{if } \overline{T}_0(\rho_k, x) \leq t - \bar{\beta}_{k+1} \end{cases} \quad (4)$$

The inf-morphism property implies that the solution of (1) subject to piecewise affine value conditions is the minimum of the viability episolutions computed for each of the affine conditions.

III. Probability distribution of a component

We are now interested in generalizing the computation framework to take into account the randomness of the value conditions. We detail the derivations in the case of an upstream or a downstream boundary condition. We indicate how the reasoning is extended to initial and internal value conditions (corresponding results are omitted for brevity). We present the way to use the derivations on each affine value conditions to compute the probability distribution of the solution subject to piecewise affine value conditions.

A. Random upstream boundary condition

We consider an upstream boundary condition γ_j, defined as before, *i.e.* defined on $[\bar{\gamma}_j, \bar{\gamma}_{j+1}] \times \{\xi\}$ by $\gamma_j(t,x) = d_j + (t - \bar{\gamma}_j)\psi(\rho_j)$. The main contribution of this article is to consider ρ_j as a random variable, with given distribution $p_{\rho_j}(\rho_j)$ and support included in $[0,\overline{\rho}_c]$. For any location (t,x), $\mathbf{M}_{\gamma_j}(t,x)$ is a random variable, the realization of which is conditioned on the realization of ρ_j. We define $\underline{\phi}_{t,x}$ on $[0,\overline{\rho}_c]$ by $\underline{\phi}_{t,x} : \rho_j \mapsto \mathbf{M}_{\gamma_j}(t,x)(\rho_j)$.

Proposition 3 (Injectivity): There exists a unique $\underline{\rho}^*(t,x) \leq \underline{\rho}_c$ and a unique $\underline{\rho}^{\diamondsuit}(t,x) \leq \underline{\rho}^*(t,x)$ such that (a) the restriction of $\underline{\phi}_{t,x}$ to $[\underline{\rho}^*(t,x),\overline{\rho}_c]$ is constant and derived from Equation (3-*ii*), (b) the restriction of $\underline{\phi}_{t,x}$ to $[0,\underline{\rho}^*(t,x)]$ is injective and is derived from (3-*i*) for $\rho \in [0,\underline{\rho}^{\diamondsuit}(t,x)]$ and from (3-*iii*) for $\rho \in [\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)]$. Moreover, we have $\frac{x-\xi}{t-\bar{\gamma}_j} \in \partial_+\psi(\underline{\rho}^*(t,x))$. We also have $\frac{x-\xi}{t-\bar{\gamma}_{j+1}} \in \partial_+\psi(\underline{\rho}^{\diamondsuit}(t,x))$ if $t \geq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat}$ and $\underline{\rho}^{\diamondsuit}(t,x) = 0$ otherwise.

Proof: In domain (*ii*), $\underline{\phi}_{t,x}$ is constant. The expression (*ii*) is valid for $t - \bar{\gamma}_j \leq \frac{x-\xi}{-u_0^-(\rho_j)}$, which restricts ρ_j algebraically. The concavity of ψ implies that $-u_0^-$ is non-increasing[1] on $[0,\underline{\rho}_c]$, with $-u_0^-(0) = \nu^\flat$ and $0 \in -u_0^-(\underline{\rho}_c)$. There exists a unique $\underline{\rho}^*(t,x) \in [0,\underline{\rho}_c]$ such that $\frac{x-\xi}{t-\bar{\gamma}_j} \in \partial_+\psi(\underline{\rho}^*(t,x))$. In particular, $\rho > \underline{\rho}^*(t,x) \Rightarrow -u_0^-(\rho) < \frac{x-\xi}{t-\bar{\gamma}_j}$ and $\rho < \underline{\rho}^*(t,x) \Rightarrow -u_0^-(\rho) > \frac{x-\xi}{t-\bar{\gamma}_j}$. The concavity of $\underline{\phi}_{t,x}$ [8] and the definition of $\underline{\rho}^*(t,x)$ imply that $\underline{\phi}_{t,x}$ is strictly increasing on $[0,\underline{\rho}^*(t,x)]$.

- If $t \geq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat}$, there exists a unique $\underline{\rho}^{\diamondsuit}(t,x) \in [0,\underline{\rho}^*(t,x)]$ such that $\frac{x-\xi}{t-\bar{\gamma}_{j+1}} \in \partial_+\psi(\underline{\rho}^{\diamondsuit}(t,x))$. In particular, $\rho > \underline{\rho}^{\diamondsuit}(t,x) \Rightarrow -u_0^-(\rho) < \frac{x-\xi}{t-\bar{\gamma}_{j+1}}$ and $\rho < \underline{\rho}^{\diamondsuit}(t,x) \Rightarrow -u_0^-(\rho) > \frac{x-\xi}{t-\bar{\gamma}_{j+1}}$. Expression (*iii*) is valid for $\rho_j \in [0,\underline{\rho}^{\diamondsuit}(t,x)]$ and expression (*i*) is valid for $\rho_j \in [\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)]$.
- If $t \leq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat}$, expression (*i*) is valid for $\rho_j \in [0,\underline{\rho}^*(t,x)]$ and we have $\underline{\rho}^{\diamondsuit}(t,x) = 0$. ∎

Proposition 4 (Bijection): The restriction of $\underline{\phi}_{t,x}$ to $[0,\underline{\rho}^*(t,x)]$ defines a bijection from $[0,\underline{\rho}^*(t,x)]$ to $[\underline{\phi}_{t,x}(0),\underline{\phi}_{t,x}(\underline{\rho}^*(t,x))]$. The expressions of $\underline{\phi}_{t,x}(0)$, $\underline{\phi}_{t,x}(\underline{\rho}^*(t,x))$ and $\underline{\phi}_{t,x}(\underline{\rho}^{\diamondsuit}(t,x))$ are computed analytically as follows:

$$\underline{\phi}_{t,x}(0) = \begin{cases} d_j & \text{if } t \leq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat} \\ d_j + (t-\bar{\gamma}_{j+1})\varphi^*\left(\frac{\xi-x}{t-\bar{\gamma}_{j+1}}\right) & \text{if } t \geq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat} \end{cases}$$

$$\underline{\phi}_{t,x}(\underline{\rho}^*(t,x)) = d_j + (t-\bar{\gamma}_j)\varphi^*\left(\frac{\xi-x}{t-\bar{\gamma}_j}\right)$$

$$\underline{\phi}_{t,x}(\underline{\rho}^{\diamondsuit}(t,x)) = d_j + (\bar{\gamma}_{j+1}-\bar{\gamma}_j)\psi(\underline{\rho}^{\diamondsuit}(t,x)) + (t-\bar{\gamma}_{j+1})\varphi^*\left(\frac{\xi-x}{t-\bar{\gamma}_{j+1}}\right)$$

Proof: The proof is derived from Proposition 3 (injectivity of $\underline{\phi}_{t,x}$ on $[0,\underline{\rho}^*(t,x)]$) and Equation (3). ∎

Proposition 5 (Differentiability): If ψ is differentiable on $[0,\underline{\rho}^*(t,x)]$, the restriction of $\underline{\phi}_{t,x}$ to $[0,\underline{\rho}^*(t,x)]$ is differentiable.

Proof: The expression of $\underline{\phi}_{t,x}$ imply that $\underline{\phi}_{t,x}$ is continuously differentiable on the intervals $[0,\underline{\rho}^{\diamondsuit}(t,x))$ and $(\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)]$. If $t \leq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat}$, this terminates the proof as $\underline{\rho}^{\diamondsuit}(t,x) = 0$. We consider the case where $t \geq \bar{\gamma}_{j+1} + \frac{x-\xi}{\nu^\flat}$

The differentiability of ψ at $\underline{\rho}^{\diamondsuit}(t,x)$ and the definition of $\underline{\rho}^{\diamondsuit}(t,x)$ imply that $\psi'(\underline{\rho}^{\diamondsuit}(t,x)) = \frac{x-\xi}{t-\bar{\gamma}_{j+1}}$. We first compute the left derivative of $\underline{\phi}_{t,x}$ at $\underline{\rho}^{\diamondsuit}(t,x)$ using expression (3-*i*). The left derivative is given by $(t-\bar{\gamma}_j)\frac{x-\xi}{t-\bar{\gamma}_{j+1}} + \xi - x$, which can be written as $(x-\xi)\frac{\bar{\gamma}_{j+1}-\bar{\gamma}_j}{t-\bar{\gamma}_{j+1}}$. Using expression (3-*iii*), we show that the right derivative is equal to the left derivative and thus $\underline{\phi}_{t,x}$ is continuously differentiable on $[0,\underline{\rho}^*(t,x)]$. ∎

Proposition 6 (Diffeomorphism): If ψ is differentiable on $[0,\underline{\rho}^*(t,x)]$, the restriction of $\underline{\phi}_{t,x}$ to $(0,\underline{\rho}^*(t,x))$ defines a diffeomorphism from $(0,\underline{\rho}^*(t,x))$ to $\left(\underline{\phi}_{t,x}(0),\underline{\phi}_{t,x}(\underline{\rho}^*(t,x))\right)$.

Proof: The proof relies on the global inversion theorem (see for example [1]). We proved that $\underline{\phi}_{t,x}$ is injective and continuously differentiable on the open interval $(0,\underline{\rho}^*(t,x))$. We show that the differential function is invertible on this interval. Since $\underline{\phi}_{t,x}$ is concave and strictly increasing on $(0,\underline{\rho}^*(t,x))$, the derivative is strictly positive on this interval, thus invertible and $\underline{\phi}_{t,x}$ defines a diffeomorphism on $(0,\underline{\rho}^*(t,x))$. ∎

To derive the expression $\underline{\phi}_{t,x}^{-1}$, we use the following definition, first introduced and proved in [7].

Definition 6 (Densities associated with v_j and g_j [7]): Let v_j in $[0,\nu^\flat]$ and g_j be in $[0,\varphi^*(-v_j)]$. Let $\rho \in [0,\rho_{\max}]$ be such that[2] $\psi(\rho) - \rho v_j = \varphi^*(-v_j)$. There exists two solutions to the equation $\psi(\rho_j) - v_j\rho_j = g_j$ on $[0,\rho_{\max}]$, denoted $\rho_1(v_j,g_j)$ and $\rho_2(v_j,g_j)$ with $\rho_1(v_j,g_j) \leq \rho \leq \rho_2(v_j,g_j)$.

Proposition 7 (Expression of $\underline{\phi}_{t,x}^{-1}$): The inverse of the diffeomorphism induced by the restriction of $\underline{\phi}_{t,x}$ to $(0,\underline{\rho}^*(t,x))$ onto its image is denoted $\underline{\phi}_{t,x}^{-1}$ and can be computed analytically as:

$$\underline{\phi}_{t,x}^{-1}(m) = \rho_1\left(\frac{x-\xi}{t-\bar{\gamma}_j}, \frac{m-d_j}{t-\bar{\gamma}_j}\right) \tag{5}$$

Proof: For $\rho_j \in [0,\underline{\rho}^{\diamondsuit}(t,x)]$, the expression of $\underline{\phi}_{t,x}(\rho_j)$ is given by (*iii*). Let m be in the image

[1] Since u_0^- may not be uniquely defined, non-increasing is understood in the following sense: $\forall(\rho,\rho') \in [0,\overline{\rho}_c]^2$ s.t. $\rho < \rho'$, $\forall u_0^-(\rho) \in -\partial_+\psi(\rho)$, $\forall u_0^-(\rho') \in -\partial_+\psi(\rho')$, then $-u_0^-(\rho) \geq -u_0^-(\rho')$.

[2] The existence of such a ρ comes from the definition of φ^*.

of $[0,\underline{\rho}^{\diamondsuit}(t,x)]$ under $\underline{\phi}_{t,x}$, there exists a unique $\rho_j \in [0,\underline{\rho}^{\diamondsuit}(t,x)]$ such that:

$$\psi(\rho_j) = \frac{m - d_j - (t-\overline{\gamma}_{j+1})\varphi^*\left(\frac{\xi - x}{t-\overline{\gamma}_{j+1}}\right)}{\bar{\gamma}_{j+1} - \bar{\gamma}_j}. \quad (6)$$

This implies that the right hand side of (6) is in $[0,q_{\max}]$ and that the unique solution is given by

$$\underline{\phi}_{t,x}^{-1}(m) = \underline{\rho}\left(\frac{m - d_j - (t-\overline{\gamma}_{j+1})\varphi^*\left(\frac{\xi - x}{t-\overline{\gamma}_{j+1}}\right)}{\bar{\gamma}_{j+1} - \bar{\gamma}_j}\right),$$

where $\underline{\rho}$ is defined in Definition 3.

For $\rho_j \in [\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)]$, the expression of $\underline{\phi}_{t,x}(\rho_j)$ is given by (i). Let m be in $\underline{\phi}_{t,x}([\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)])$. There exists a unique $\rho_j \in [\underline{\rho}^{\diamondsuit}(t,x),\underline{\rho}^*(t,x)]$ such that

$$\frac{m-d_j}{t-\bar{\gamma}_j} = \psi(\rho_j) - \rho_j \frac{x-\xi}{t-\bar{\gamma}_j}.$$

We know that $\frac{x-\xi}{t-\bar{\gamma}_j} \in [0,-u_0^-(\rho_j)]$ and in particular, $\frac{x-\xi}{t-\bar{\gamma}_j} \in [0,v^\flat)$. The existence of ρ_j and the definition of φ^* also imply that $\frac{m-d_j}{t-\bar{\gamma}_j} \leq \varphi^*(-\frac{x-\xi}{t-\bar{\gamma}_j})$. We define $v_j = \frac{x-\xi}{t-\bar{\gamma}_j}$ and $g_j = \frac{m-d_j}{t-\bar{\gamma}_j}$. Let ρ be such that $\psi(\rho) - \rho v_j = \varphi^*(-v_j)$, then $v_j \in \partial_+\psi(\rho)$. We know that $-u_0^-(\underline{\rho}^*(t,x)) > v_j$. The concavity of ψ implies that $\rho > \underline{\rho}^*(t,x)$. According to Definition 6, there exists two solutions $\rho_1(v_j,g_j)$ and $\rho_2(v_j,g_j)$ to the equation $\psi(\rho_j) - \rho_j v_j = g_j$, satisfying $\rho_1(v_j,g_j) \leq \rho \leq \rho_2(v_j,g_j)$. Since $\rho > \underline{\rho}^*(t,x)$, only the first solution is possible which yields (5). ■

We use the previous results to derive the probability distribution of $\mathbf{M}_{\gamma_j}(t,x)$. We define

$$w(t,x) = \int_{\underline{\rho}^*(t,x)}^{\overline{\rho}_c} p_{\rho_j}(\rho_j)\, d\rho_j \quad \text{and} \quad \mathbf{M}_{\gamma_j}^{\text{init}} = d_j + (t-\overline{\gamma}_j)\varphi^*\left(\frac{\xi-x}{t-\overline{\gamma}_j}\right).$$

Proposition 8 (Probability distribution of $\mathbf{M}_{\gamma_j}(t,x)$*):* If ψ is continuously differentiable on $[0,\underline{\rho}^*(t,x)]$, the probability distribution of $\mathbf{M}_{\gamma_j}(t,x)$ is given by

$$p_{\mathbf{M}_{\gamma_j}(t,x)}(m) = w(t,x)\delta(m - \mathbf{M}_{\gamma_j}^{\text{init}}) + (1-w(t,x))\left|\frac{d}{dm}\left(\underline{\phi}_{t,x}^{-1}(m)\right)\right| p_{\rho_j}(\underline{\phi}_{t,x}^{-1}(m)).$$

Proof: Using the law of total probability, we have

$$p_{\mathbf{M}_{\gamma_j}(t,x)}(m) = w(t,x) p_{\mathbf{M}_{\gamma_j}(t,x)|\rho_j}(m|\rho_j \in [\underline{\rho}^*(t,x),\underline{\rho}_c]) + (1-w(t,x)) p_{\mathbf{M}_{\gamma_j}(t,x)|\rho_j}(m|\rho_j \in [0,\underline{\rho}^*(t,x)]).$$

Given the event "$\rho_j \in [\underline{\rho}^*(t,x),\underline{\rho}_c]$", the value of $\mathbf{M}_{\gamma_j}(t,x)$ is deterministic, and equal to $\mathbf{M}_{\gamma_j}^{\text{init}}$. The probability distribution of $\mathbf{M}_{\gamma_j}(t,x)$ conditioned on the event "$\rho_j \in [\underline{\rho}^*(t,x),\underline{\rho}_c]$" is a Dirac Delta distribution (mass probability) at $\mathbf{M}_{\gamma_j}^{\text{init}}$, which we write $p_{\mathbf{M}_{\gamma_j}(t,x)|\rho_j}(m|\rho_j \in [\underline{\rho}^*(t,x),\underline{\rho}_c]) = \delta(m - \mathbf{M}_{\gamma_j}^{\text{init}})$.

Given the event "ρ_j is in $[0,\underline{\rho}^*(t,x)]$" and given that the restriction of $\underline{\phi}_{t,x}$ on this interval induces a diffeomorphism, the probability distribution of $\mathbf{M}_{\gamma_j}(t,x)$ is derived from the probability distribution of ρ_j using the change of variable $\rho_j = \underline{\phi}_{t,x}^{-1}(m)$. ■

B. Random downstream boundary condition

We derive the probability distribution of a component associated with a random downstream boundary condition. For the sake of brevity, we do not detail the proofs which are similar to the proofs from Section III-A.

We consider a downstream boundary condition β_k, defined on $[\bar{\beta}_k,\bar{\beta}_{k+1}] \times \{\chi\}$ by $\beta_k(t,x) = f_k + (t - \bar{\beta}_k)\psi(\rho_k)$, for which the parameter ρ_k is a random variable, with given distribution $p_{\rho_k}(\rho_k)$. For any location (t,x), we define $\overline{\phi}_{t,x}$ on the interval $[\underline{\rho}_c,\rho_{\max}]$ by $\overline{\phi}_{t,x} : \rho_k \mapsto \mathbf{M}_{\beta_k}(t,x)(\rho_k)$.

Proposition 9 (Injectivity): There exists a unique $\overline{\rho}^*(t,x) \geq \underline{\rho}_c$ and a unique $\overline{\rho}^{\diamondsuit}(t,x) \geq \overline{\rho}^*(t,x)$ such that (a) the restriction of $\overline{\phi}_{t,x}$ to $[\overline{\rho}_c,\overline{\rho}^*(t,x)]$ is constant and derived from Equation (4-*ii*), (b) the restriction of $\overline{\phi}_{t,x}$ to $[\overline{\rho}^*(t,x),\rho_{\max}]$ is injective and is derived from (4-*i*) for $\rho \in [\overline{\rho}^*(t,x),\overline{\rho}^{\diamondsuit}(t,x)]$ and from (4-*iii*) for $\rho \in [\overline{\rho}^{\diamondsuit}(t,x),\rho - \max]$. Moreover, we have $\frac{\chi-x}{t-\bar{\beta}_k} \in -\partial_+\psi(\overline{\rho}^*(t,x))$. We also have $\frac{\chi-x}{t-\bar{\beta}_{k+1}} \in -\partial_+\psi(\overline{\rho}^{\diamondsuit}(t,x))$ if $t \geq \bar{\beta}_{k+1} + \frac{\chi-x}{v^\sharp}$ and $\overline{\rho}^{\diamondsuit}(t,x) = \rho_{\max}$ otherwise.

Proof: The proof is similar to the proof of Proposition 3 and omitted for brevity. ■

Proposition 10 (Bijection): The restriction of $\overline{\phi}_{t,x}$ to $[\overline{\rho}^*(t,x),\rho_{\max}]$ defines a bijection from $[\overline{\rho}^*(t,x),\rho_{\max}]$ to $[\overline{\phi}_{t,x}(\rho_{\max}),\overline{\phi}_{t,x}(\overline{\rho}^*(t,x))]$. The expressions of $\overline{\phi}_{t,x}(\rho_{\max})$, $\overline{\phi}_{t,x}(\overline{\rho}^{\diamondsuit}(t,x))$ and $\overline{\phi}_{t,x}(\overline{\rho}^*(t,x))$ are computed analytically as follows:

$$\overline{\phi}_{t,x}(\rho_{\max}) = \begin{cases} f_k & \text{if } t \leq \bar{\beta}_{k+1} + \frac{\chi-x}{v^\sharp} \\ f_k + (t-\bar{\beta}_{k+1})\varphi^*\left(\frac{\xi-x}{t-\bar{\beta}_{k+1}}\right) & \text{if } t \geq \bar{\beta}_{k+1} + \frac{\chi-x}{v^\sharp} \end{cases}$$

$$\overline{\phi}_{t,x}(\overline{\rho}^*(t,x)) = f_k + (t-\bar{\beta}_k)\varphi^*\left(\frac{\xi-x}{t-\bar{\beta}_k}\right)$$

$$\overline{\phi}_{t,x}(\overline{\rho}^{\diamondsuit}(t,x)) = f_k + (\bar{\beta}_{k+1}-\bar{\beta}_k)\psi(\overline{\rho}^{\diamondsuit}(t,x)) + (t-\bar{\beta}_{k+1})\varphi^*\left(\frac{\xi-x}{t-\bar{\beta}_{k+1}}\right)$$

Proof: The proof is derived from Proposition 9 (injectivity of $\overline{\phi}_{t,x}$) and Equation (3). ■

Proposition 11 (Differentiability): If ψ is differentiable on $(\overline{\rho}^*(t,x),\rho_{\max}]$, the restriction of $\overline{\phi}_{t,x}$ to $(\overline{\rho}^*(t,x),\rho_{\max}]$ is differentiable.

Proof: The proof is readily adapted from the proof of Proposition 5. We use the expression of $\overline{\phi}_{t,x}$ for $\rho_k \in [\overline{\rho}^*, \overline{\rho}^{\diamondsuit}(t,x))$ and for $\rho_k \in (\overline{\rho}^{\diamondsuit}(t,x), \rho_{\max}]$ given in (4) to show the differentiability on each of the two intervals and the continuity of the differential at $\rho = \overline{\rho}^{\diamondsuit}(t,x)$. ■

Proposition 12 (Diffeomorphism): If ψ is differentiable on $(\overline{\rho}^*(t,x), \rho_{\max}]$, the restriction of $\overline{\phi}_{t,x}$ to $(\overline{\rho}^*(t,x), \rho_{\max})$ defines a diffeomorphism from $(\overline{\rho}^*(t,x), \rho_{\max})$ to $\left(\overline{\phi}_{t,x}(\rho_{\max}), \overline{\phi}_{t,x}(\overline{\rho}^*(t,x))\right)$.

Proof: As for the proof of Proposition 6, the proof relies on the global inversion theorem and uses the injectivity and differentiability of $\overline{\phi}_{t,x}$ on the open interval $(\overline{\rho}^*(t,x), \rho_{\max})$, as well as the invertibility of the differential on this interval. ■

Proposition 13 (Expression of $\overline{\phi}_{t,x}^{-1}$): The inverse of the diffeomorphism induced by the restriction of $\overline{\phi}_{t,x}$ to $(\overline{\rho}^*(t,x), \rho_{\max})$ onto its image is denoted $\overline{\phi}_{t,x}^{-1}$ and can be computed analytically as

$$\overline{\phi}_{t,x}^{-1}(m) = \rho_2\left(\frac{\chi - x}{t - \bar{\beta}_k}, \frac{m - f_k}{t - \bar{\beta}_k}\right) \tag{7}$$

Proof: The proof is similar to the proof of Proposition 7 and omitted for brevity. ■

We use the previous results to derive the probability distribution of $\mathbf{M}_{\beta_k}(t,x)$. We define

$$\text{and} \quad \begin{array}{rcl} w(t,x) & = & \int_{\overline{\rho}_c}^{\overline{\rho}^*(t,x)} p_{\rho_k}(\rho_k)\, d\rho_k \\ \mathbf{M}_{\beta_k}^{\text{init}} & = & f_k + (t - \overline{\beta}_k)\varphi^*\left(\frac{\xi - x}{t - \overline{\beta}_k}\right) \end{array}.$$

Proposition 14 (Probability distribution of $\mathbf{M}_{\beta_k}(t,x)$): If ψ is continuously differentiable on $[\overline{\rho}^*(t,x), \rho_{\max}]$, the probability distribution of $\mathbf{M}_{\beta_k}(t,x)$ is given by

$$\begin{array}{l} p_{\mathbf{M}_{\beta_k}(t,x)}(m) = w(t,x)\delta(m - \mathbf{M}_{\beta_k}^{\text{init}}) \\ \quad + (1 - w(t,x))\left|\frac{d}{dm}\left(\overline{\phi}_{t,x}^{-1}(m)\right)\right| p_{\rho_k}(\overline{\phi}_{t,x}^{-1}(m)). \end{array}$$

Proof: The proof is similar to the proof of Proposition 8 and uses the law of total probability and the change of variable $\rho_k = \overline{\phi}_{t,x}^{-1}(m)$ for $m \in [\overline{\phi}_{t,x}(\rho_{\max}), \overline{\phi}_{t,x}(\overline{\rho}^*(t,x))]$. ■

C. Probability distribution of the solution

In Sections III-A and III-B, we computed the probability distribution of a solution associated with an affine upstream or downstream boundary condition. Similar derivations are performed to compute the probability distribution of the solution associated with an affine initial or internal value condition: we use the deterministic solution derived in [7] and define appropriate conditioning and change of variables to derive the probability distribution of the solution. To compute the probability distribution of the solution associated with piecewise affine initial, upstream, downstream and internal value conditions we use the inf-morphism property (Proposition 1). We consider I random value conditions $\mathbf{c}_i$ and denote by $p_{\mathbf{M}_{\mathbf{c}_i}(t,x)}$ the probability distribution of the corresponding component i at each location x and time t. We assume that the value conditions are independent, and thus the random variables $\mathbf{M}_{\mathbf{c}_i}(t,x)$ are independent. Let $\mathbf{c}$ be the minimum of the value conditions $\mathbf{c}_i$, $i \in I$, the probability distribution of the solution at time t and location x associated with the random value condition $\mathbf{c}$ is denoted $\mathbf{M}_{\mathbf{c}}(t,x)$. For any realization of the random value condition, the solution satisfies the inf-morphism property. The random variable $\mathbf{M}_{\mathbf{c}}(t,x)$ is the minimum of the random variables $\mathbf{M}_{\mathbf{c}_i}(t,x)$:

$$\mathbf{M}_{\mathbf{c}}(t,x) = \min_{i \in I} \mathbf{M}_{\mathbf{c}_i}(t,x)$$

We denote by $\mathbf{P}_{\mathbf{M}_{\mathbf{c}_i}(t,x)}$ the cumulative probability distribution of the random variable $\mathbf{M}_{\mathbf{c}_i}(t,x)$ associated with the random value condition $\mathbf{c}_i$. It is defined by: $\mathbf{P}_{\mathbf{M}_{\mathbf{c}_i}(t,x)} = \int_{-\infty}^{m} p_{\mathbf{M}_{\mathbf{c}_i}(t,x)}(\tilde{m})\, d\tilde{m}$.

Proposition 15 (Probability distribution of $\mathbf{M}_{\mathbf{c}}(t,x)$): The probability distribution of the solution $\mathbf{M}_{\mathbf{c}}(t,x)$, corresponding to the random value condition $\mathbf{c}$ is computed from the probability distribution of the components $\mathbf{M}_{\mathbf{c}_i}(t,x)$ associated with the affine value conditions $\mathbf{c}_i$ as follows:

$$p_{\mathbf{M}_{\mathbf{c}}(t,x)}(m) = \sum_{i \in I} p_{\mathbf{M}_{\mathbf{c}_i}(t,x)}(m) \prod_{j \neq i} (1 - \mathbf{P}_{\mathbf{M}_{\mathbf{c}_j}(t,x)}(m))$$

Remark 1 (Parameter estimation): The probability distribution of the solution of the HJ-PDE at time t and location x is parametric. The parameters characterize the probability distribution of the initial, upstream, downstream and internal value conditions. They can be estimated from (noisy) measurements of the solution, using likelihood maximization for example.

IV. Numerical illustration

We illustrate the derivations through a simulation of probabilistic traffic flows. The Moskowitz HJ-PDE was first introduced in [15], and was further studied in the literature [16], [10]. The Moskowitz function is a possible macroscopic description of traffic flow, in which the traffic is described by a surface representing the so-called *cumulative number of vehicles* [16]. Under this model, isolines of the Moskowitz function represent trajectories of the vehicles. The Hamiltonian ψ is referred to as flux function or fundamental diagram, usually assumed to be concave.

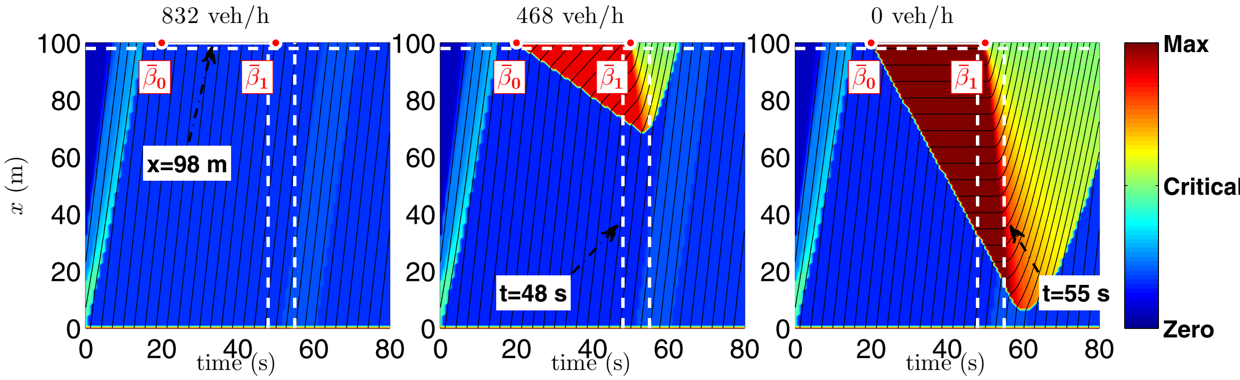

Fig. 1. Deterministic solution of the Moskowitz HJ-PDE under given initial and upstream boundary conditions and with three different values for the capacity reduction (from left to right, the outflow is limited to 832, 468 and 0 vehicles per hour respectively). The color scale represents the spatial derivative of the solution (density). Black lines represent the isolines of the solution (vehicle trajectories). The different values of the maximum outflow influence the formation of a queue upstream of the capacity reduction.

We are given a concave Hamiltonian ψ, initial and upstream boundary conditions, specified in the form of two piecewise affine functions taking finite values on the domains $\{0\} \times [\xi, \chi]$ and $[0, T] \times \{\xi\}$ respectively. During time interval $[\bar{\beta}_0, \bar{\beta}_1]$, we simulate a reduction of the output capacity at $x = \chi$, leading to the potential formation of a queue. The reduction of the output capacity is represented by a random variable ρ_0 with support in $[\underline{\rho}_c, \rho_{\max}]$, corresponding to the output density. The randomness of the downstream boundary condition leads to randomness in the queue formation, which we illustrate numerically.

We consider a *Greenshields* Hamiltonian, parameterized by the maximum density $\rho_{\max} = 0.1$ veh/m and the maximum flow $q_{\max} = 1300$ veh/h. It is defined on $[0, \rho_{\max}]$ by $\psi(\rho) = 4\frac{q_{\max}}{\rho_{\max}^2}\rho(\rho_{\max} - \rho)$. We compute the solution of the HJ-PDE on the domain $[0, T] \times [\xi, \chi]$ with $T = 80$ s, $\xi = 0$ m and $\chi = 100$ m. We choose $\bar{\beta}_0 = 20$ s and $\bar{\beta}_1 = 50$ s and impose a random capacity reduction during the time interval $[\bar{\beta}_0, \bar{\beta}_1]$ at $x = \xi$. During this time interval, the output density is a random variable, ρ_0 unifomly distributed on $[0.08, 0.1]$.

Figure 1 represents the deterministic solution of the Moskowitz HJ-PDE for output densities ρ_0 equal to 0.08, 0.09 and 0.1 veh/m, corresponding to maximum output flows $\psi(\rho_0)$ equal to 832, 468 and 0 veh/h respectively. We display isolines of the Moskowitz function and a colormap of the spatial partial derivative, which is a common two dimensional representation of the solution. The figure illustrates the differences in the solution of the HJ-PDE, under different downstream boundary conditions and underlines the importance to study the probability distribution of the solution when the conditions are noisy or cannot be estimated accurately. Depending on the importance of the capacity reduction, the solution may exhibit shock-waves, corresponding to discontinuities in the density ρ (and in the flow $\psi(\rho)$). In the context of transportation, it is common to refer to these shock-waves as queue formations or queue dissipations. Note that a flow of 832 veh/h does not create any queue formation because the capacity at time t and location χ is greater than the flow imposed by the initial and upstream boundary conditions at time t and location χ. As the maximum flow decreases, a queue forms with a speed of formation depending on the importance of the capacity reduction. At the end of the capacity reduction, the queue dissipates.

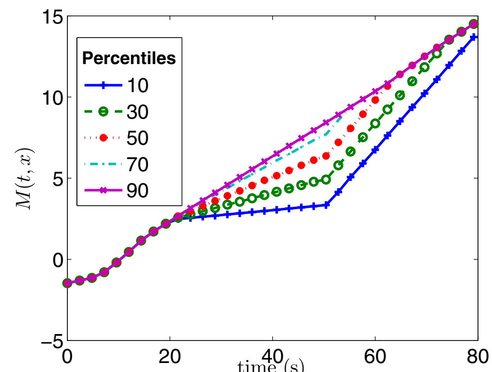

Fig. 2. Distribution of the solution of the Moskowitz HJ-PDE at a fixed location $x = 98$ meters (2 meters upstream of the capacity reduction). The value at $t = 0$ represents the label of the vehicle at $x = 98$ at the initial time, which is determined up to a constant chosen by the initialization of the Moskowitz function at $(\xi, 0)$.

We compute the probability distribution of the solution according to the derivations of Section III, which we illustrate on Figures 2 and 3 using percentiles. For the random variable $\mathbf{M}(t,x)$, the n^{th}-percentile, denoted $\mathbf{M}^n(t,x)$ for $n \in [0, 100]$, satisfies $p_{\mathbf{M}(t,x)}(m \leq \mathbf{M}^n(t,x)) = n/100$. Percentiles are commonly used to represent prob-

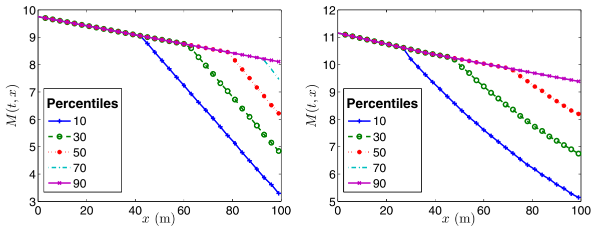

Fig. 3. Distribution of the solution of the Moskowitz HJ-PDE at a fixed time. **Left:** Solution at $t = 48$ seconds, 28 seconds after the beginning of the capacity reduction. **Right:** Solution at $t = 55$ seconds, 5 seconds after the end of the capacity reduction.

ability distributions. Figure 2 illustrates the probability distribution at a fixed location $x = 98$ m (upstream of the end of the segment), as it evolves over time. The location is indicated on Figure 1, with a dashed line labeled $x = 98$ m. The slope of each curve represents the flow at the corresponding time and location. Points where the curve is not differentiable correspond to the presence of a shock-wave at the corresponding time and location. At the beginning of the capacity reduction, the flow decreases when the capacity reduction is sufficient to cause the formation of a queue. This creates a shock-wave, *i.e.* a non-differentiability of the solution. At the end of the capacity reduction, the queue dissipates (second shock-wave), corresponding to another non-differentiability of the solution. The duration of the congestion varies depending on the importance of the capacity reduction. In Figure 3, we represent the probability distribution of the solution at two time instances, $t = 48$ s and $t = 55$ s. The time instances are indicated in Figure 1, with dashed lines labeled $t = 48$ s and $t = 55$ s respectively. The figure illustrates the distribution of the queue at the specified times. The slope of each curve corresponds to the density of the solution at the specified time and location.

V. Conclusion and discussion

The present article derives the probability distribution of the solution to a Hamilton-Jacobi partial differential for which the prescribed value conditions are probabilistic. The derivations allow for the analysis of the effects of the randomness of the value conditions on the solution, as illustrated in Section IV. Another application of this approach is the estimation of the parameters characterizing the probability distribution of the value conditions, through maximum-likelihood estimation for example. The article also introduces the necessary derivations to statistically estimate the parameters characterizing the distribution of value conditions. The derivations lead to an instantaneous computation of the distribution given the distribution of the value condition without the need for sampling or simulation.

This work has important applications for systems described by a Hamilton-Jacobi equation for which the value conditions are noisy or cannot be measured accurately. We illustrate the applicability in the context of traffic flows with random capacity reduction, leading to probabilistic congestion and queue formations.

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