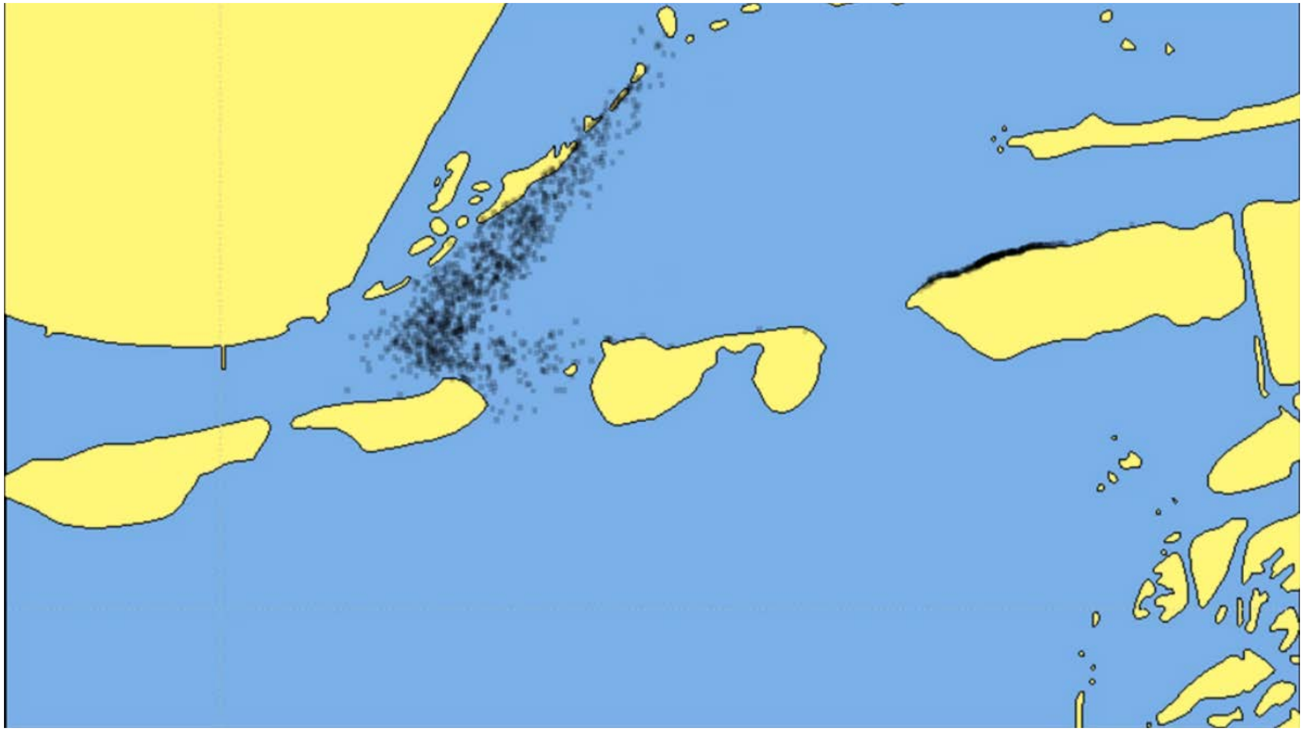


CE 291 Project:

Oil Spill Simulation in Ocean

Student: Zhi Long Liu, Daniel Shen

Instructor: Alex Bayen

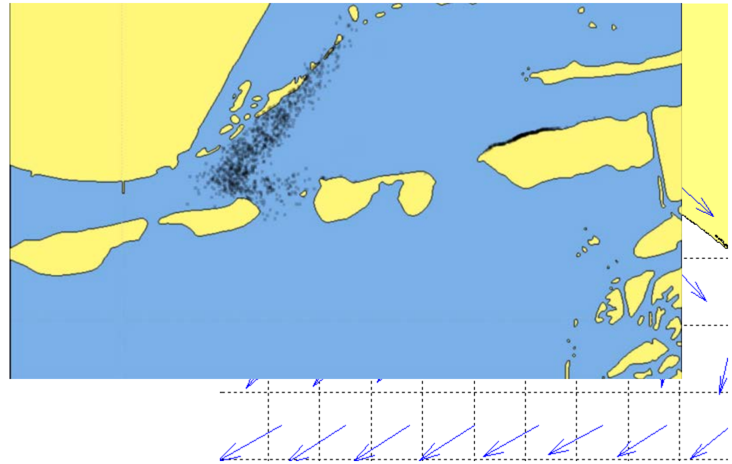
Advisor: Raja Sengupta

TABLE OF CONTENTS

- Background
- Problem Statement
- Simulations
- Error Analysis
- Analytical Model
- Conclusion
- Future Works

BACKGROUND: OIL SPILL PROCESS

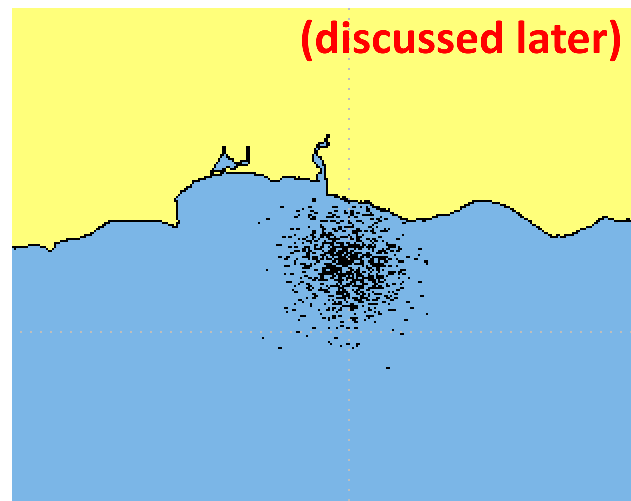
- Interaction with
  - ocean water & wind (velocity fields)
  - shoreline (boundary conditions)
- Processes Involved:
  - advection
  - horizontal diffusion
  - ~~evaporation~~
  - ~~dissolution~~
  - ~~emulsification~~
  - ~~change in density & kinematic viscosity~~
  - ~~etc.~~



ocean current field

random motion

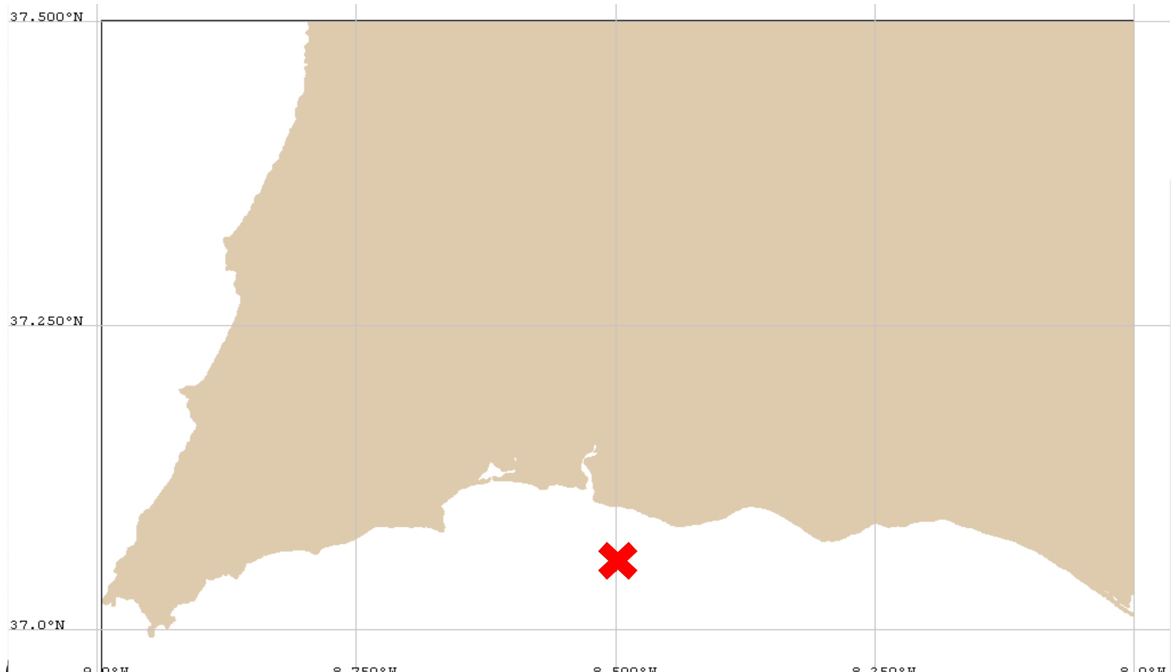
(discussed later)



PROBLEM STATEMENT

- SPILL LOCATION: (south side of Portugal)

37.05 ° N
8.50 ° W



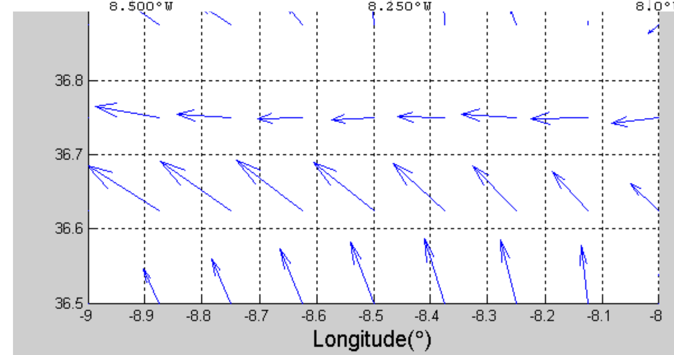
- DIFFUSION

D = 10 m^2/sec

- ADVECTION

Current Velocity Field data from NOAA
(National Oceanic & Atmospheric Administration)

duration:	4 days
space accuracy:	$1/8^{\circ}$
time accuracy:	6 hours

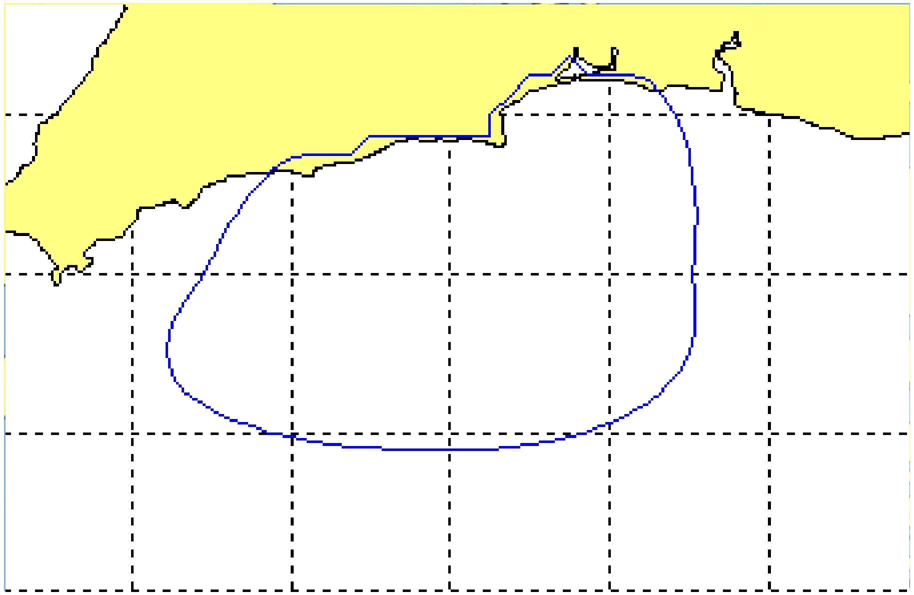


MODELING METHODS

- GNOME (General NOAA Operational Modeling Environment)
- MATLAB
  - Analytical Solution of DE w/ the Mapping Toolbox
  - Level Set Toolbox

    by Ian Mitchell

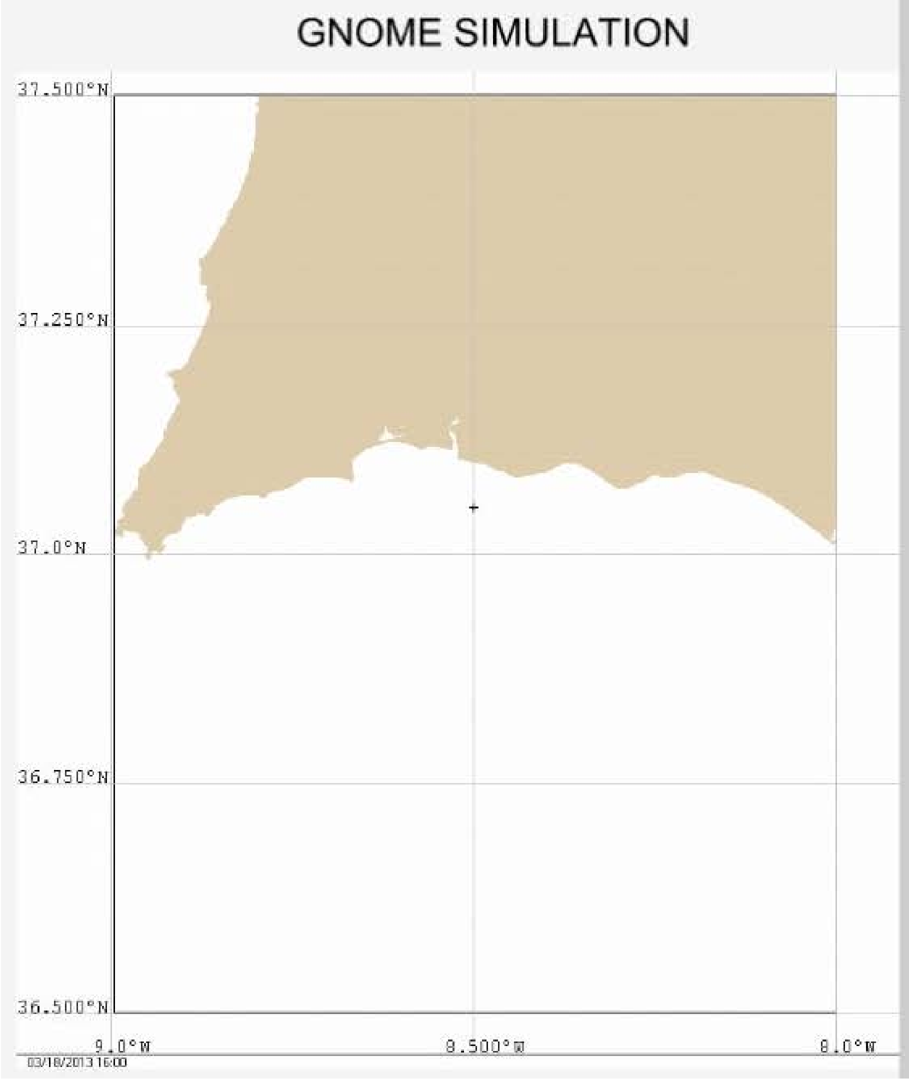
    Computer Science, UBC

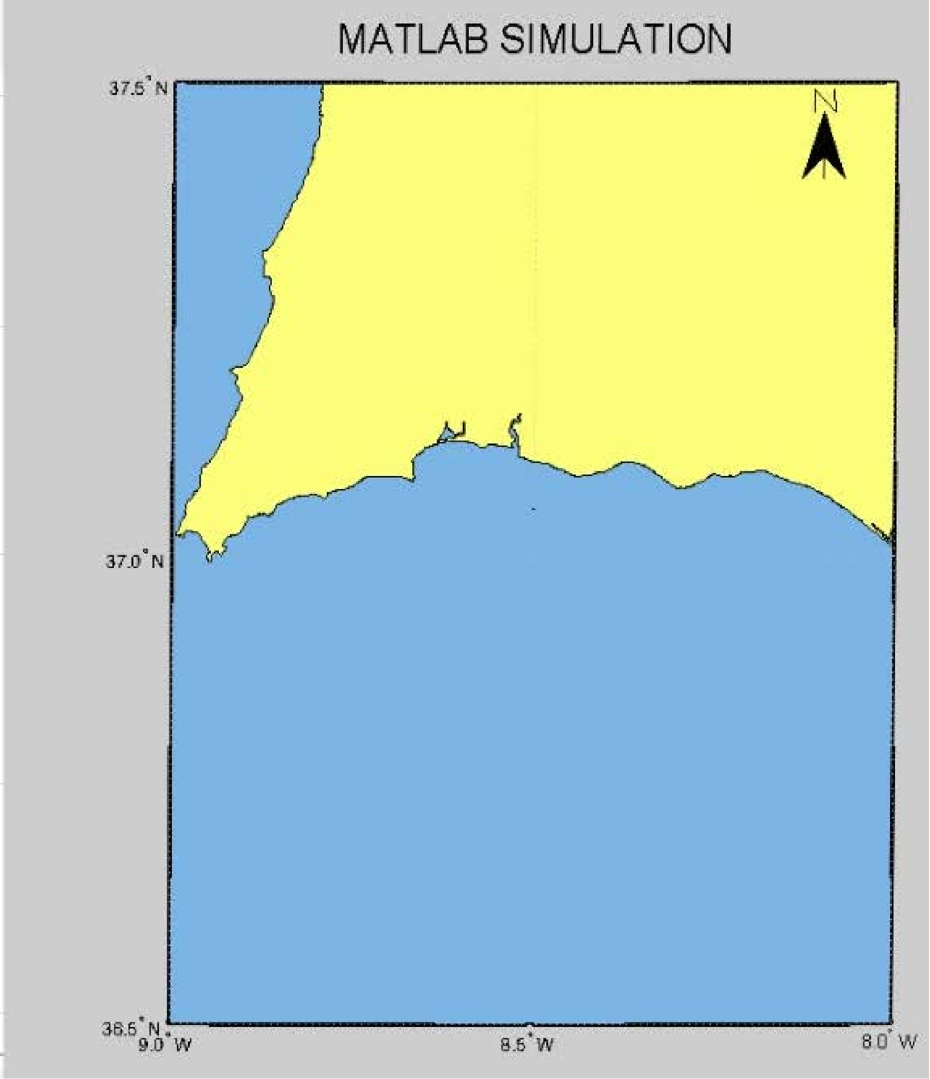


Analytical Model

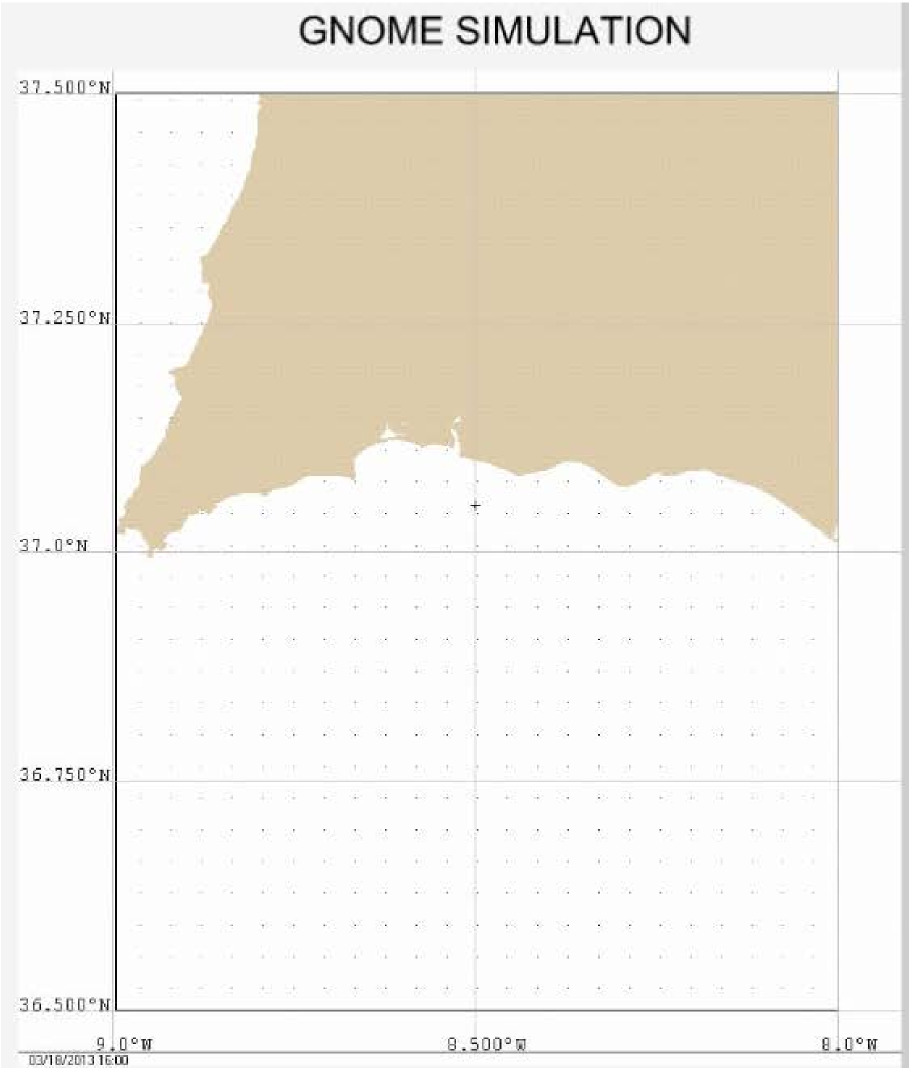
SIMULATIONS

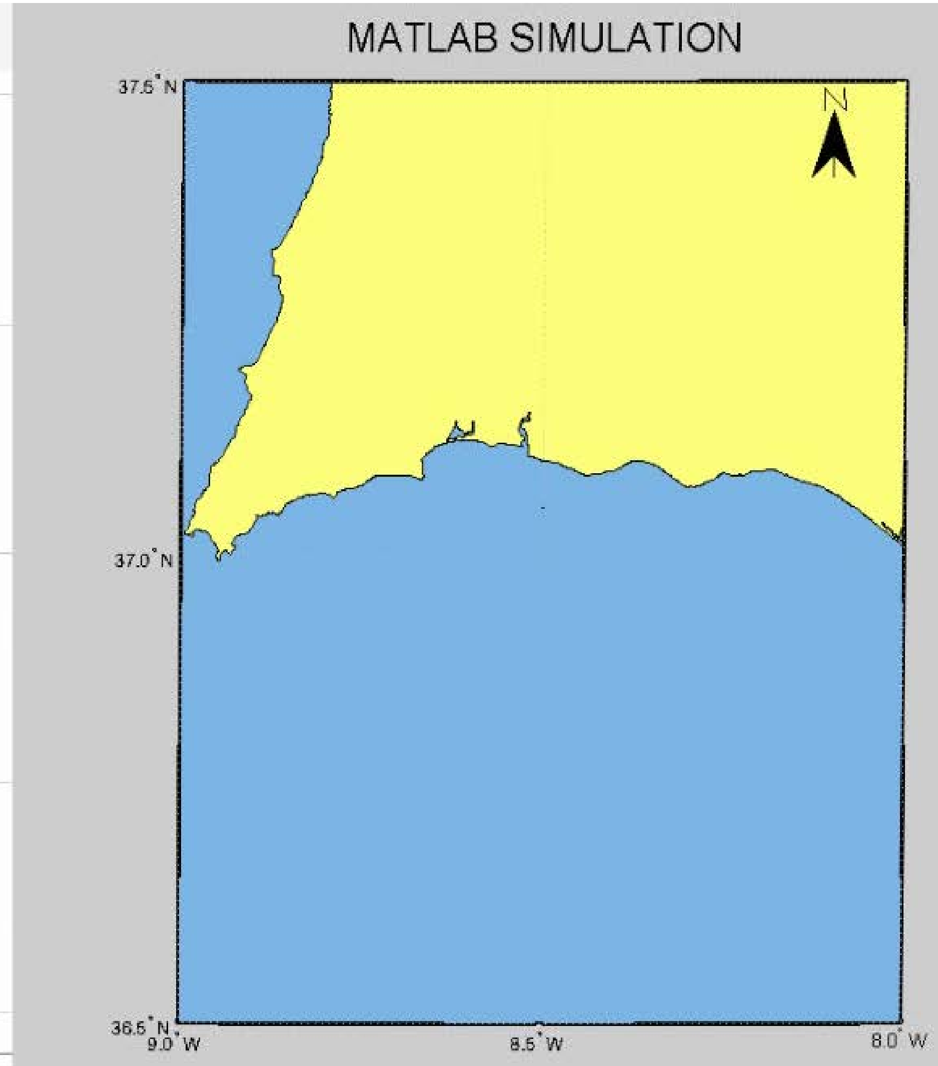
ANALYTICAL RESULTS –DIFFUSION ONLY





ANALYTICAL RESULTS – ADVECTION & DIFFUSION

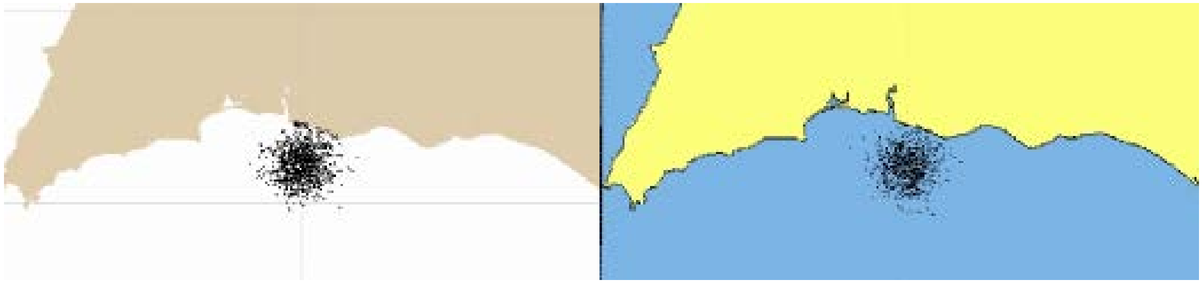




ERROR ANALYSIS - ANALYTICAL

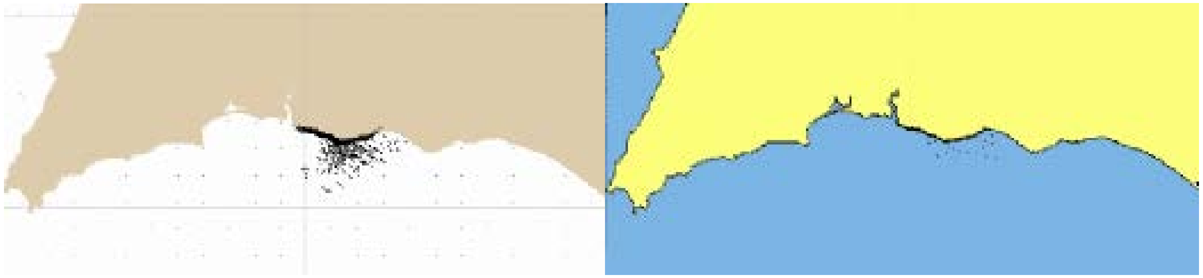
- Diffusion

  matches well



Diffusion

- Advection



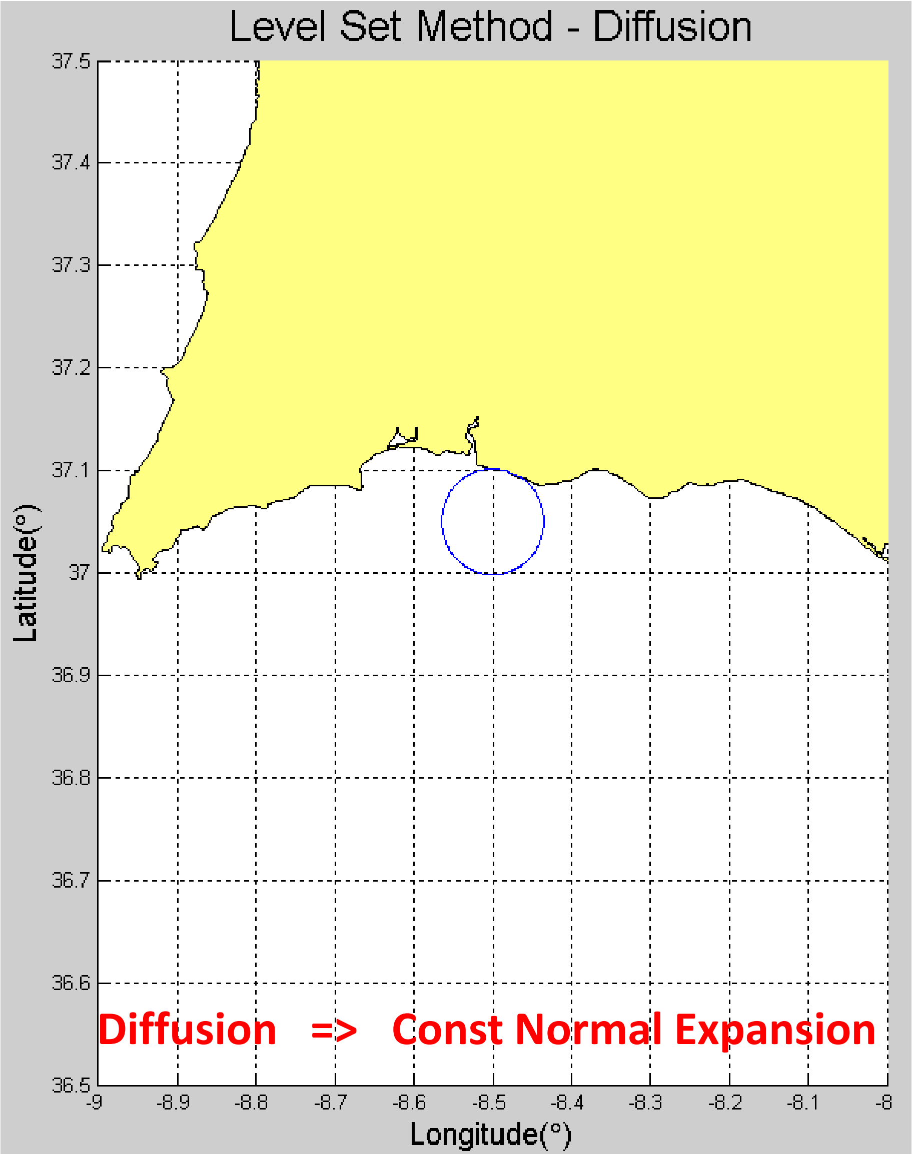
  behave differently

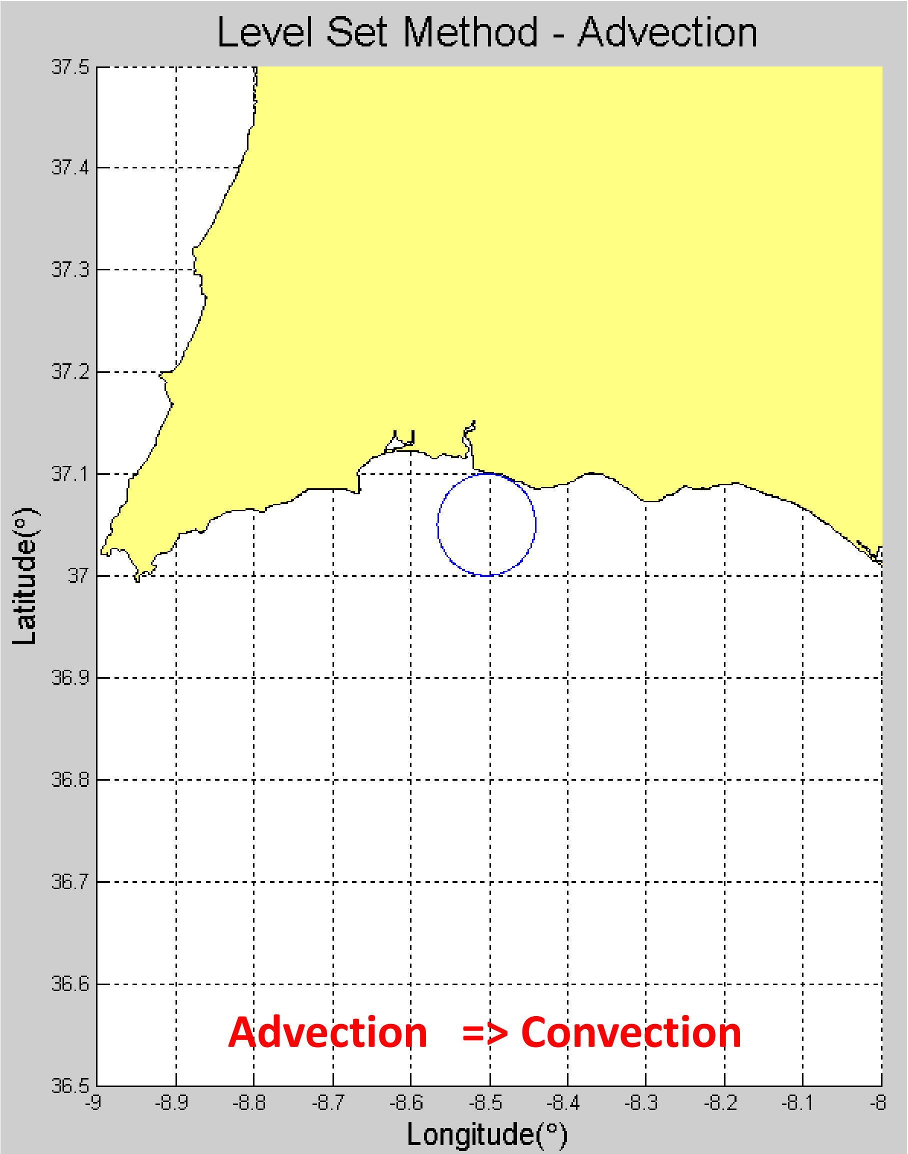
Advection + Diffusion

1. different interpolation technique of current field

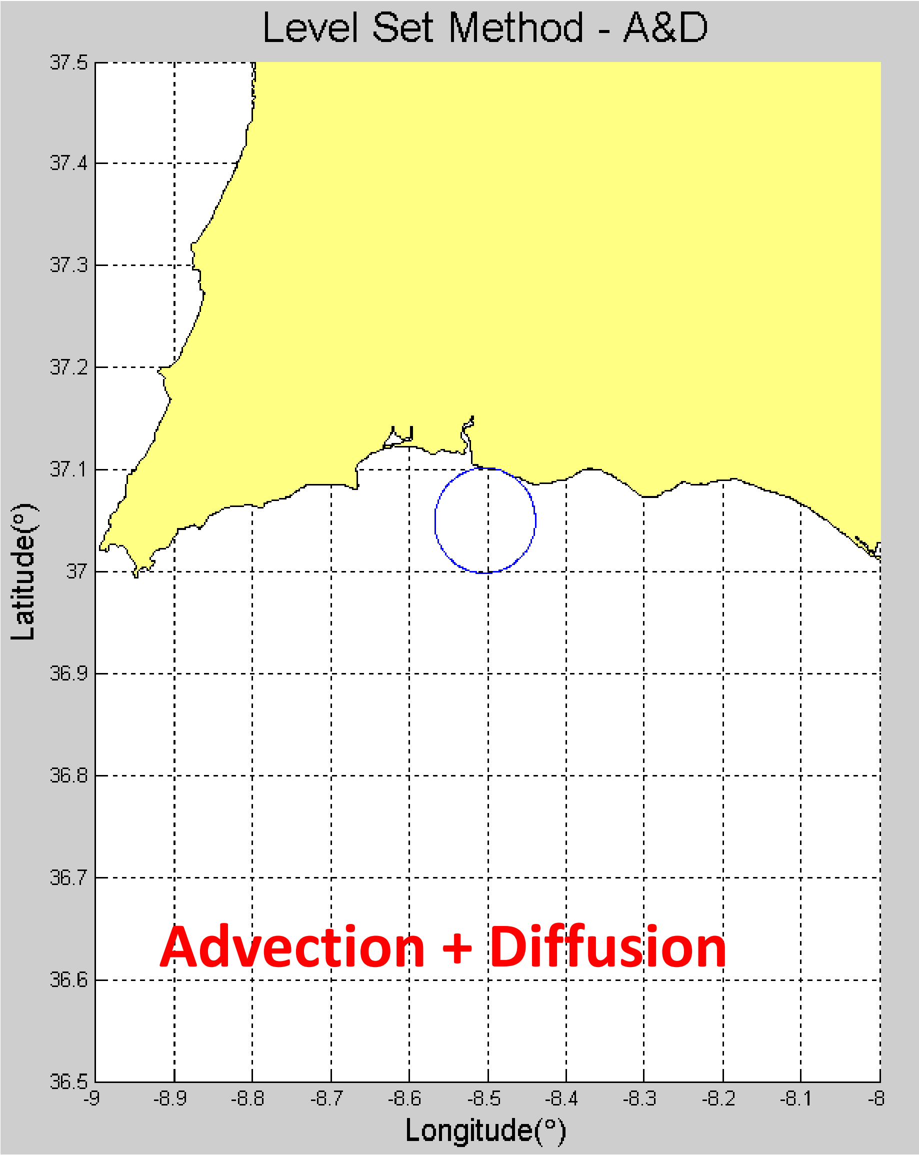
2. small deviation in map file (boundary conditions)

PRELIM. RESULTS – LEVEL SET METHOD





PRELIM. RESULTS – LEVEL SET METHOD



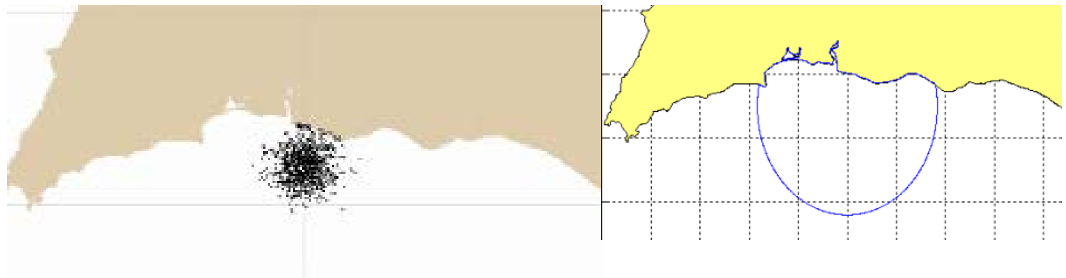
ERROR ANALYSIS – LEVEL SET METHOD

- Diffusion

  Diffusion ≠ Const Expansion

  1^{o} Longitude ≠ 1^{o} Latitude

  ➔ distorted

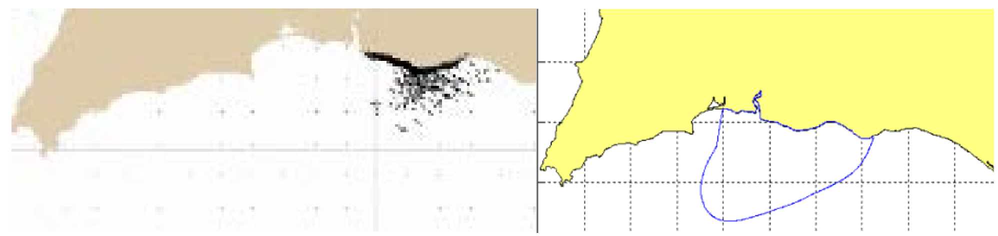


Diffusion

- Advection

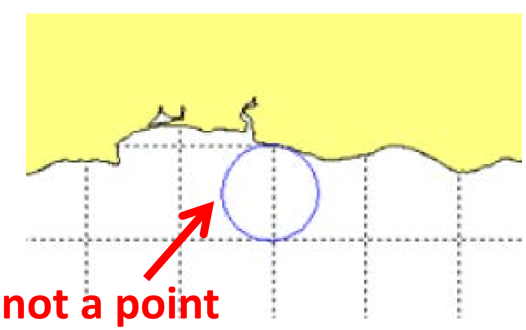
  no direct comparison

  should be fine



Advection + Diffusion

- Point Source – Instability



THEORY

THE PDE'S

The ADE:

advection component

diffusion component

$$\boxed{\frac{\partial C}{\partial t} + \vec{v} \cdot \nabla C = D\nabla^2 C}$$

W/O the advection part ($\vec{v} = 0$)

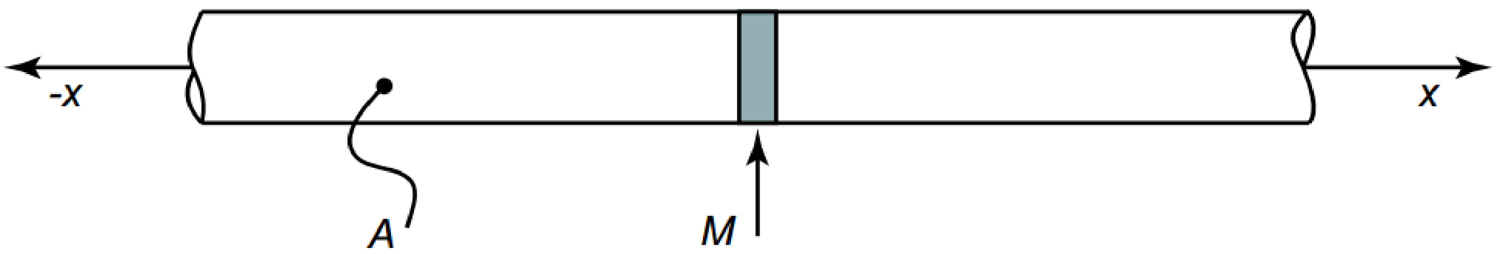
$$\vec{v} \cdot \nabla C = 0$$

the Diffusion Equation (DE) is

$$\boxed{\frac{\partial C}{\partial t} = D\nabla^2 C}$$

SOLUTION TO 1-D DE PDE

- Problem Statement:



$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}$$ ... (PDE)

no oil @ ∞

no boundary

$$\boxed{C(\pm\infty, t) = 0}$$ ... (BC)

oil is concentrated

@ a point @ t = 0

$$\boxed{C(x,0) = (M/A)\delta(x)}$$ ... (IC)

mass is constant

over all volume

$$\boxed{\int_V C(x,t)dV = M}$$ ... (mass cons.)

CONVERT PDE TO ODE

- STEP 1: Vashy-Buckingham (π) Theorem.
- STEP 2: Use the Chain Rule & Plug into the DE PDE.
- STEP 3: Boundary Condition
- STEP 4: Initial Condition
- STEP 5: Mass Conservation

CONVERT PDE TO ODE

- STEP 1: Vashy-Buckingham (π) Theorem.

$$A = \begin{array}{c} \\ M \\ L \\ T \end{array} \begin{array}{c} \begin{array}{ccccc} C & M/A & D & x & t \end{array} \\ \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ -3 & -2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

$$N - rank = 2$$

$$\pi_1 = \frac{C}{M/\left(A\sqrt{Dt}\right)}$$

$$\pi_2 = \frac{x}{\sqrt{Dt}}$$

$$\boxed{C = \frac{M}{A\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)}$$

Set similarity variable

$$\boldsymbol{\eta = \pi_2 = \frac{x}{\sqrt{Dt}}}$$

then

$$\boldsymbol{C = \frac{M}{A\sqrt{Dt}} f(\eta)}$$

CONVERT PDE TO ODE

- STEP 2: Use the Chain Rule & Plug into the DE PDE.

$$\begin{cases} C = \dfrac{M}{A\sqrt{Dt}} f(\eta) \\[2ex] \dfrac{\partial \mathrm{C}}{\partial \mathrm{t}} = D \dfrac{\partial^2 \mathrm{C}}{\partial x^2} \end{cases}$$

$$\frac{\partial C}{\partial \mathrm{t}} = \frac{\partial}{\partial \mathrm{t}}\left[\frac{M}{A\sqrt{Dt}} f(\eta)\right] \overset{\textbf{chain rule}}{=} \cdots = -\frac{M}{2At\sqrt{Dt}}\left(f(\eta) + \eta \frac{\partial f}{\partial \eta}\right)$$

$$\frac{\partial^2 C}{\partial \mathrm{x}^2} = \frac{\partial}{\partial \mathrm{x}}\left[\frac{\partial}{\partial \mathrm{x}}\left(\frac{M}{A\sqrt{Dt}} f(\eta)\right)\right] \overset{\textbf{chain rule}}{=} \cdots = \frac{M}{ADt\sqrt{Dt}} \frac{\partial^2 f}{\partial \eta^2}$$

$$\Longrightarrow \quad \boxed{\frac{d^2 f}{\mathrm{d}\eta^2} + \frac{1}{2} f(\eta) + \frac{1}{2}\eta \frac{df}{d\eta} = 0} \quad \textbf{ODE}$$

CONVERT PDE TO ODE

- STEP 3 & 4: Boundary Conditions & Initial Conditions

Boundary Conditions

$$\begin{cases} C = \dfrac{M}{A\sqrt{Dt}} f\left(\dfrac{x}{\sqrt{Dt}}\right) \\ C(\pm\infty, t) = 0 \end{cases}$$

$$\Rightarrow \quad \left.\frac{M}{A\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)\right|_{x=\pm\infty} = 0$$

$$\Rightarrow \quad \boxed{f(\pm\infty) = 0}$$

Initial Conditions

$$\begin{cases} C = \dfrac{M}{A\sqrt{Dt}} f\left(\dfrac{x}{\sqrt{Dt}}\right) \\ C(x, 0) = (M/A)\delta(x) \end{cases}$$

$$\Rightarrow \quad \left.\frac{M}{A\sqrt{Dt}} f\left(\frac{x}{\sqrt{Dt}}\right)\right|_{t=0} = (M/A)\delta(x)$$

$$\Rightarrow \quad \left. f\left(\frac{x}{\sqrt{Dt}}\right)\right|_{t=0} = \delta(x)\sqrt{Dt}$$

$$\Rightarrow \quad \boxed{f(\pm\infty) = 0}$$

BC's & IC's are the same.

CONVERT PDE TO ODE

- STEP 5: Mass Conservation

$$\boxed{\int_V C(x,t)dV = M}$$

$$\eta = \frac{x}{\sqrt{Dt}} \quad \Rightarrow \quad x = \eta\sqrt{Dt} \quad \Rightarrow \quad dx = \sqrt{Dt}\,d\eta$$

$$\begin{aligned}\int_V C(x,t)dV &= \int_{-\infty}^{+\infty}\left[\frac{M}{A\sqrt{Dt}}f(\eta)\right](Adx)\\ &= \int_{-\infty}^{+\infty}\frac{M}{\sqrt{Dt}}f(\eta)\left(\sqrt{Dt}d\eta\right)\\ &= M\int_{-\infty}^{+\infty}f(\eta)d\eta = M\end{aligned}$$

$$\boxed{\int_{-\infty}^{+\infty}f(\eta)d\eta = 1}$$

THE ODE PROBLEM & SOLUTION

- The ODE Problem

$$\boxed{\begin{gathered}\frac{d^2 f(\eta)}{\mathrm{d}\eta^2} + \frac{1}{2} f(\eta) + \frac{1}{2}\eta \frac{df(\eta)}{d\eta} = 0, \\ f(\pm\infty) = 0, \\ \int_{-\infty}^{+\infty} f(\eta) d\eta = 1\end{gathered}}$$

- Solution to the ODE

$$\boxed{C(x,t) = \frac{M/A}{\sqrt{4\pi D t}} \exp\left(-\frac{x^2}{4Dt}\right)}$$

MODELING DIFFUSION

- Solution to the 1-D Diffusion Equation

$$\boxed{C(x,t) = \frac{M/A}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right)}$$

- Let $\sigma^2 = 2Dt$, then the 1-D Diffusion Equation becomes

$$\frac{C(\sigma,x)}{M/A} = \boxed{\frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)}$$

Gaussian distribution with variance $\boldsymbol{\sigma^2}$.

- MATLAB command **normrnd(μ, σ)** is used to model diffusion

$\mu = 0$ → mean

$\sigma = \sqrt{2D\Delta t}$ → standard deviation

Diffusion => Normal distribution

MODELING ADVECTION

- Solution to the 1-D Diffusion Equation

$$\boxed{C(x,t) = \frac{M/A}{\sqrt{4\pi Dt}} \exp\left(-\frac{{x_d}^2}{4Dt}\right)}$$

- For the 1-D Advection-Diffusion Equation:

shifted by a velocity term

$$\boxed{C(x,t) = \frac{M/A}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x_{a\&d} - \boxed{v_x t})^2}{4Dt}\right)}$$

$$x_d = x_{a\&d} - v_x t$$

$$\boxed{x_{a\&d} = x_d + v_x t}$$

Advection => position increments

MATLAB IMPLEMENTATION - ANALYTICAL

- Particle Movement

```
myLon(k) = myLon(k-1)
    + normrnd(0,sqrt(2*D*dt))/(MtoDEG*cosd(myLat(k-1)))
    + (dx*dt)/(MtoDEG*cosd(myLat(k-1)));
myLat(k) = myLat(k-1)
    + normrnd(0,sqrt(2*D*dt))/MtoDEG
    + (dy*dt)/MtoDEG;
```

current position

previous position

diffusion component

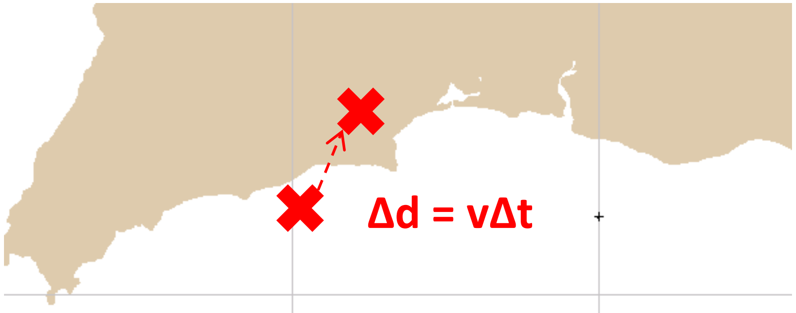
advection component

convert factor from meter to degree

MATLAB IMPLEMENTATION - ANALYTICAL

- Boundary Condition at **coastline**

```
% if the oil particle is on land, stop the iteration.
if myLand(k-1) == true
    myLon(k:end) = myLon(k-1);
    myLat(k:end) = myLat(k-1);
    myLand(k:end) = myLand(k-1);
    break;
end
```



$\Delta d \rightarrow 0$ means $\Delta t \rightarrow 0$ (Small time increments)

MATLAB IMPLEMENTATION – LEVEL SET

- Surface Movement

```
%% Set up spatial approximation for advection
k = 1;
advectionData.velocity{1,1} = v{1,k};
advectionData.velocity{2,1} = v{2,k};
advectionFunc = @termConvection;
advectionData.grid = g;

%% Set up motion in the normal direction
diffusionData.speed = aValue;
diffusionFunc = @termNormal;
diffusionData.grid = g;

%% Combine components of motion.
schemeFunc = @termSum;
schemeData.innerFunc = { diffusionFunc; advectionFunc };
schemeData.innerData = { diffusionData; advectionData };
```

MATLAB IMPLEMENTATION – LEVEL SET

- Boundary Condition at <u>**coastline**</u>

High resolution binary mask

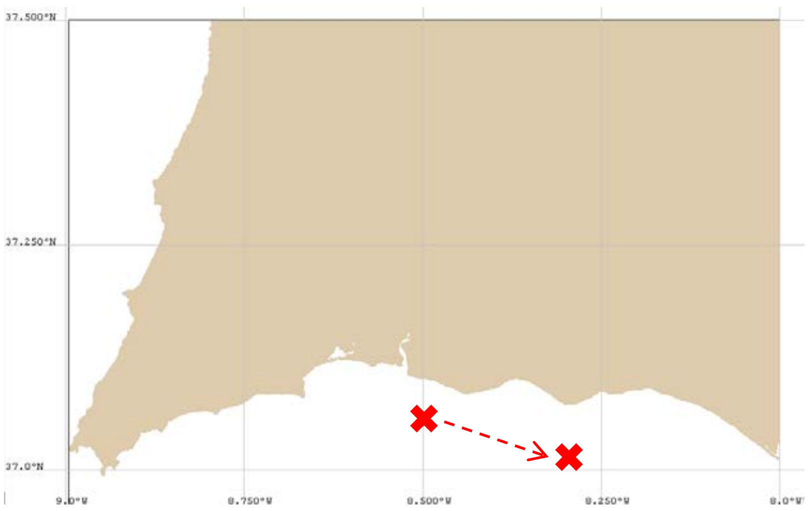
$\Delta x \rightarrow 0$ & $\Delta y \rightarrow 0$ (small space increments)

CONCLUSION

- Analytical
  - Oil Spill → Lagrangian Particles
  - Diffusion → Random Motion (Gaussian Distribution)
  - Advection → Velocity Field
  - B.C. → High resolution in **time** (small Δt)
- Level Set
  - Oil Spill → Stretchable Surface
  - Diffusion → Const Normal Expansion
  - Advection → Velocity Field
  - B.C. → High resolution in **space** (small Δx & Δy)

FUTURE WORKS

- Level Set
  - Diffusion Coefficient ←→ Const Diffusion Expansion Rate
  - Diffusion Scaling in Vertical Dimension
- Possibly Moving Source?



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[2]. 2012. General NOAA Operational Modeling Environment (GNOME) Technical Documentation. Office of Response and Restoration. Emergency Response Division.

[3]. Performing a Numerical Simulation of an Oil Spill. Mathworks.

[4]. "A Toolbox of Level Set Methods." Ian M. Mitchell. UBC Department of Computer Science Technical Report TR-2007-11 (June 2007).

THE ADE PDE

- The Advection-Diffusion Equation (ADE) :

advection component

diffusion component

$$\boxed{\frac{\partial C}{\partial t} + \boxed{\nabla \cdot (\vec{v}C)} = \boxed{D\nabla^2 C}}$$

0 (incompressible for oil)

$$(\cancel{\nabla \vec{v}})C + \vec{v}(\nabla C)$$

where:

$$C = \text{concentration/density of fluid (oil)}$$
$$\vec{v} = \text{velocity of current field}$$
$$D = \text{diffusion coefficient}$$

- For incompressible fluid, $\nabla \vec{v} = 0$, by the chain rule

$$\boxed{\frac{\partial C}{\partial t} + \boxed{\vec{v} \cdot \nabla C} = D\nabla^2 C}$$