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Boundary controllability and asymptotic stabilization of a nonlocal traffic flow model

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Dedicated to Enrique Zuazua on the occasion of his 60th birthday

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Abstract We study the exact boundary controllability of a class of nonlocal conservation laws modeling traffic flow. The velocity of the macroscopic dynamics depends on a weighted average of the traffic density ahead and the averaging kernel is of exponential type. Under specific assumptions, we show that the boundary controls can be used to steer the system towards a target final state or out-flux. The regularizing effect of the nonlocal term which leads to the uniqueness of weak solutions enables us to prove that the exact controllability is equivalent to the existence of weak solutions to the backwards-in-time problem. We also study steady states and the long-time behavior of the solution under specific boundary conditions.

Keywords Conservation laws, nonlocal flux, traffic flow, exact controllability, boundary controllability, stabilization, characteristics.

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1 Introduction

Conservation laws with nonlocal fluxes arise in many areas of applications and have thus attracted much attention in recent years: e.g., traffic flow [46, 24, 22, 23, 21, 57], crowd dynamics [2, 33, 31, 32], sedimentation phenomena [13], slow erosion of granular matter [4, 29, 28], materials with fading memory effects [18], biological and industrial models [34]. However, there are few papers dealing with the controllability and long-time behaviour of solutions of nonlocal conservation laws. For a very specific nonlocal model describing manufacturing systems, where the velocity is strictly positive and the nonlocal term is independent of the spatial domain, some results on state and out-flux controllability and asymptotic exponential stabilization have been obtained in [41, 35]. In [57] for nonlocal dynamics with also backwards looking nonlocal term the authors study the well-posedness and stability of classical solutions on a ring road, i.e., with periodic boundary datum by using a finite difference approximation of solutions. They also succeed in showing the exponential stability of the solution to the steady state (constant) solution and give a counter example that for constant nonlocal kernel stability cannot be expected. This is due to the existence of traveling wave solutions and heavily depends on the used kernel (compare also [54]). Indeed, in [54] again for periodic solutions the author succeed in showing the exponential stability for every monotone kernel and a linear velocity function to the steady state solution.

In this contribution, we investigate the controllability of a class of nonlocal conservation laws with explicit space-dependent nonlocal term (on a one-dimensional bounded domain) modeling traffic flow: the velocity of the density at a given space-time point depends on a weighted average of the traffic density ahead (see Fig. 1.1). We are motivated by the question whether it is possible to steer the traffic state on a road towards a target end-state or as to reach a given out-flux.

The model is quite similar to the problems considered in [57, 54], however, we do not consider periodic solutions but instead a control at the entrance point of the road in terms of the density and at the exit point of the road in terms of the velocity (realized by an appropriate definition of the nonlocal term). This precise model had been introduced in [60] and been studied for its analytical properties like existence and uniqueness of weak solutions and maximum principle.

It is described by the following initial-boundary value problem.

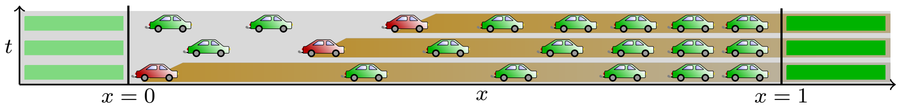

Fig. 1.1 The nonlocal impact (in gold) in traffic flow modeling. The red car looks ahead within the golden region and, taking into account the presence of a high car density far away, adjusts its velocity. Inflow and the speed of the cars leaving the road segment are located at $x = 0$ and $x = 1$, respectively. The green areae at the left and right side of the road segment visualize the respective boundary datum, u_ℓ and u_r.

Definition 1.1 (The nonlocal dynamics) We consider the following initial-boundary value problem:

$$\partial_t\rho(t,x) + \partial_x\big(V(\mathcal{W}[\rho](t,x))\rho(t,x)\big) = 0, \qquad (t,x) \in \Omega_T, \tag{1.1}$$

$$\rho(0,x) = \rho_0(x), \qquad x \in (0,1), \tag{1.2}$$

$$V(\mathcal{W}[\rho](t,0))\rho(t,0) = V(\mathcal{W}[\rho](t,0))u_\ell(t), \quad t \in (0,T) \tag{1.3}$$

with $\Omega_T \coloneqq (0,T) \times (0,1)$, supplemented by the nonlocal operator

$$\mathcal{W}[\rho](t,x) \coloneqq \tfrac{1}{\eta}\int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}} \left(\begin{cases}\rho(t,y) & \text{if } y < 1\\ u_r(t) & \text{if } y \geq 1\end{cases}\right) \mathrm{d}y, \tag{1.4}$$

for $(t,x) \in \Omega_T$ and $\rho : \bar{\Omega}_T \to [0,1]$. Here, $\rho_0 : (0,1) \to [0,1]$ is the initial datum, $u_\ell : (0,T) \to [0,1]$ is the (entering) boundary datum at $x = 0$; $u_r : (0,T) \to [0,1]$ is the right hand side nonlocal impact; $V : [0,1] \to \mathbb{R}_{\geq 0}$ is the velocity; $\eta \in \mathbb{R}_{>0}$ is the nonlocal average parameter; and the exponential function in the nonlocal operator is the weight of the nonlocal term.

At the left-hand side of the road, the entry point, we prescribe an in-flux boundary condition which can be interpreted as an on-ramp of a road and can be used to control the dynamics as an on-ramp metering. The function u_r in the nonlocal operator $\mathcal{W}$ in Eq. (1.4) can be interpreted as a parameter which influences the velocity with which the right hand side boundary datum leaves at $x = 1$. Due to the nonlocal term, it also influences the velocity on the entire link. The need for u_r being defined outside of the considered domain Ω_T stems from the fact that the nonlocal term – dependent on the nonlocal weight – requires input from $(1,\infty)$. It can be used to model traffic lights: if $u_r = 1$, no density leaves (red light); if $u_r = 0$, the adjacent road is fully evacuated and density can leave as fast as possible (green light). Thus, for any control purpose, both the left-hand side boundary datum u_ℓ as well as the right-hand term u_r can be used (compare [9] for the corresponding local dynamics on bounded domains).

The choice of an averaging kernel of exponential type is motivated by the analysis in [16], where it has been shown that – under specific additional assumptions on input datum and velocity V – the solution of the Cauchy problem converges to the entropy solution of the corresponding conservation law as the nonlocal impact η approaches zero. In traffic models however, the exponential kernel is not really standard: indeed, one would consider a kernel with compact support as cars look ahead within a finite space horizon. However, this kind of behavior is nicely approximated by an exponentially decaying kernel. The claimed results should hold true for more general kernels (compare [59]); we will not detail this in the present contribution.

Moreover, due to the choice of an exponential kernel, the boundary condition which is prescribed on the flux in Eq. (1.3) can be given directly in terms of density, i.e. as

$$\rho(t,0) = u_{\ell}(t), \quad t \in [0,T],$$

as long as $\min\{\|\rho_0\|_{L^1((0,1))}, u_r(t)\} < 1$ for all $t \in [0,T]$. Indeed, in this case, the velocity V is never null at the boundary (and also nowhere else) so that the boundary datum always enters the domain and is thus always attained.

1.1 Literature on the control of conservation laws

The previous boundary control results on nonlocal conservation laws are only focused on a simpler version of the equation presented in Definition 1.1 that was introduced in [8] to model semiconductor manufacturing systems:

$$\begin{aligned} \partial_t\rho(t,x) + \partial_x(\rho(t,x)V(W[\rho](t))) &= 0, && (t,x) \in \Omega_T,\\ \rho(0,x) &= \rho_0(x), && x \in (0,1),\\ V(W(t))\rho(t,0) &= u(t), && t \in (0,T), \end{aligned} \tag{1.5}$$

with a strictly positive velocity function $V \in C^1(\mathbb{R};\mathbb{R}_{>0})$, namely

$$V(s) = \tfrac{1}{1+s},\ s \geq 0, \quad \text{and} \quad W[\rho](t) = \int_0^1 \rho(t,x)\,\mathrm{d}x.$$

The difficulties in comparison to the latter model are due to the fact that, for the nonlocal conservation law we consider in Definition 1.1, the nonlocal term depends also on the spatial component, i.e. the spatial dependency is not integrated out as in Eq. (1.5). In [40,72,35,41,42,20], for the open-loop system, the authors study an optimal control problem, state controllability and out-flux controllability; and for the closed-loop system, they use a Lyapunov function approach to prove some exponential stabilization results. These results were generalized in [25], where (local) state controllability and out-flux controllability results were established for a space-dependent velocity (but still space-independent nonlocal term) $V(x, W[\rho](t))$. In [57,54] the authors consider indeed the dynamics in Definition 1.1, however on a ring road which makes boundary datum in Eq. (1.3) and u_r as part of the nonlocal operator in Eq. (1.4) unnecessary. They show that for specific kernels stability cannot be expected and for monotonically decreasing kernel and linear velocity function exponential stability to the constant steady state solution holds (see also Section 1).

The optimal control problem for an analogous system of scalar nonlocal conservation laws on networks that models a highly re-entrant multi-commodity manufacturing system was analyzed in [50].

A more general conservation law with explicitly space dependent nonlocal flow describing a supply chain model and the description of pedestrian flows was considered in [30].

On the other hand, questions related to the control of local scalar conservation laws and systems have received much attention throughout the past several decades and, consequently, the literature is vast. We refer the reader to [37,51,61] and references therein for an overview of controllability results for hyperbolic conservation laws, in the case of solutions without shocks; and to [1,7,6,5,65] in the case of solutions developing shocks. For asymptotic stabilization of hyperbolic systems, see [64,69,36,38,39,74,75]. The nodal profile controllability for quasi-linear hyperbolic systems has been considered in [49, 48,62,63].

1.2 Outline of the paper

This paper is organized as follows. In Section 2, we recall some preliminary results on the well-posedness for the class of nonlocal conservation laws introduced in Definition 1.1.

In Section 3 we prove that any end state can be reached from accordingly defined initial and boundary datum on a sufficiently small time horizon. This phenomenon is exclusively "nonlocal": for local conservation laws, the reachable targets are characterized by the Oleinik entropy condition [67,3]. We also provide some numerical examples to illustrate the result.

In Section 4, we discuss the exact controllability to a given end-state or out-flux of the nonlocal model with boundary controls on the left (in-flux) and on the right (out-flux) of the domain. We prove that it is equivalent to the existence of a solution of the corresponding backwards-in-time nonlocal conservation law. This is related to the fact that weak solutions of the nonlocal conservation law are unique so that we have no loss of information over time.

We study the long-time behavior of solutions in Section 5, when constant boundary conditions are prescribed and the initial condition is suitably chosen. We show that the solution converges to the corresponding constant steady state. Some numerical simulations verify the results and suggest that the results should hold for every initial datum.

In Section 6, we state existence and uniqueness of steady state solutions for constant but not necessarily identical constant boundary datum $u_\ell \neq u_r$.

Finally, in Section 7, we conclude this contribution and present some open problems.

2 Preliminaries and basic results

We first recall some well-known results on the existence and uniqueness of solutions to the initial-boundary value problem described in Definition 1.1. To this end, we introduced the following (regularity) assumptions.

Assumption 1 (Assumption on the data of Definition 1.1) *For* $T \in \mathbb{R}_{>0}$ *we assume*

Nonlocal parameter: $\eta \in \mathbb{R}_{>0}$
Initial datum: $\rho_0 \in L^{\infty}((0,1);[0,1])$*;*
Velocity: $V \in W^{1,\infty}((0,1);\mathbb{R}_{\geq 0}):\ V' \leqq 0, V' \not\equiv 0,\ \big(V(s)=0 \iff s=1\big)$*;*
Boundary: $(u_{\ell}, u_{r}) \in L^{\infty}((0,T);[0,1])^2$.

Following [60, Definition 2.4], we give the following definition of solutions:

Definition 2.1 (Weak solutions) We say that $\rho \in C([0,T];L^1((0,1))) \cap L^{\infty}((0,T);L^{\infty}((0,1)))$ is a *weak solution* to the initial-boundary value problem introduced in Definition 1.1 if for every $\varphi \in W^{1,\infty}((0,T)\times(0,1))$ with $\varphi(T,\cdot)=0$ and $\varphi(\cdot,1)=0$, we have that

$$
\begin{aligned}
0 &= \iint_{\Omega_T} \Big(\partial_t\varphi(t,x) + V(\mathcal{W}[\rho])(t,x)\partial_x\varphi(t,x)\Big)\rho(t,x)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^1 \rho_0(x)\varphi(0,x)\,\mathrm{d}x + \int_0^T \varphi(t,0)V(\mathcal{W}[\rho])(t,0)u_{\ell}(t)\,\mathrm{d}t.
\end{aligned}
\tag{2.1}
$$

Existence and uniqueness of weak solutions have been investigated in [60]. We recall the main well-posedness result in the following theorem.

Theorem 2.1 (Existence, uniqueness, maximum principle) *Given Assumption 1, the nonlocal initial-boundary value problem introduced in Definition 1.1 admits a unique weak solution* $\rho \in C([0,T];L^1((0,1))) \cap L^{\infty}((0,T);L^{\infty}((0,1)))$ *in the sense of Definition 2.1. Moreover, the solution can be stated in terms of characteristics for* $(t,x) \in \Omega_T$ *as*

$$
\rho(t,x) = \begin{cases} \rho_0(\xi_{w^*}(t,x;0))\,\partial_2\xi_{w^*}(t,x;0), & x \geq \xi_{w^*}(0,0;t) \\ u(\xi_{w^*}[t,x]^{-1}_{\max}(0))\,\partial_2\xi_{w^*}(t,x;\xi_{w^*}[t,x]^{-1}_{\max}(0)) & x \leq \xi_{w^*}(0,0;t) \end{cases}
\tag{2.2}
$$

where $\xi : [0,T]\times[0,1]\times[0,T] \to \mathbb{R}_{\geq 0}$ *is the characteristic curve that satisfies*

$$
\xi_{w^*}(t,x;\tau) = x + \int_t^{\tau} V(w^*(s,\xi_{w^*}(t,x;s)))\,\mathrm{d}s,\ \ (t,x,\tau) \in \overline{\Omega_T}\times[0,T],
\tag{2.3}
$$

$\xi^{-1}_{\max}[t,x]$ *denotes the time-inverted characteristics tracing back the points* $(t,x) \in \{(t,x) \in \Omega_T : x \leq \xi_{\tilde{w}}(0,0;t)\}$ *to the boundary ([60, Definition 2.5, Equation (2.3)]) and* $w^* \in L^{\infty}((0,T);W^{1,\infty}((0,1)))$ *is the unique solution of a fixed-point equation on* $(t,x) \in \Omega_T$ *given in [60, Theorem 3.1, Eq. (3.2)]. In addition, the following maximum principle holds for a.e.* $(t,x) \in \Omega_T$

$$
0 \leq \rho(t,x) \leq \max\{\|\rho_0\|_{L^{\infty}((0,1))}, \|u_{\ell}\|_{L^{\infty}((0,t))}, \|u_{r}\|_{L^{\infty}((0,t))}\}.
\tag{2.4}
$$

Proof The proof can be found in [60, Theorem 3.1, Theorem 4.2, Corollary 5.9] for a compactly supported nonlocal kernel being monotonically decreasing. However, the exponential kernel considered in Definition 1.1 actually simplifies the analysis and the results can be obtained analogously. We omit the details. □

We remark that, in contrast to local conservation laws (see [44, 15, 53]), nonlocal models of this kind do not require an entropy condition to select a unique solution. Having stated these fundamental results, we study the controllability properties of the nonlocal weak solutions.

3 Reachability for sufficiently small times

In this section, we show that, for any given function in $L^\infty((0,1);[0,1])$, we can find suitable boundary and initial datum so that the solution of the corresponding nonlocal conservation law reaches the target at a (sufficiently small) time $T > 0$. The key idea for the proof is to consider the backward-in-time problem, which solvability is equivalent to the controllability of the given forward problem. Thanks to the results in [58], the backward problem is solvable for any terminal data for sufficiently small time horizon. This is due to the fact that the nonlocal velocity function is for small time Lipschitz-continuous (independently of the specific nonlocal weight and area of integration as long as the initial datum is essentially bounded).

This result is in contrast to local conservation laws where the attainable set necessarily needs to satisfy an Oleinik inequality [3, 67] also for arbitrary small time preventing the dynamics to have jumps downwards (in the case where the flux function is strictly concave) and thus postulating a rarefaction wave getting less steep over time.

Theorem 3.1 (Exact controllability on small time horizon) *For every $\rho_{des} \in L^\infty((0,1);[0,1))$ with $\|\rho_{des}\|_{L^\infty((0,1))} < 1$ there exist a time $T \in \mathbb{R}_{>0}$, admissible controls $u_\ell, u_r \in L^\infty((0,T);[0,1))$ and admissible initial datum $\rho_0 \in L^\infty((0,1);[0,1))$ such that the corresponding weak solution*

$$\rho \in C\big([0,T]; L^1((0,1))\big) \cap L^\infty\left((0,T); L^\infty((0,1))\right)$$

to the conservation law in Definition 1.1 satisfies

$$\rho(T,\cdot) \equiv \rho_{des}.$$

Proof For $u_r \equiv c$ with $c \in [0,1)$ there exists – as shown in [58, Theorem 2.20] – a sufficiently small time-horizon $T \in \mathbb{R}_{>0}$ such that the following auxiliary end-value problem

$$\begin{aligned} \partial_t p(t,x) + \partial_x\big(V(\mathcal{W}[p](t,x))p(t,x)\big) &= 0, && (t,x) \in (0,T) \times \mathbb{R}, \\ p(T,x) &= \rho_{\mathrm{des}}(x), && x \in (0,1), \\ p(T,x) &= c, && x \in \mathbb{R} \setminus (0,1), \end{aligned} \tag{3.1}$$

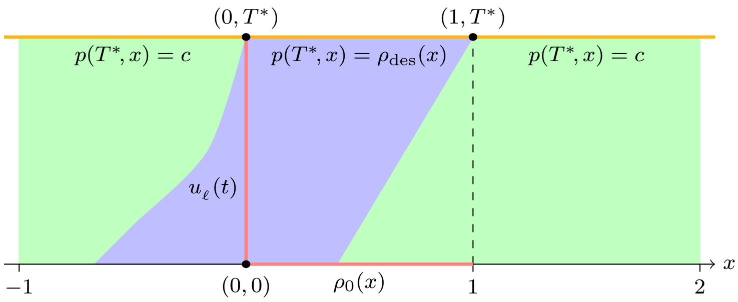

Fig. 3.1 Transformation of the end boundary value problem into a backward in time Cauchy problem on $\mathbb{R}$. In gold the given desired state and the "boundary" data, in red the corresponding datum ρ_0 and u_ℓ which gives – forward in time – the desired state ρ_{des}.

with

$$\mathcal{W}[p](t,x) := \tfrac{1}{\eta}\int_x^{\infty} \exp\left(\tfrac{x-y}{\eta}\right) p(t,y)\,\mathrm{d}y, \tag{3.2}$$

admits a unique solution $p \in C([0,T]; L^1_{\text{loc}}(\mathbb{R}))$. Moreover, by [58, Lemma 2.6, Item 2] there exists $d \in \mathbb{R}_{\geq 0}$ (depending on $\eta, \rho_{\text{des}}, c$ and V) such that

$$\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq \max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}\,,\, c\}e^{d(T-t)}.$$

The key idea of interpreting the control problem as a Cauchy problem on $\mathbb{R}$ backwards in time is illustrated in Fig. 3.1. Thus, for

$$T \leq \tfrac{1}{d}\log\left(\max\{\|\rho_{\text{des}}\|_{L^\infty((0,1))}\,,\, c\}^{-1}\right),$$

we obtain $\|p(t,\cdot)\|_{L^\infty(\mathbb{R})} \leq 1$ for all $t \in [0,T]$. Consequently, by choosing

$$u_\ell(t) = p(t,0), \qquad t \in (0,T), \tag{3.3}$$

$$u_r(t) = c, \qquad t \in (0,T), \tag{3.4}$$

$$\rho_0(x) = p(0,x), \qquad x \in (0,1), \tag{3.5}$$

the boundary and initial data are admissible and the solution to the corresponding IVP (1.1) satisfies $\rho(T,\cdot) \equiv \rho_{\text{des}}$ on $(0,1)$. Note that u_ℓ is given by $p(\cdot,0)$ which can be evaluated as an L^1 function at $x = 0$ as the backwards "velocity" is not zero. □

Remark 3.1 (Surjectivity of state to control map over small times) We remark that the statement in Theorem 3.1 amounts to

$$\bigcup_{t\in(0,T]} \bigcup_{\substack{u_\ell \in L^\infty((0,T);[0,1))\\ u_r \in L^\infty((0,T);[0,1))\\ \rho_0 \in L^\infty((0,1);[0,1))}} \rho[\rho_0, u_\ell, u_r](t,\cdot) = L^\infty((0,1);[0,1)),$$

where $\rho[\rho_0, u_\ell, u_r] \in C([0,T]; L^1((0,1))) \cap L^\infty((0,T); L^\infty((0,1)))$ denotes the weak solution of the nonlocal conservation law in Definition 1.1 with initial datum ρ_0, left hand side boundary datum u_ℓ and nonlocal right hand side u_r. It thus states the surjectivity of the control to state map when uniting over sufficiently small times. Note that this is in contrast to the local dynamics as pointed out before, where for strictly concave flux – due to Oleinik's entropy condition [67,3] – the jumps downwards are "smoothed" over time.

Example 3.1 (Numerical example for exact controllability on a sufficiently small time horizon) We consider a target function

$$\rho_{\text{des}} :\equiv \tfrac{1}{2} + \tfrac{1}{4}\chi_{\left(\frac{1}{4},\frac{1}{2}\right)} - \tfrac{1}{4}\chi_{\left(\frac{1}{2},\frac{3}{4}\right)}$$

We verify numerically that we can find suitable initial ρ_0 and boundary data mu_ℓ, u_r such that $\rho(T,\cdot) \equiv \rho_{\text{des}}$ for this sufficiently small time horizon $T = 0.6$ (see Fig. 3.2). We remark that, for local conservation laws, Oleinik's entropy condition would prevent the reachability of this state. It can also be observed the important role of the nonlocal parameter $\eta \in \mathbb{R}_{>0}$. The smaller the η (here $\eta \in \{1, 0.9, 0.8\}$ in the given example the more the solution increases backwards over time. This is illustrated in the first three rows of Fig. 3.2 and in particular the boundary datum so that for $\eta = 0.8$, the backward solution has already exceeded 1 and is thus not admissible for $T = 0.6$. The fourth row in Fig. 3.2 represents the solution and control for a sufficiently smaller $\eta = 0.1$. Here, the final time needs to be chosen much smaller, $T = 0.05$ and even then the backwards solution already reaches the bound 1 and would cease to exist if we would consider it on a sufficiently larger time horizon. Due to the short time horizon in the fourth row, the significant changes in the desired datum ρ_{des} are tackled mainly by the initial datum and the boundary datum is almost constant. Due to the short time it can also be seen in the middle pictures the relation between desired state and initial state where the initial state is moved slightly before the desired state but also needs to compensate for the nonlocal term, necessitating the peaks at the jump discontinuities.

4 Exact boundary controllability and time-inverted dynamics

In this section, we consider two control problems:

- steering a given initial state towards a prescribed target state,
- achieve a prescribed out-flux on the right-hand side of the road.

In both cases, we show that exact controllability holds if and only if the corresponding backwards-in-time dynamics admits a weak solution and satisfies some bounds. This result is essentially due to the fact that, for nonlocal conservation laws, there is no loss of information (with regard to initial and boundary data), i.e. initial and left side boundary data can be uniquely identified from a given final state, right boundary datum and right side nonlocal impact. This is also related to the key result that weak solutions of nonlocal conservation laws are per se unique and no entropy condition is required (see [58,60]).

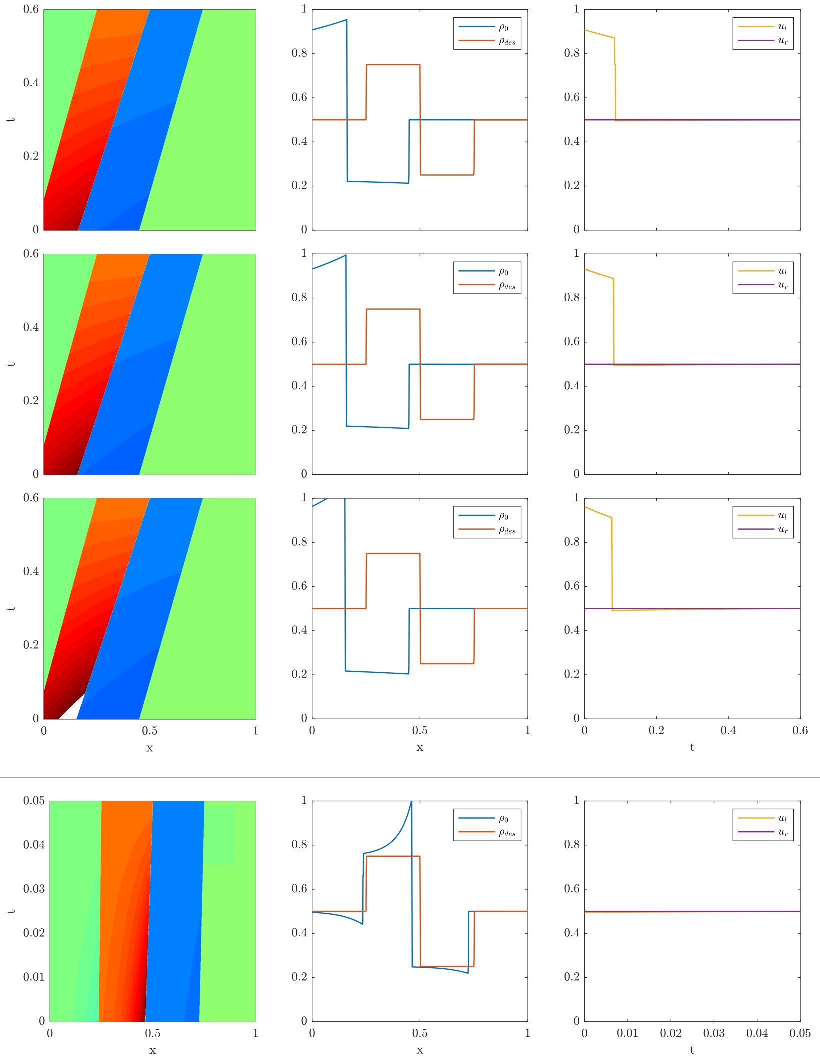

Fig. 3.2 Illustration of Example 3.1 for different $\eta \in \{0.8, 0.9, 1, 0.1\}$. **Left:** The solution with the proper boundary and initial datum to reach the desired state $\rho_{\text{des}} \equiv \frac{1}{2} + \frac{1}{4}\chi_{(0.25,0.5)} - \frac{1}{4}\chi_{(0.5,0.75)}$ for $T = 0.6$. **Middle:** Desired state ρ_{des} and the corresponding initial state ρ_0 to steer the system to ρ_{des}. **Right:** Boundary data, i.e. u_ℓ, u_r, to steer the system to ρ_{des}. **Colorbar:** 0 [colorbar] 1

Our approach is reminiscent of the one used to obtain an exact controllability result for the linear transport equation (see [38, Section 2.1])

$$\begin{aligned} \partial_t \rho(t,x) + \partial_x \rho(t,x) &= 0, && (t,x) \in (0,T) \times (0,1), \\ \rho(0,x) &= \rho_0(x), && x \in (0,1), \\ \rho(t,0) &= u(t), && t \in (0,T), \end{aligned}$$

namely, given $\rho_0 \in L^p((0,1))$ with $p \in \mathbb{R}_{\geq 1} \cup \{\infty\}$ and a target profile $\rho_{\text{des}} \in L^p((0,1))$, a control $u \in L^p((0,1))$ exists so that $\rho(T,\cdot) = \rho_{\text{des}}$ if and only if $T \geq 1$. The key of the proof is observing that the solution of the initial-boundary value problem is given explicitly by

$$\rho(t,x) := \begin{cases} \rho_0(x-t), & (t,x) \in (0,1) \times (0,T),\ t \leq x,\\ u(t-x), & (t,x) \in (0,1) \times (0,T),\ t > x; \end{cases}$$

therefore, if $T \geq 1$, we can choose

$$u(t) := \begin{cases} \rho_{\text{des}}(T-t), & t \in (T-1,T),\\ 0, & t \in (0,T-1), \end{cases}$$

and then the solution satisfies for $(t,x) \in (0,T) \times (0,1)$ $\rho(t,x) = u(T-x) = \rho_{\text{des}}(x)$. In other words, after the initial data (which moves along characteristics) has left the domain, we can inject the solution of the backward-in-time problem having ρ_{des} as initial data in the left-hand boundary.

Since the waves of hyperbolic equations have finite speed of propagation and the control is applied at the boundary, an exact controllability result requires that the time T horizon must be sufficiently large. Similarly, in our nonlocal model, the first crucial step is to know that the initial state leaves the domain in finite time. This seems very natural when prescribing as right-hand side boundary term a density $u_r \in [0,1)$, which then necessarily pulls out the initial data and for non-zero velocities. However, contrary to the linear case, for the nonlocal conservation law considered here the initial datum – even after leaving the domain – has still an impact on the solution as it has changed the shape of the solution stemming from the boundary datum through the nonlocal term.

The result about initial datum leaving the domain is detailed in the following Lemma and is illustrated in Fig. 4.1.

Lemma 4.1 (Initial datum leaving domain in finite time) *Given Assumption 1 and a large enough $T \in \mathbb{R}_{>0}$, assume that $\|u_r\|_{L^\infty((0,T))} < 1$. Then, the initial datum – evolving with the dynamics in Definition 1.1 – leaves the domain in finite time, i.e. the corresponding characteristics ξ as in Eq.* (2.3) *emanating from* $(0,0)$ *satisfies*

$$\exists! T^* \in (0,T] : \xi_{\tilde{w}}(0,0;T^*) = 1 \text{ with } T^* \leq V\Big(1 - \tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\Big)^{-1}. \tag{4.1}$$

Proof We show that the zero characteristics moves with positive speed which is bounded away from zero. To this end, we use the maximum principle in Theorem 2.1 and estimate the nonlocal operator in Eq. (1.4) as follows for

$(t,x) \in \Omega_T$:

$$\begin{aligned}
\mathcal{W}[q](t,x) &= \tfrac{1}{\eta}\int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}} \left(\begin{cases}\rho(t,y) & \text{if } y<1\\ u_r(t) & \text{if } y\geq 1\end{cases}\right)\mathrm{d}y\\
&\leq \tfrac{1}{\eta}\int_0^\infty \mathrm{e}^{-\frac{y}{\eta}} \left(\begin{cases}1 & \text{if } y<1\\ u_r(t) & \text{if } y\geq 1\end{cases}\right)\mathrm{d}y\\
&= \tfrac{1}{\eta}\int_0^1 \mathrm{e}^{-\frac{y}{\eta}}\,\mathrm{d}y + \tfrac{u_\ell(t)}{\eta}\int_1^\infty \mathrm{e}^{-\frac{y}{\eta}}\,\mathrm{d}y\\
&= 1-\mathrm{e}^{-1} + u_r(t)\mathrm{e}^{-1} = 1-\tfrac{1-u_r(t)}{\mathrm{e}} \leq 1-\tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}.
\end{aligned}$$

Using this estimate, which is uniform in $(t,x)\in\Omega_T$, and the monotonicity of V in Assumption 1, we can bound the zero characteristics in Eq. (2.3) from below:

$$\xi_{w^*}(0,0;t) = \int_0^t V(\mathcal{W}[q](s,\xi[0,0](s)))\,\mathrm{d}s \tag{4.2}$$

$$\geq \int_0^t V\Big(1-\tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\Big)\,\mathrm{d}s = tV\Big(1-\tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\Big). \tag{4.3}$$

As V is non-zero at $1-\tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}$ (again by Assumption 1), we end up with the upper bound

$$\tilde{T} = V\Big(1-\tfrac{1-\|u_r\|_{L^\infty((0,T))}}{\mathrm{e}}\Big)^{-1} \tag{4.4}$$

when the initial datum has necessarily left the domain Ω_T. This also explains the assumption of T being sufficiently large, as we require $T\geq\tilde{T}$. As $\xi_{w^*}(0,0;\cdot)\in C([0,T])$, i.e. is continuous a $T^*\in(0,T]$ satisfying $\xi_{\tilde{w}}(0,0;T^*)=1$ indeed exists. As $t\mapsto\xi_{w^*}(0,0;t)$ is also strictly monotone, such a T^* is unique. $\square$

Remark 4.1 (Improved upper bounds on T^* for affine linear velocities) In particular, in the case of an affine linear Greenshields velocity function (see [47]), i.e. $V(s)\equiv 1-s$, we obtain an improved estimate on the bound in Eq. (4.4). We make the same ansatz as in Eq. (4.3) and write for the time derivative of the zero characteristics

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t}\xi_{w^*}(0,0;t)\\
&= V\big(\mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\big) = 1-\mathcal{W}[\rho,u_r](t,\xi_{w^*}(0,0;t))\\
&= 1-\tfrac{1}{\eta}\int_{\xi_{w^*}(0,0;t)}^\infty \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\left(\begin{cases}\rho(t,y) & \text{if } y\leq 1\\ u_r(t) & \text{if } y\geq 1\end{cases}\right)\mathrm{d}y
\end{aligned}$$

taking advantage of the maximum principle (2.4) in Theorem 2.1

$$
\begin{aligned}
&\geq 1-\tfrac{1}{\eta}\int_{\xi_{w^*}(0,0;t)}^{\infty} \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\left(\begin{cases}\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\} & \text{if } y\leq 1\\ u_r(t) & \text{if } y\geq 1\end{cases}\right)\mathrm{d}y\\
&=1-\tfrac{\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}}{\eta}\int_{\xi_{w^*}(0,0;t)}^{1}\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\,\mathrm{d}y-\tfrac{u_r(t)}{\eta}\int_1^{\infty}\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-y}{\eta}}\,\mathrm{d}y\\
&=1+\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\Big(\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}-1\Big)-u_r(t)\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\\
&=\mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\big(\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}-u_r(t)\big)\\
&\quad+1-\max\{\|\rho_0\|_{L^\infty((0,1))},u_r(t)\}\\
&\geq \mathrm{e}^{\frac{\xi_{w^*}(0,0;t)-1}{\eta}}\max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\}\\
&\quad+1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.
\end{aligned}
$$

Recalling that $\xi(0,0;0)=0$ and solving the previous differential inequality in the case of equality, we obtain as solution y the following expression

$$
\begin{aligned}
y_\eta(t)&=1+bt-\eta\ln\Big(a\big(1-\mathrm{e}^{\frac{bt}{\eta}}\big)+b\mathrm{e}^{\frac{1}{\eta}}\Big)+\eta\ln(b),\\
a&\coloneqq\max\{\|\rho_0\|_{L^\infty((0,1))}-\|u_r\|_{L^\infty((0,T))},0\},\\
b&\coloneqq 1-\max\{\|\rho_0\|_{L^\infty((0,1))},\|u_r\|_{L^\infty((0,T))}\}.
\end{aligned}
$$

Solving for $T^*\in\mathbb{R}_{>0}$ such that $y(T^*)=1$ gives the upper bound on T^*

$$
T^*_{\text{improved}}(\eta)=\tfrac{\eta}{b}\ln\Big(\tfrac{a+b\exp(\frac{1}{\eta})}{a+b}\Big). \tag{4.5}
$$

Let us compare the results in Lemma 4.1 with the improved estimate in this remark. The velocity function is required to satisfy $V(x)=1-x,\ x\in[0,1]$ and we assume that $\rho_0\equiv\frac{1}{2}$ and $u_r\equiv 0$. Then, we obtain for the estimate in Eq. (4.1) the upper bound on T^*(which we call T_1^* and which is given by

$$
T_1^*=\tfrac{1}{1-(1-\frac{1}{\mathrm{e}})}=\mathrm{e}.
$$

Applying in comparison Eq. (4.5), we obtain for $\eta\in\mathbb{R}_{>0}$

$$
T^*_{\text{improved}}(\eta)\leq 2\eta\ln\big(\tfrac{1}{2}(1+\mathrm{e}^{\frac{1}{\eta}})\big).
$$

which is illustrated for $\eta\in(0,2)$ in Fig. 4.1. Clearly the improved estimate on T^*, i.e. $T^*_{\text{improved}}(\eta),\ \eta\in\mathbb{R}_{>0}$ is sharper. Particularly, it also depends on the nonlocal reach $\eta\in\mathbb{R}_{>0}$. As the right hand side datum is minimal here, it is expected that with rising η the nonlocal term $W(t,\xi[0,0](t))$ becomes smaller as more and more the nonlocal right hand side u_r influences the nonlocal term. Thus, the velocity is higher and the upper bound on the time when the initial datum has left becomes smaller.

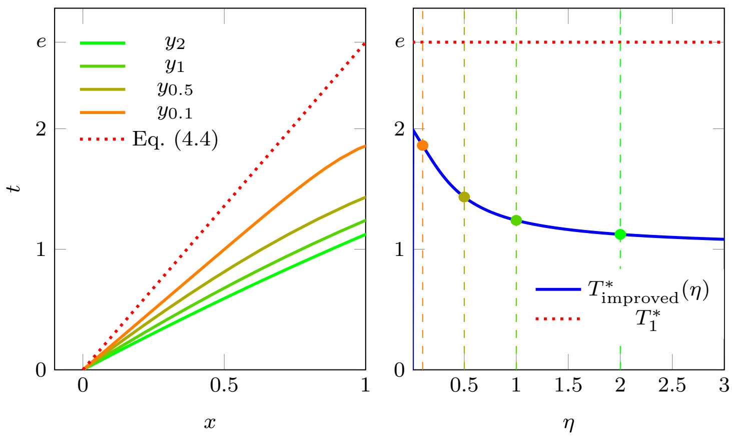

Fig. 4.1 Illustration of the different upper bounds for the initial datum leaving the domain. We chose $u_r = 0$ and $\rho_0 \equiv \frac{1}{2}$. **Left:** The different upper bounds for the zero characteristics $t \mapsto \xi(0,0;t)$. The dashed red line is the – rather coarse – estimate uniform in η given in Lemma 4.1. The solid lines, which represent the improved upper bounds on T^* for affine linear velocity functions, are with higher accuracy. **Right:** The improved bounds on T^* for different values of the nonlocal reach η. As η becomes larger, the upper bound becomes smaller as we have initialized the right hand side u_r as being zero so that for larger η this zero becomes more and more dominant and leads to a increased velocity approaching 1 in the limit and consequently $\lim_{\eta\to\infty} T^*(\eta) = 1$.

Remark 4.2 (Remark 4.1 for $\eta \to 0$, i.e. local conservation laws) The upper bound $T^*_{\text{improved}}(\eta)$ Eq. (4.5) on T^* as in Eq. (4.1) (the time, when the initial datum has left the domain) is a function of $\eta \in \mathbb{R}_{>0}$. For $\eta \to 0$ we formally obtain the local conservation law. For specific cases, i.e. for the Cauchy problem on $\mathbb{R}$, affine velocities and initial datum bounded away from zero it has been proved that the solution of the Cauchy problem associated to the nonlocal conservation law converges to the entropy solution of the corresponding local conservation law [16] (see also [27] for the nonlocal-to-local limit of the same equation with additional (vanishing) viscosity effects). Although this cannot easily be extended to the initial boundary value problem considered in this work, it is still interesting to compute the limit for $\eta \to 0$ of the upper bound $T^*_{\text{improved}}(\eta),\ \eta \in \mathbb{R}_{>0}$. We obtain

$$\lim_{\eta\to 0} T^*_{\text{improved}}(\eta) = \lim_{\eta\to 0} \tfrac{\eta}{b} \ln\left(\tfrac{a+b\exp(\frac{1}{\eta})}{a+b}\right) = \tfrac{1}{b},$$

(compare also Fig. 4.1 for $b = \frac{1}{2}$). Indeed, this is then an upper bound for the time the local conservation law needs to transport the mass of the initial datum, i.e., $\int_0^1 \rho_0(x)\,\mathrm{d}x$, out of the domain. For constant initial datum and constant right hand side this estimate is actually sharp.

Having shown that for a reasonable nonlocal right hand side u_r the initial datum leaves the domain in finite time we can state our main result in this section:

Theorem 4.1 (Equivalence controllability/time-inverted dynamics) *Let Assumption 1 and the following hold:*

- $\rho_0 \in L^\infty((0,1);[0,1])$
- $\rho_{des} \in L^\infty((0,1);[0,1))$
- $u_r \in L^\infty(\mathbb{R}_{>0};[0,c])$, $c \in [0,1)$
- $\rho_r \in L^\infty((0,\infty);[0,1])$

Define

$$T^* := T^*_{\rho_0,u_r} := \operatorname*{arg\text{-}min}_{t\in\mathbb{R}_{\geq 0}} \{\xi[\rho_0,u_r](0,0;t)=1\}, \tag{4.6}$$

$$\varXi_{\rho_0,u_r} := \{(t,x)\in \varOmega_{T^*} : \xi[\rho_0,u_r](0,0;t) < x < 1\}, \tag{4.7}$$

$$v[\rho,u_r](t,x) := \begin{cases} \rho(t,x) & \textit{if } (t,x)\in \varXi_{\rho_0,u_r},\\ u_r(t) & \textit{if } x>1,\\ 0 & \textit{otherwise}, \end{cases} \qquad (t,x)\in \varXi \cup [0,T^*]\times\mathbb{R}_{>1}, \tag{4.8}$$

$$\tilde{\mathcal{W}}[p,v](t,x) := \tfrac{1}{\eta}\int_x^\infty \mathrm{e}^{\frac{x-y}{\eta}} \left(\begin{cases} p(t,y) & (t,y)\in \varOmega_{T^*}\setminus \varXi_{\rho_0,u_r}\\ v[\rho,u_r](t,y) & \textit{otherwise}\end{cases}\right)\mathrm{d}y,\ (t,x)\in\varOmega_{T^*}. \tag{4.9}$$

Then, the following two results hold:

1. *There exists $u_\ell \in L^\infty((0,T^*);[0,1])$ such that $\rho(T^*,\cdot)\equiv\rho_{des}$ if and only if the backward nonlocal balance law*

$$\partial_t p(t,x) = -\partial_x\big(V(\tilde{\mathcal{W}}[p,v[\rho,u_r]](t,x)p(t,x)\big), \qquad (t,x)\in \varOmega_{T^*}\setminus \varXi_{\rho_0,u_r}, \tag{4.10}$$

$$p(T^*,x) = \rho_{des}(x), \qquad x\in(0,1) \tag{4.11}$$

with $v[\rho,u_r]$ as in Eq. (4.8) and $\tilde{\mathcal{W}}$ as in Eq. (4.9) admits a solution satisfying $\|p\|_{L^\infty((0,T^*);L^\infty((0,1)))}\leq 1$.

2. *There exist $T\in[T^*,\infty)$, and $u_\ell \in L^\infty((0,T);[0,1])$ such that $\rho(t,1)\equiv \rho_r(t)$ for a.e. $t\in(T^*,T)$ if and only if the backward nonlocal balance law*

$$\partial_t p(t,x) = -\partial_x\big(V(\tilde{\mathcal{W}}[p,v[\rho,u_r]](t,x)p(t,x)\big) \tag{4.12}$$

$$p(t,1) = \rho_r(t), \qquad t\in(T^*,T), \tag{4.13}$$

$$p(T,x) = 0, \qquad x\in[0,1], \tag{4.14}$$

admits a solution, satisfying $\|p\|_{L^\infty((0,T);L^\infty((0,1)))}\leq 1$.

Proof First, we mention that T^* as in Eq. (4.6) exists and is unique according to Lemma 4.1. We prove only the result in Item 1, since the other can be obtained analogously.

Let us assume that we can control the system to the desired end state/boundary state. Then, we can time-invert the dynamics; the solution to the corresponding backwards-in-time system exists and satisfies Eqs. (4.10) to (4.11).

Conversely, let us assume that the backward system admits a weak solution. Then, we can evaluate the solution at $x = 0$ to obtain the proper boundary data which serves indeed as control to steer the system towards the desired state. The regularity needed for this to hold is $C([0,1]; L^1((0,T)))$. Although such regularity generally does not hold (compare also [60, Remark 5.6]), we have it as long as the corresponding velocity is bounded away from zero which is true in the underlying case as also illustrated in Lemma 4.1 as longs as $\|u_r\|_{L^\infty((0,T))} < 1$. □

Remark 4.3 (Explanation of Theorem 4.1) The backwards in time nonlocal conservation laws and the suitable domain on which they need to be solved are illustrated in Fig. 4.2. The red lines indicate the data which needs to be

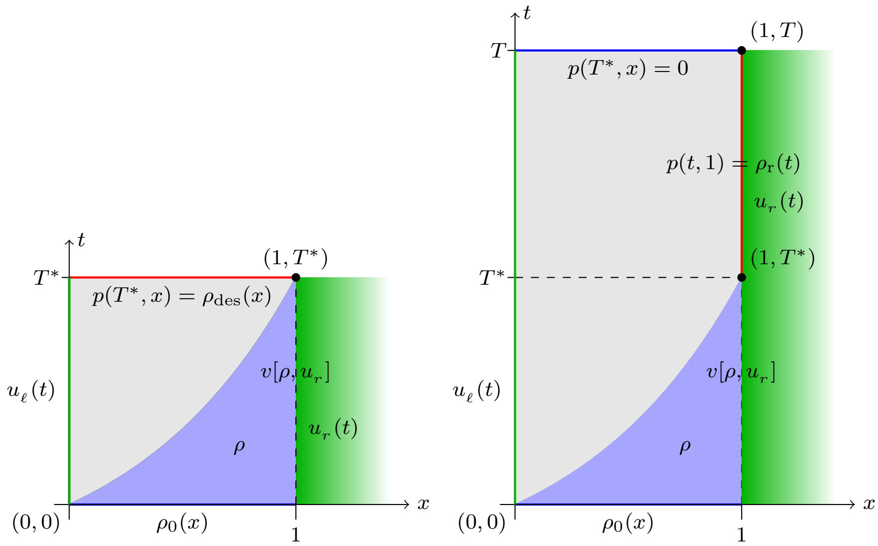

Fig. 4.2 **Left**: Illustration of the statement in Theorem 4.1, Item 1. The desired end value (in red) we want to control the system to and the known quantities in blue. The green colors indicate functions that we want to control to reach the desired state $\rho(T^*; \cdot)$. The backward in time equation is considered on the grey area. **Right:** Illustration of the statement in Theorem 4.1, Item 2. Red again indicates (here) the boundary value we would like to control the system to, in blue we have the quantities which are given (In particular, the end value can be chosen arbitrarily, and in green the quantity we can use to control the system, i.e. left hand side boundary datum and right hand side nonlocal impact. The backward system is considered on the grey area with explicit boundary conditions from $(1, T^*)$ to $(1, T)$.

fit, the blue areae illustrate datum which is given by prescribed initial datum and right hand side nonlocal impact. The backwards problem is – for both cases – considered on the grey area/domain.

Remark 4.4 (Controlling to target state and out-flux simultaneously) The previous result can in a straightforward way be generalized to the case where we seek left hand side boundary datum u_ℓ and nonlocal right hand side u_r so that on a significantly large time $T \in \mathbb{R}_{>T^*}$ the end state satisfies

$$\rho(T, \cdot) \equiv \rho_{\mathrm{des}}$$

and the boundary state

$$\rho(\cdot, 1) \equiv \rho_{\mathrm{r}}.$$

We do not go into details.

As the previous result is not explicit in the way that we cannot "a priori" tell to which final states we can control the system to we show in the following that a constant state can always be reached in a sufficiently large time when also controlling u_r.

Lemma 4.2 (Controllability to constant state) *Let* $\rho_0 \in L^\infty((0,1);[0,1])$ *and* $\rho_{des} \equiv c \in [0,1)$ *be given. Then,*

$$\exists T \in \mathbb{R}_{>0}\ (u_\ell, u_r) \in L^\infty((0,T);[0,1])^2 :\ \rho(T,\cdot) \equiv \rho_{des}$$

where ρ *denotes the solution of the Definition 1.1 for boundary datum* u_ℓ*, right hand side nonlocal term* u_r *and initial datum* ρ_0*.*

Proof We prove this result by introducing different steps in which we control the solution to specific datum.

First, following Lemma 4.1 there exists $T^* \in \mathbb{R}_{>0}$ so that for $u_\ell(t) = u_r(t) = 0$ the initial datum has left the domain and we thus have

$$\rho(T^*, \cdot) \equiv 0,$$

i.e. the road is fully evacuated. Second, we show that the initial state zero can be controlled in finite time to the steady state $\varepsilon \in \mathbb{R}_{>0}$ for ε sufficiently small and continue this process until we have reached the constant state. We take advantage of Theorem 4.1 and consider the following sequences of surrogate problems for $n \in \mathbb{N}_{\geq 1}$ backwards in time and $(t,x) \in \Omega_{T_n^*} \setminus \Xi_{\varepsilon(n-1),\varepsilon(n-1)}$

$$\begin{aligned} &\partial_t p_n(t,x) = -\partial_x\big(V(\tilde{\mathcal{W}}[p_n, v[\varepsilon\cdot(n-1), \varepsilon\cdot(n-1)]](t,x)p_n(t,x)\big),\\ &p_n(T_n^*, x) = n\varepsilon, \end{aligned} \tag{4.15}$$

and $T_n^* := \sum_{k=1}^n T^*_{(k-1)\varepsilon,(k-1)\varepsilon}$ as in Eq. (4.6). As we will stop when we have found $n* \in \mathbb{N}_{\geq 1} :\ n\varepsilon = c$ (we can always chose ε so that this holds), we can immediately give an uniform upper bound on T_n^* by invoking Lemma 4.1

$$T^*_{(n-1)\varepsilon,(n-1)\varepsilon} \leq \tfrac{1}{V\left(1-\frac{1-c}{\mathrm{e}}\right)} \quad \text{and} \quad T_n^* \leq \tfrac{n}{V\left(1-\frac{1-c}{\mathrm{e}}\right)} \quad \forall n \in \mathbb{N}_{\geq 1} \tag{4.16}$$

thanks to the monotonicity of V in Assumption 1. Now, we show that for sufficiently small ε the system in Eq. (4.15) admits a solution on the entire time horizon $\frac{1}{V(1-\frac{1-c}{\mathrm{e}})}$. To this end, we study how at a given space time point $(t,x) \in (T^*_{n-1}, T^*_n) \setminus \Xi_{\varepsilon(n-1),\varepsilon(n-1)}$ a maximum evolves backwards in time. So assume that at such a (t,x) the solution is maximal, parametrized on the characteristics, i.e., $p_n(t,\xi(T^*,x;t)) = \|p_n(t,\cdot)\|_{L^\infty(\mathbb{R})}$ (and thus also $\partial_2 p_n(t,\xi(^*,x;t)) = 0$) we estimate

$$
\begin{aligned}
&- \tfrac{\mathrm{d}}{\mathrm{d}t} p_n(t,\xi(T^*_n,x;t)) \\
&= V'(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1),\varepsilon(n-1)]](t,\xi(T^*_n,x;t)) \\
&\quad \cdot \partial_2 \tilde{\mathcal{W}}[p, v[\varepsilon(n-1),\varepsilon)n-1)](t,\xi(T^*_n,x;t)) \\
&= V'(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1),\varepsilon(n-1)]](t,\xi(T^*_n,x;t)) \\
&\quad \cdot \tfrac{1}{\eta}(\tilde{\mathcal{W}}[p_n, v[\varepsilon(n-1),\varepsilon)n-1)](t,\xi(T^*_n,x;t)) - p_n(t,\xi(T^*_n,x;t))) \\
&\leq \tfrac{1}{\eta}\|V'\|_{L^\infty((0,1))} p_n(t,\xi(T^*_n,x;t))^2.
\end{aligned}
$$

Integrating the previous differential inequality from backwards from T^*_n to t gives as upper bound

$$
\|p_n(t,\cdot)\|_{L^\infty((0,1))} \leq \frac{1}{\frac{n}{\varepsilon} - \frac{1}{\eta}\|V'\|_{L^\infty((0,1))}(T^*_n - t)} \overset{!}{\leq} 1. \tag{4.17}
$$

For admissibility we need to make sure that the previous $\|p_n(t,\cdot)\|_{L^\infty((0,1))}$ is less or equal one, which is satisfied if

$$
t \geq T^*_n - \eta \frac{1-n\varepsilon}{\varepsilon\|V'\|_{L^\infty((0,1))}} \geq T^*_n - \eta \frac{1-\kappa}{\varepsilon\|V'\|_{L^\infty((0,1))}} \tag{4.18}
$$

However, for $\varepsilon \in \mathbb{R}_{>0}$ sufficiently small we obtain the well-posedness of the backwards system Eq. (4.15) on every time horizon and thus, particularly on the time horizon required for the initial datum to leave, i.e. Eq. (4.16). As the estimates are uniform in $n \in \mathbb{N}_{\geq}$ we can then pick as many sequences as needed to control iteratively the zero initial datum to ε, 2ε,... until we have reached the constant state c. This concludes the proof.

Remark 4.5 (Extensions of Lemma 4.2) The previous Lemma 4.2 can be generalized. For instance, the solution can also be steered to monotonically increasing ρ_{des} by first controlling it as is possible to the sufficiently large constant state $\mathbb{R} \ni c \geq \|\rho_{\mathrm{des}}\|_{L^\infty((0,1))}$ and then showing that the backward in time system does not blow up. However, due to the monotonicity this cannot happen. Another extension might consist of disturbing the constant ρ_{des} slightly with regard to the L^∞ norm and to still achieve controllability (compare also Remark 3.1). We do not go into details.

Example 4.1 (Controllability and lack of controllability in minimal time) We present some examples related to the state controllability result in Theorem 4.1. In Fig. 4.3, we consider three cases: $\rho_{\mathrm{des}}(x) = \frac{1}{2}(1-x)$, $\rho_{\mathrm{des}}(x) = \frac{1}{2}x$, and $\rho_{\mathrm{des}} \equiv \frac{1}{2}$, with initial and right boundary data given by $\rho_0(x) = \frac{1}{2}(1+x)$ and $u_r = \frac{1}{2}$. In Fig. 4.4, the left hand side boundary datum u_ℓ for achieving the desired final state ρ_{des} in minimal time are also shown. As can be observed is that on the left three pictures in Fig. 4.3 the initial datum leaves faster. This is due to the fact that η is larger so that the nonlocal right hand side $u_r = \frac{1}{2}$ has a higher impact on the velocity of the entire road. Another remarkable fact is that for smaller η and end-datum $\rho_{\mathrm{des}} = \frac{1}{2}x,\ x \in [0,1]$ (see the fifth pictures in Fig. 4.3 or the maxmimum of the yellow dotted curve in Fig. 4.4) actually becomes larger than the desired state and then smaller again to compensate for the later velocity of the system. This clearly indicates that generally not every end state can be tracked as the corresponding control could exceed 1 and would thus not be admissible.

Finally, all pictures indicate what is also described in Theorem 4.1 and which is apparent from the dynamics that the solution below the characteristics emanating from $(0,0)$, i.e. the solution which only depends on initial datum and the right hand side nonlocal impact u_r stays the same independent of the boundary datum. Clearly, the time where the initial datum has left is thus the same.

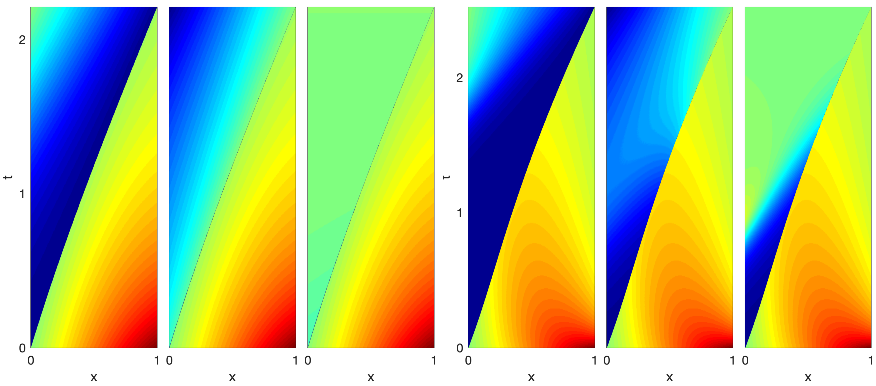

Fig. 4.3 The left three images correspond to $\eta = 1$, the right three to $\eta = 0.1$. In the left images of both triples, $\rho_{\mathrm{des}}(x) = \frac{1}{2}(1-x)$, in the middle $\rho_{\mathrm{des}}(x) = \frac{1}{2}x$ and in the right images $\rho_{\mathrm{des}} \equiv \frac{1}{2}$. In all images, the initial datum is given by $\rho_0(x) = \frac{1}{2}(1+x)$ and the right boundary $u_r = \frac{1}{2}$. **Colorbar:** 0 ▬ 1

5 Long-time behavior

In this section, we are concerned with the long-time behavior of the solution to Definition 1.1 when prescribing constant $(u_\ell, u_r) \in [0,1)^2$. Under the assumption that the initial datum is uniformly less or equal or greater or equal $u_\ell = u_r$, we can show that the solution converges to a given constant. We detail this in the following Theorem.

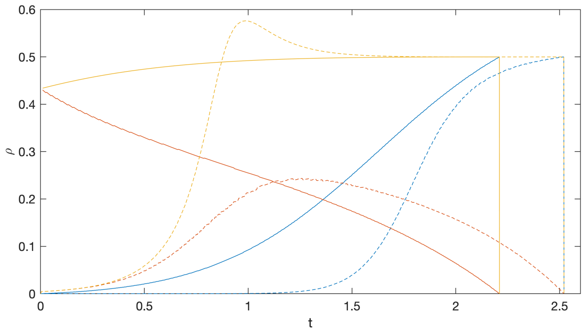

Fig. 4.4 Corresponding to Fig. 4.3 in Example 4.1 the left hand side boundary datum u_ℓ for achieving the desired final state ρ_{des} in minimal time. Solid lines represent the boundary datum for $\eta = 1$, dashed lines for $\eta = 0.1$. The colors represent the related desired state ρ_{des} which we want to achieve: We have for $x \in [0,1]$ in red $\rho_{\text{des}}(x) = \frac{1}{2}x$, in blue $\rho_{\text{des}}(x) = \frac{1}{2}(1-x)$ and in yellow $\rho_{\text{des}}(x) = \frac{1}{2}$.

Theorem 5.1 (Long-time behavior) *Suppose that $\kappa \in (0,1)$ is given, let Assumption 1 hold and assume $u_r \equiv \kappa$, $u_\ell \equiv \kappa$ and the derivative of V may satisfy $V'(s) < 0\ \forall s \in [0,1)$. In addition, let*

$$\text{(5.1)} \qquad \Big(\rho_0 \geqq \kappa \text{ on } (0,1)\Big) \quad \text{or} \quad \Big(\rho_0 \leqq \kappa \text{ on } (0,1)\Big).$$

Then, the corresponding solution ρ converges exponentially in time to κ

$$\|\rho(t,\cdot)-\kappa\|_{L^1((0,1))} \leq \eta\Big(\exp\Big(\tfrac{\|\rho_0-\kappa\|_{L^1((0,1))}}{\eta}\Big)-1\Big)\mathrm{e}^{\frac{K(\eta)}{\eta}t} \quad \forall t \in \mathbb{R}_{\geq 0}$$

with

$$\begin{aligned}\bar{\kappa} &\coloneqq (1-\kappa)(1-\exp(-\eta^{-1}))\\ K(\eta) &\coloneqq (1-\kappa)\kappa \sup_{s\in\langle\kappa,\bar{\kappa}\rangle} V'(s)\exp\big(-\eta^{-1}\big) < 0\\ \langle a,b\rangle &\coloneqq \big(\min\{a,b\},\max\{a,b\}\big),\ (a,b)\in\mathbb{R}^2.\end{aligned}$$

Proof Let us define the difference between $\rho(t,\cdot)$ and κ in the integral sense for $t \in [0,T]$

$$M(t) \coloneqq \int_0^1 (\rho(t,x)-\kappa)\,\mathrm{d}x.$$

As we want to compute the time-derivative of $M(t)$ we first need to show that $t \mapsto M(t)$ is differentiable. However, this can be achieved by taking advantage of the solution formula in terms of characteristics Eq. (2.2). Assuming that

$T^* \in \mathbb{R}_{>0}$ is so that $\xi(0,0;T^*)=1$ and we can write for $t \in [0,T^*]$

$$\begin{aligned}M(t) &= \int_0^{\xi(0,0;t)} u(\xi_{w^*}[t,x]_{\max}^{-1}(0))\,\partial_2\xi_{w^*}(t,x;\xi_{w^*}[t,x]_{\max}^{-1}(0))\,\mathrm{d}x\\ &\quad+\int_{\xi(0,0;t)}^1 \rho_0(\xi(t,x;0))\partial_2\xi(t,x;0)\,\mathrm{d}x-\kappa\\ &=\int_0^t u(z)V(\mathcal{W}[\rho](z,0))\,\mathrm{d}z+\int_0^{\xi(t,1;0)}\rho_0(z)\,\mathrm{d}z-\kappa\end{aligned}$$

which is clearly differentiable with regard to t sufficiently small. Taking the time derivative we obtain

$$\begin{aligned}M'(t) &= u(t)V(\mathcal{W}[\rho](t,0))+\rho_0(\xi(t,1;0))\partial_1\xi(t,1;0)\\ &=\kappa V(\mathcal{W}[\rho](t,0))-\rho(t,1)V(\mathcal{W}[\rho](t,1)).\end{aligned}\tag{5.2}$$

As the previous result does not depend explicitly on the initial datum (we have replaced the part of the initial datum by the general expression of the solution ρ), this result holds for every time $t \in \mathbb{R}_{>0}$. Assume for now that $\rho_0 \geq \kappa$ so that we know thanks to the maximum principle in Eq. (2.4) in Theorem 2.1

$$\rho(t,x) \geq \kappa \quad \forall (t,x) \in [0,T]\times(0,1) \text{ a.e.}\tag{5.3}$$

it follows directly

$$1 \geq M(t) \geq 0 \quad \forall t \in [0,T].\tag{5.4}$$

The upper bound 1 is a consequence of the maximum principle and the fact that also $\kappa \in [0,1]$ by assumption. Then, we estimate the nonlocal term as follows

$$\begin{aligned}\mathcal{W}[\rho](t,0) &:= \tfrac{1}{\eta}\int_0^\infty \exp\left(-\tfrac{s}{\eta}\right)\left(\begin{cases}\rho(t,s) & s<1\\ \kappa & s\geq 1\end{cases}\right)\mathrm{d}s,\\ &\overset{(5.3)}{\geq} \tfrac{\kappa}{\eta}\int_0^{1-M(t)}\exp\left(-\tfrac{s}{\eta}\right)\,\mathrm{d}s+\tfrac{1}{\eta}\int_{1-M(t)}^1\exp\left(-\tfrac{s}{\eta}\right)\,\mathrm{d}s+\tfrac{\kappa}{\eta}\int_1^\infty\exp\left(-\tfrac{s}{\eta}\right)\,\mathrm{d}s\\ &=\tfrac{\kappa}{\eta}\int_0^\infty \exp\left(-\tfrac{s}{\eta}\right)\,\mathrm{d}s+\tfrac{1-\kappa}{\eta}\int_{1-M(t)}^1\exp\left(-\tfrac{s}{\eta}\right)\,\mathrm{d}s\\ &=\kappa+(1-\kappa)\Big(\exp\big(-\tfrac{1-M(t)}{\eta}\big)-\exp\big(-\tfrac{1}{\eta}\big)\Big)\\ &=\kappa+\exp\big(-\tfrac{1}{\eta}\big)(1-\kappa)\Big(\exp\big(\tfrac{M(t)}{\eta}\big)-1\Big),\end{aligned}$$

from which we can continue the estimate on $M(t)$ in Eq. (5.2) and obtain – again recalling that $\rho(t,1)\geq\kappa\ \forall t\in[0,T]$ –

$$M'(t) \overset{V'\leqq 0}{\leq} V\Big(\kappa+\exp\big(-\tfrac{1}{\eta}\big)(1-\kappa)\Big(\exp\big(\tfrac{M(t)}{\eta}\big)-1\Big)\Big)\kappa-\underbrace{V(\kappa)\rho(t,1)}_{\geq V(\kappa)\kappa}$$

using the mean value theorem and defining $\bar{\kappa} \coloneqq (1-\kappa)(1-\exp(-\eta^{-1})) < 1$

$$\begin{aligned}&\leq \sup_{s\in(\kappa,\bar{\kappa})} V'(s)\Big(\exp\big(-\tfrac{1}{\eta}\big)(1-\kappa)\Big(\exp\big(\tfrac{M(t)}{\eta}\big)-1\Big)\Big)\kappa\\ &\leq (1-\kappa)\kappa \sup_{s\in(\kappa,\bar{\kappa})} V'(s)\exp\big(-\tfrac{1}{\eta}\big)\Big(\exp\big(\tfrac{M(t)}{\eta}\big)-1\Big).\end{aligned}$$

We solve the previous differential inequality for equality, call the solution $M_{=}(t)$ and obtain for it

$$\begin{aligned}M_{=}(t) &= -\eta\ln\Big(\big(\mathrm{e}^{-\frac{M(0)}{\eta}}-1\big)\mathrm{e}^{\frac{K(\eta)t}{\eta}}+1\Big)\\ K(\eta) &\coloneqq (1-\kappa)\kappa\sup_{s\in(\kappa,\bar{\kappa})} V'(s)\exp\big(-\tfrac{1}{\eta}\big) < 0\end{aligned} \tag{5.5}$$

From this, we obtain

$$0 \leq M(t) \leq M_{=}(t) \quad \forall t \in \mathbb{R}_{\geq 0} \tag{5.6}$$

For proving the rate of convergence, we use $\ln(x) \leq x-1\ \forall x \in \mathbb{R}_{>0}$ and have

$$M_{=}(t) = \eta\ln\Big(\Big(1-\big(1-\mathrm{e}^{-\frac{M(0)}{\eta}}\big)\mathrm{e}^{\frac{K(\eta)t}{\eta}}\Big)^{-1}\Big) \tag{5.7}$$

$$\leq \eta\cdot\frac{\big(1-\mathrm{e}^{-\frac{M(0)}{\eta}}\big)\mathrm{e}^{\frac{K(\eta)t}{\eta}}}{1-\big(1-\mathrm{e}^{-\frac{M(0)}{\eta}}\big)\mathrm{e}^{\frac{K(\eta)t}{\eta}}} \leq \eta\big(\mathrm{e}^{\frac{M(0)}{\eta}}-1\big)\mathrm{e}^{\frac{K(\eta)}{\eta}t}. \tag{5.8}$$

For initial datum $\rho_o(x) \leq \kappa$ for a.e. $x \in (0,1)$, the results follow by performing similar estimates with opposite sign. ☐

Remark 5.1 (Theorem 5.1 for $\kappa = 0$ and $\kappa = 1$) The previous result with the given constants does not give exponential stability for $\kappa = 0 \vee 1$ as then $K(\eta) = 0$ for $\eta \in \mathbb{R}_{>0}$.

However, for $\kappa = 0$, the boundary contribution to the solution is zero and, by Lemma 4.1, we know that the initial data leaves the domain in finite time $T^* \in \mathbb{R}_{>0}$. Afterwards, the solution remains identically zero so that we have stability to the zero solution in finite time and particularly exponentially.

For $\kappa = 1$, the situation is slightly more delicate. We look at the change of the L^1 norm of the solution in time,

$$\partial_t\|\rho(t,\cdot)\|_{L^1((0,1)} = -\int_0^1 \partial_x\big(\rho(t,y)V(\mathcal{W}[\rho,u_r](t,y))\big)\,\mathrm{d}y \tag{5.9}$$

$$= \rho(t,0)V(\mathcal{W}[\rho,u_r](t,0)) - \rho(t,1)V(\mathcal{W}[\rho,u_r](t,1)) \tag{5.10}$$

$$= V(\mathcal{W}[\rho,u_r](t,0)) \tag{5.11}$$

$$\geq V\bigg(\tfrac{1}{\eta}\int_0^{\|\rho(t,\cdot)\|_{L^1((0,1))}} 1\cdot\exp\big(\tfrac{-y}{\eta}\big)\,\mathrm{d}y + \exp(-\tfrac{1}{\eta})\bigg) \tag{5.12}$$

$$= V\bigg(1-\exp\Big(\tfrac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}\Big) + \exp\big(-\tfrac{1}{\eta}\big)\bigg) \tag{5.13}$$

and using the mean value theorem, $\exists \zeta \in (0,1)$ s.t.

$$= V(1) - V'(\zeta) \cdot \left(e^{\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}} - e^{-\frac{1}{\eta}} \right) \tag{5.14}$$

$$\geq - \sup_{s\in(0,1)} V'(s) \cdot \left(e^{\frac{-\|\rho(t,\cdot)\|_{L^1((0,1))}}{\eta}} - e^{-\frac{1}{\eta}} \right), \tag{5.15}$$

and consequently

$$\|\rho(t,\cdot)\|_{L^1((0,1)} \geq 1 + \eta \log \left(\left(e^{\frac{\|\rho_0\|_{L^1((0,1)}-1}{\eta}} - 1 \right) \exp\left(t \sup_{s\in(0,1)} V'(s) \frac{e^{-\frac{1}{\eta}}}{\eta} \right) + 1 \right).$$

As $\ln(x+1) \geq x(x+1)^{-1}\ \forall x > \mathbb{R}_{>-1}$ we can continue our estimate

$$\begin{aligned} &\geq 1 + \eta \frac{\left(e^{\frac{\|\rho_0\|_{L^1((0,1)}-1}{\eta}} - 1 \right) \exp\left(t \sup_{s\in(0,1)} V'(s) \frac{e^{-\frac{1}{\eta}}}{\eta} \right)}{\left(e^{\frac{\|\rho_0\|_{L^1((0,1)}-1}{\eta}} - 1 \right) \exp\left(t \sup_{s\in(0,1)} V'(s) \frac{e^{-\frac{1}{\eta}}}{\eta} \right) + 1} \\ &\geq 1 + \tfrac{\eta}{2} \left(e^{\frac{\|\rho_0\|_{L^1((0,1)}-1}{\eta}} - 1 \right) \exp\left(t \sup_{s\in(0,1)} V'(s) \frac{e^{-\frac{1}{\eta}}}{\eta} \right). \end{aligned}$$

This is the exponential convergence to the steady state solution in the case that $\kappa = 1$, i.e., that the road is blocked at the right hand side and $u_\ell \equiv 1$. For the statement to hold we require

$$\sup_{s\in(0,1)} V'(s) < 0.$$

In the case that this assumption does not hold but only Assumption 1 we can still show that the solution converges to the 1 solution for $t \to \infty$, however without any order of convergence. The convergence is then due to the fact that the mapping $t \mapsto \|\rho(t,\cdot)\|_{L^1((0,1))}$ is monotonically increasing in t and bounded from above by 1. Then, we know that a limit point for this sequence exists, i.e., $\exists A \in (0,1] : \lim_{t\to\infty} \|\rho(t,\cdot)\|_{L^1((0,1))} = A$. Thanks to Eq. (5.13), the time derivative of $\|\rho(t,\cdot)\|_{L^1((0,1))}$ is nonnegative and only zero for $\|\rho(t,\cdot)\|_{L^1((0,1))} = 1$ which implies $A = 1$.

Example 5.1 (Long-time behavior and comparison to steady-state solutions) In Fig. 5.1, we present some numerical simulations related to Theorem 5.1. We assume that

$$V \equiv 1 - \cdot,\ u_r \equiv \tfrac{1}{2},\ u_\ell \in \left\{\tfrac{1}{4}, \tfrac{1}{2}, \tfrac{3}{4}\right\},\ \eta \in \{0.1, 1\},\ \rho_0 \equiv \tfrac{1}{2}\chi_{(\frac{1}{3},\frac{2}{3})}. \tag{5.16}$$

One remarkable feature which can be seen in all pictures is that after the initial datum has left, the solution does not change a lot anymore and seems to become stationary. Although we are not able to show this in the general case it seems like all solutions converge to the corresponding steady state, anticipating the existence and uniqueness of steady state solutions in Theorem 6.1. Indeed, this is also illustrated in Fig. 5.2, where – in the left hand side picture – the solutions are plotted at $t \in \{2, 4, 8\}$ and in the right picture the steady state solution in comparison with the corresponding solution at time $t = 8$.

Worth mentioning is also the impact of the size of the nonlocal parameter $\eta \in \mathbb{R}_{>0}$. As the initial datum's L^1 mass is smaller than $u_r = \frac{1}{2}$ in the present case the initial datum leaves faster when η is larger. The different η also affects how the solution later on will evolve which can be seen in particular at the two rightmost pictures. Larger η decreases the spatial derivative of the solution which is clear from the fact that for larger η there is more averaging of the velocity.

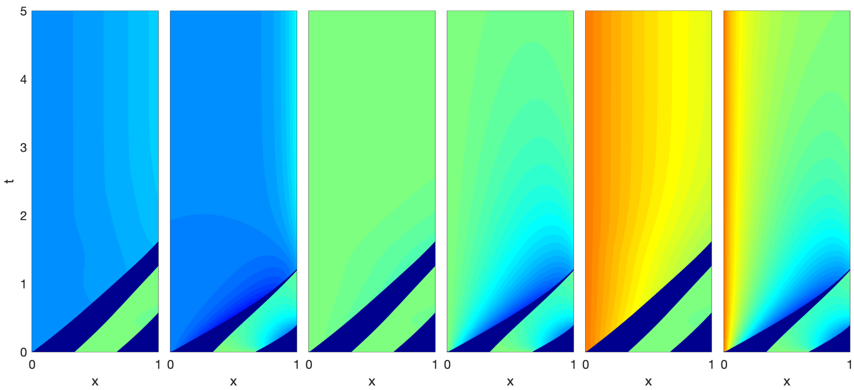

Fig. 5.1 The pictures are ordered from left to right. **First:** $u_\ell = \frac{1}{4}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Second:** $u_\ell = \frac{1}{4}$, $u_r = \frac{1}{2}$, $\eta = 0.1$, **Third:** $u_\ell = \frac{1}{2}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Fourth:** $u_\ell = \frac{1}{2}$, $u_r = \frac{1}{2}$, $\eta = 0.1$, **Fifth:** $u_\ell = \frac{3}{4}$, $u_r = \frac{1}{2}$, $\eta = 1$, **Sixth:** $u_\ell = \frac{3}{4}$, $u_r = \frac{1}{2}$, $\eta = 0.1$ with $\rho_0 \equiv \frac{1}{2}\chi_{(\frac{1}{3},\frac{2}{3})}$ in every case. **Colorbar:** 0 1

6 Steady states and long-time behavior of the linearized system

In the literature, steady state solutions for nonlocal conservation laws on a bounded domain have not been studied yet. On the real axis, traveling wave solutions have been discussed in [70].

On the other hand, for local conservation laws, this topic has been largely discussed over the past few decades. In [43], Dafermos (inspired by the previous analysis and numerical experiment of [71]) used the method of generalized characteristics to study the long-time behavior of solutions for the initial-boundary value problem for conservation law with spatial inhomogeneity. The analysis of entropic steady states of the initial-boundary value problem for a conservation law with a source term was later carried out in [66]. In [26], the

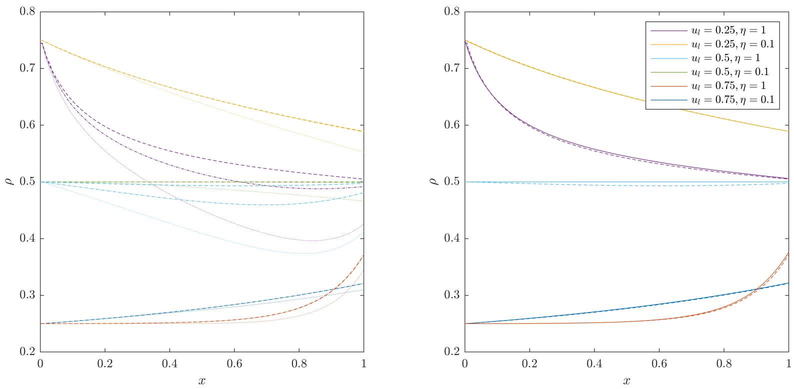

Fig. 5.2 Left: Illustrations of the evolution of solutions at different time snippets $t \in \{2, 4, 8\}$ (dotted $t = 2$, dash-dotted $t = 4$ and dashed $t = 8$). The different colors represent the six different cases in Fig. 5.1 for different u_ℓ, η as described in the legend of the right picture and fixed $u_r = 0.5$. **Right:** Comparison of the different solutions at $t = 8$ and the corresponding steady state solutions as in Theorem 6.1.

existence of stationary solutions for a scalar conservation law was obtained in the case of a nonlocal source term. In [45], the authors considered stationary scalar conservation laws with a damping term and showed the existence and uniqueness of entropy solutions as time-asymptotic limits of solutions of the corresponding (evolutionary) hyperbolic conservation laws. In the case of periodic data (and boundary conditions), the time-asymptotic decay properties of entropy solutions were showed in [73, 68, 19]. For the asymptotic behavior of the closely related equations of Hamilton-Jacobi type, we refer the reader to [11, 56, 55, 10, 10] and references therein. For nonlinear hyperbolic systems of balance laws, there is a large body of literature dealing with the existence of global (classical) solutions around an equilibrium (see, e.g., [52, 76, 14, 12]).

In the following theorem, we prove the existence and uniqueness of steady state solutions on a bounded domain when prescribing constant left hand side boundary datum and constant nonlocal right hand side datum.

Theorem 6.1 (Steady state solutions on bounded domains) *In the setting of Definition 1.1, we have that, for every* $u_\ell \equiv \text{const} \in [0, 1], u_r \equiv \text{const} \in [0, 1]$*, there exists a unique and monotone* $\bar{\rho} \in W^{1,\infty}((0, 1); [\min\{u_\ell, u_r\}, \max\{u_\ell, u_r\}])$ *satisfying*

$$\text{(6.1)} \qquad \tfrac{\mathrm{d}}{\mathrm{d}x}\Big(\bar{\rho}(x) V\big(\mathcal{W}[\bar{\rho}, u_r](x)\big)\Big) = 0, \qquad x \in [0, 1]$$

$$\text{(6.2)} \qquad \bar{\rho}(0) = u_\ell,$$

$$\text{(6.3)} \qquad \mathcal{W}[\bar{\rho}, u_r](t, x) = \tfrac{1}{\eta} \int_x^\infty \mathrm{e}^{-\frac{y-x}{\eta}} \left(\begin{cases} \bar{\rho}(y) & \text{if } y < 1 \\ u_r & \text{if } y \geq 1 \end{cases} \right) \mathrm{d}y, \qquad x \in [0, 1].$$

In addition, if $V \in C^\infty([0, 1])$*, then* $\bar{\rho} \in C^\infty([0, 1])$.

We remark that, for $u_\ell = u_r$, a solution is given by $\bar{\rho} \equiv u_r$, which can be checked by just plugging it into Eqs. (6.1) to (6.2). However, even in this simplear case we need to prove that this is the only solution and that one and only one solution exists in case $u_r \neq u_\ell$, which is carried out in the following proof.

Proof As a first step, we show the existence of solutions by a Schauder fixed-point argument.

A solution of Equations (6.1) to (6.2) can be interpreted as the solution of the following fixed-point problem with the fixed-point mapping

$$\boldsymbol{F}:\begin{cases}\Omega & \to \Omega\\ \bar{\rho} & \mapsto \Big(x\mapsto u_\ell \frac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))}\Big)\end{cases} \tag{6.4}$$

with a proper $\Omega \subset C([0,1])$ yet to be defined. We distinguish two different cases: $u_r \leq u_\ell$ and $u_r \leq u_\ell$.

If $u_r \leq u_\ell$, we define

$$\begin{aligned}\Omega &\coloneqq \big\{\bar{\rho}\in W^{1,\infty}((0,1)) : \big(u_r \leq \bar{\rho}(x) \leq u_\ell\big) \wedge \big(\mathcal{A}\leq \bar{\rho}'(x)\leq 0\big)\ \forall x\in[0,1]\big\}\\ \mathcal{A} &\coloneqq -u_\ell \tfrac{V(u_r)\|V'\|_{L^\infty((u_r,u_\ell))}(u_\ell-u_r)}{\eta V(u_\ell)^2}\end{aligned} \tag{6.5}$$

and show that $\boldsymbol{F}$ is a self-mapping on Ω, i.e. $\boldsymbol{F}[\Omega]\subseteq\Omega$. To this end, we take $\bar{\rho}\in\Omega$ and compute for $x\in[0,1]$

$$\tfrac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}](x) = -u_\ell \tfrac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))^2} V'(\mathcal{W}[\bar{\rho},u_r](x))\partial_x\mathcal{W}[\bar{\rho},u_r](x). \tag{6.6}$$

Since $V' \leqq 0$, we need to show that $\partial_x\mathcal{W}[\bar{\rho},u_r] \leqq 0$, which we do with the following manipulations for $x\in[0,1]$:

$$\partial_x\mathcal{W}[\bar{\rho},u_r](x) = \partial_x\Big(\tfrac{1}{\eta}\int_x^1 \mathrm{e}^{\frac{x-y}{\eta}}\bar{\rho}(y)\,\mathrm{d}y\Big) + \partial_x\Big(\tfrac{1}{\eta}u_r\int_1^\infty \mathrm{e}^{\frac{x-y}{\eta}}\,\mathrm{d}y\Big) \tag{6.7}$$

$$= -\tfrac{1}{\eta}\bar{\rho}(x) + \tfrac{1}{\eta^2}\int_x^1 \mathrm{e}^{\frac{x-y}{\eta}}\bar{\rho}(y)\,\mathrm{d}y + \tfrac{1}{\eta^2}u_r\int_1^\infty \mathrm{e}^{\frac{x-y}{\eta}}\,\mathrm{d}y \tag{6.8}$$

$$= \tfrac{1}{\eta}\big(\mathcal{W}[\bar{\rho},u_r](x) - \bar{\rho}(x)\big). \tag{6.9}$$

As $\bar{\rho}$ is monotonically decreasing and $\bar{\rho}\geqq u_r$, we obtain that $\partial_x\mathcal{W}[\bar{\rho},u_r]\leqq 0$ and thus $\partial_x\boldsymbol{F}[\bar{\rho}]\leqq 0$. From the monotonicity it also follows that $\boldsymbol{F}[\bar{\rho}]\leqq u_\ell$. It remains to be checked that $\boldsymbol{F}[\bar{\rho}]\geq u_r$. To this end, let us assume, for the sake of finding a contradiction, that $\exists x^*\in(0,1) : \boldsymbol{F}[\bar{\rho}](x^*) < u_r$. As $\boldsymbol{F}[\bar{\rho}]$ is monotonically decreasing, we know that $\boldsymbol{F}[\bar{\rho}](x) < u_r \forall x\in(x^*,1]$. Similarly to the computations above, we also obtain that $\mathcal{W}[\boldsymbol{F}[\bar{\rho}],u_r]$ is monotonically decreasing if $\boldsymbol{F}[\bar{\rho},u_r]$ is monotonically decreasing (which we have proven). However, for $x=1$, we obtain $\mathcal{W}[\boldsymbol{F}[\bar{\rho}],u_r](1) = u_r$ but $\mathcal{W}[\boldsymbol{F}[\bar{\rho},u_r]](x) < u_r \forall x\in[x^*,1)$, a contradiction to the monotonicity of $\mathcal{W}[\bar{\rho},u_r]$. Thus, we conclude that

$$\boldsymbol{F}[\bar{\rho}](x)\geq u_r \qquad \forall x\in[0,1].$$

Next, we prove the lower bound on $\frac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}]$. Recalling Eq. (6.6), estimate

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}x}\boldsymbol{F}[\bar{\rho}](x) &\geq -u_{\ell}\tfrac{V(\mathcal{W}[u_r,u_r](0))\|V'\|_{L^\infty((u_r,u_\ell))}(u_\ell-\mathcal{W}[u_r,u_r](0))}{\eta V(\mathcal{W}[u_\ell,u_r](0))^2}\\ &= -u_\ell\tfrac{V(u_r)\|V'\|_{L^\infty((u_r,u_\ell))}(u_\ell-u_r)}{\eta V(u_\ell)^2} = \mathcal{A}.\end{aligned}$$

Altogether, we have shown that $\boldsymbol{F}[\Omega] \subset \Omega$. To show the existence of solutions, we apply Schauder's fixed-point theorem, which requires the following assumptions to be satisfied:

- $\boldsymbol{F} : \Omega \to \Omega$ is continuous in a proper topology. Indeed, by choosing $C([0,1])$ with the natural maximum norm, $\boldsymbol{F}$ is continuous.
- The set Ω is closed in $C([0,1])$ and it is convex. The closedness is due to the fact that we have uniform constraints on $\bar{\rho}$ in the definition of Ω and the convexity is obvious.
- Ω is compact in $C([0,1])$. This is due to the fact that the derivatives of the sets in Ω have as upper bound 0 and as lower bound $\mathcal{A}$ which is uniform. Therefore, the functions in Ω are uniform Lipschitz-continuous with Lipschitz-constant $\mathcal{A}$. Thus, they are also equi-continuous and we can apply Ascoli-Arzelà [17, Theorem 4.25] which guarantees the claimed compactness, i.e. $\Omega \overset{\mathrm{c}}{\hookrightarrow} C([0,1])$.

Using Schauder's fixed-point theorem in the version in [77, Corollary 2.13], we conclude that there exists a solution of (6.4) lying in Ω as defined in (6.5).

If $u_r \leq u_\ell$, the proof of existence is almost identical to the case $u_r \geq u_\ell$ when exchanging the monotonicity in Ω from decreasing to increasing. We do not go into details.

For the uniqueness, we rewrite the steady state equation in (6.1) as a system of ODEs by introducing $g(x) = \mathcal{W}[\bar{\rho}, u_r](x),\ x \in [0,1]$. Then, we obtain on $x \in [0,1]$ and

$$\begin{aligned}\bar{\rho}'(x) &= -\tfrac{\bar{\rho}(x)V'(g(x))\frac{1}{\eta}(g(x)-\bar{\rho}(x))}{V(g(x))}, & \bar{\rho}(0) &= u_\ell\\ g'(x) &= \tfrac{1}{\eta}(g(x)-\bar{\rho}(x)), & g(1) &= u_r.\end{aligned}\tag{6.10}$$

As this is a system of ODEs with initial datum for one equation and end datum for the other equation, the uniqueness can not directly been addressed by a Picard-Lindelöf type of argument. For this reason, we rewrite (6.10) as the following initial-value problem

$$\begin{aligned}\bar{\rho}'(x) &= -\tfrac{\bar{\rho}(x)V'(g(x))\frac{1}{\eta}(g(x)-\bar{\rho}(x))}{V(g(x))}, & \bar{\rho}(0) &= u_\ell\\ g'(x) &= \tfrac{1}{\eta}(g(x)-\bar{\rho}(x)), & g(0) &= c_\ell\end{aligned}\tag{6.11}$$

for $u_\ell \leq u_r$, where $c_\ell \in [c_\ell^*, u_r]$ such that $c_\ell^* = \mathcal{W}[\tilde{\rho}, u_r](0)$ and $\tilde{\rho}$ is a solution to the fixed-point equation in (6.4). Then, the right-hand side of the system of ODEs in (6.11) is locally Lipschitz-continuous, the solution is thus locally unique and satisfies per constructionem the proposed bounds as instantiated in Ω in Eq. (6.5). It is thus globally Lipschitz-continuous on $[0,1]$ and admits a unique solution.

A similar proof can be made for $u_\ell \geq u_r$ by changing the initial boundary value problem to the corresponding end value problem and using again the existence of solutions as obtained by the previous Schauder argument.

To establish the higher regularity of solutions, we recall the fixed-point problem in Eq. (6.4) which has a unique solution by the argument above. Differentiating gives for

$$\bar{\rho}'(x) = -u_\ell \tfrac{V(\mathcal{W}[\bar{\rho},u_r](0))}{V(\mathcal{W}[\bar{\rho},u_r](x))^2} V'(\mathcal{W}[\bar{\rho},u_r](x))\partial_x \mathcal{W}[\bar{\rho},u_r](x), \quad x \in [0,1]. \tag{6.12}$$

We know that $\bar{\rho} \in W^{1,\infty}((0,1))$ and as $\partial_x \mathcal{W}[\bar{\rho},u_r]$ is by Eqs. (6.7) to (6.9) again Lipschitz-continuous, the entire right hand side of Eq. (6.12) is again Lipschitz-continuous and thus also $\bar{\rho}'$. This can be iterated arbitrarily, and we obtain the claimed regularity. □

7 Conclusions and future work

In this contribution, we have obtained the first results on the controllability of nonlocal conservation laws on bounded domains when the nonlocal term is explicitly space-dependent and a maximum principle holds. We have also studied the long-time behavior of the solutions and established their convergence towards steady states under suitable assumptions.

There are many problems left open in this line of research. Amongst them, we mention the following ones.

1. Proving the main theorems for a general monotonically decreasing kernel and not as in this work only for the exponential kernel. In this case, according to [60, Corollary 5.9], the solution to the corresponding nonlocal balance law still exists and is unique and satisfies a maximum principle; however, the proofs of some of our results seem to present many more technical difficulties.
2. Studying in detail the relationship between the controllability of nonlocal conservation laws and the controllability of the corresponding local equations. A first attempt in this direction is made in Remark 4.2.
3. Extending the results in Theorem 5.1 to the case when initial datum does not satisfy the lower/upper bounds in Eq. (5.1). As pointed out before, numerical simulations suggest that these results should hold for general initial datum.
4. Extending the results in Theorem 5.1 when considering constant boundary data such that $u_\ell \neq u_r$. In this case, we expect the dynamics then to converge to the steady state solutions of Eq. (1.1) with the corresponding initial and boundary data (see Theorem 6.1). This is also again suggested by the corresponding numerical simulations Fig. 5.2.

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