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Nonlinear Stabilization of a Viscous Hamilton-Jacobi PDE

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***Abstract*— We consider the boundary stabilization problem for the non-uniform equilibrium profiles of a viscous *Hamilton-Jacobi* (HJ) *Partial Differential Equation* (PDE) with parabolic concave Hamiltonian. We design a nonlinear full-state feedback control law, assuming Neumann actuation, which achieves an arbitrary rate of convergence to the equilibrium. Our design is based on a feedback linearizing transformation which is locally invertible. We prove local exponential stability of the closed-loop system in the H^1 norm, by constructing a Lyapunov functional, and provide an estimate of the region of attraction.**

I. Introduction

Boundary control of nonlinear parabolic PDEs is an important research problem because such systems are common in applications such as, for example, fluids [12], plasma systems [9], [10], and chemical reactors [13]. Viscous Hamilton-Jacobi PDEs, a particular class of semilinear parabolic PDEs, appear in optimal control of stochastic systems [4], and, more importantly, constitute approximations of traffic flow dynamics, commonly modeled by (inviscid) Hamilton-Jacobi PDEs [14], [15]. The Hamilton-Jacobi formulation (for traffic or other systems) is obtained from conservation laws (describing traffic or other physical systems), such as, for example, the (inviscid) Burgers PDE, after applying a change of variables [14]. Yet, the latter formulation is different from the former and requires the development of different tools for control [1], [6]. Few results exist dealing with the boundary control [28], and estimation [14], [15], [16], [17], of Hamilton-Jacobi PDEs, which is a different problem than the one considered here.

In the present article we consider the problem of nonlinear boundary control of a specific viscous Hamilton-Jacobi PDE, which can be viewed as the counterpart of the boundary control problem of the viscous Burgers PDE, for which explicit design approaches exist in the literature [3], [11], [12], [23], [25], [26], [29], [33]. Results dealing with the nonlinear boundary stabilization of more general classes of nonlinear parabolic PDE systems also exist [8], [27], [30], [31], [35], [36], [37]. In particular, the control design methodologies introduced in [25], [26], [36], [37] are inspired from techniques originally developed for control of finite-dimensional nonlinear systems, namely, feedback linearization [21] and backstepping [24].

We design a nonlinear full-state feedback control law for the boundary stabilization of the non-uniform stationary profiles of a viscous Hamilton-Jacobi PDE with parabolic concave Hamiltonian (a.k.a. *Greenshields* Hamiltonian [7], [19]) and Neumann actuation, which we show that are not asymptotically stable in open-loop. Our design is based on a linearizing change of variables, inspired from the Hopf-Cole transformation [18], [20], which, together with the choice of the control laws, transform the system to a linear diffusion-advection system (see also [25], [26] for the design of feedback linearizing control laws in the case of the viscous Burgers equation). We stabilize the linearized system using backstepping [32], achieving an arbitrary decay rate. We prove local exponential stability of the closed-loop nonlinear system in the H^1 norm, by constructing a Lyapunov functional, with the aid of which we provide an estimate of the region of attraction. A nonlinear collocated static output-feedback control design, as in the case with nonlinear "radiation boundary conditions" ([3], [23], [29] for the case of Burgers equation), is also possible, yet, without achieving an arbitrary decay rate of the closed-loop solutions.

In Section II we introduce the problem of stabilization of a viscous Hamilton-Jacobi PDE system with Neumann actuation and explain its relation to traffic modeling. In Section III we prove that the open-loop system is not asymptotically stable. We design a full-state feedback linearizing controller in Section IV and prove local exponential stability of the closed-loop system in Section V.

Notation: We use the common definition of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions from [21]. For a function $u \in L^2(0,1)$ we denote by $\|u(t)\|_{L^2}$ the norm $\|u(t)\|_{L^2} = \left(\int_0^1 u(x,t)^2 dx\right)^{\frac{1}{2}}$. For a function $u \in H^1(0,1)$ we denote by $\|u(t)\|_{H^1}$ the norm $\|u(t)\|_{H^1} = \left(\int_0^1 u(x,t)^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 u_x(x,t)^2 dx\right)^{\frac{1}{2}}$. For a function $u \in H^2(0,1)$ we denote by $\|u(t)\|_{H^2}$ the norm $\|u(t)\|_{H^2} = \left(\int_0^1 u(x,t)^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 u_x(x,t)^2 dx\right)^{\frac{1}{2}} + \left(\int_0^1 u_{xx}(x,t)^2 dx\right)^{\frac{1}{2}}$. Norms in time and space are given by $\|u\|_{H_T^{2,0}} = \left(\int_0^T \|u(t)\|_{H^2}^2 dt\right)^{\frac{1}{2}}$, $\|u\|_{H_T^{2,1}} = \|u\|_{H_T^{2,0}} + \|u_t\|_{H_T^{2,0}}$, and we denote $H^{2,0} = H_\infty^{2,0}$ and $H^{2,1} = H_\infty^{2,1}$. We denote by $C^j(A)$ the space of functions that have continuous derivatives of order j on A.

II. Hamilton-Jacobi PDE with Greenshields Hamiltonian and its Relation to Traffic

We consider the following system

$$
\begin{aligned}
u_t(x,t) &= \epsilon u_{xx}(x,t) - u_x(x,t)\left(1 + u_x(x,t)\right) && (1)\\
u_x(0,t) &= U_0(t) && (2)\\
u_x(1,t) &= U_1(t), && (3)
\end{aligned}
$$

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where $x \in [0,1]$ is the spatial domain, $\epsilon > 0$ is a viscosity coefficient, and U_0, U_1 are control variables.

System (1)–(3) is the viscous version of a macroscopic description of the dynamics of traffic flow on a highway, in which u represents the so-called Moskowitz function [2], [14]. The value of the Moskowitz function $M = u(x,t)$ is interpreted as the "label" of a given vehicle x and t along a road segment [34]. The inviscid version of system (1)–(3) is a Hamilton-Jacobi PDE which is originated from a first-order hyperbolic PDE describing a conservation law, with a Greenshields flux function (which becomes a Greenshields Hamiltonian in the Hamilton-Jacobi description), for the traffic density [5], and is obtained after applying a change of variables on the density [14]. The problem of stabilization of the inviscid version of system (1)–(3) is a different problem which we do not consider in the present article, but it is investigated in [28], and in [1], [6] in conservation law form.

We find next the equilibrium profile of system (1)–(3). The equilibrium y of system (1)–(3) satisfy the following ODE

$$\epsilon y(x)'' - y'(x)\left(1 + y'(x)\right) = 0, \tag{4}$$

which gives

$$y'(x) = -\frac{1}{1 + c^* e^{-\frac{x}{\epsilon}}}, \tag{5}$$

where $c^* \in \mathbb{R}$ is arbitrary, and hence,

$$y(x) = y(0) - x - \epsilon \log\left|\frac{1 + c^* e^{-\frac{x}{\epsilon}}}{1 + c^*}\right|. \tag{6}$$

We stabilize the equilibrium profile (6) for any $y(0) \in \mathbb{R}$ and for c^* such that $-1 < c^*$ or $c^* < -e^{\frac{1}{\epsilon}}$, which guarantees that y' is continuous for all $x \in [0,1]$ (and hence, so is y'' according to (4)). Setting $c^* = \sigma e^{\frac{1}{2\epsilon}}$ we write (6) as

$$y(x) = y(0) - x - \epsilon \log\left(\frac{1 + \sigma e^{-\frac{x - \frac{1}{2}}{\epsilon}}}{1 + \sigma e^{\frac{1}{2\epsilon}}}\right). \tag{7}$$

Although c^* can take negative values, the choice $c^* \geq 0$ in (6) has an interesting interpretation. Relation (5) for $c^* \geq 0$ guarantees that $-1 \leq y'(x) \leq 0$, $\forall x \in [0,1]$. This implies that the traffic density at equilibrium, which is equal to minus the spatial derivative of the Moskowitz function u [14], [34], is bounded below from zero and above by one. This is consistent with the fact that in traffic models the density varies on the interval between the roots of the Hamiltonian which in the present case are 0 and -1 [14].

The equilibrium profile (7) for $y(0) = 1$ and $\sigma = 1$, as well as its derivative, for several values of the viscosity coefficient ϵ are shown in Fig. 1. One can observe that as ϵ converges to zero, the equilibrium profile of u becomes non-differentiable, with the singularity located at $x = \frac{1}{2}$ (one could change the location of this singularity by choosing a different c^*). The non-differentiable profile is the equilibrium profile of the inviscid version of (1)–(3) which we do not consider here.

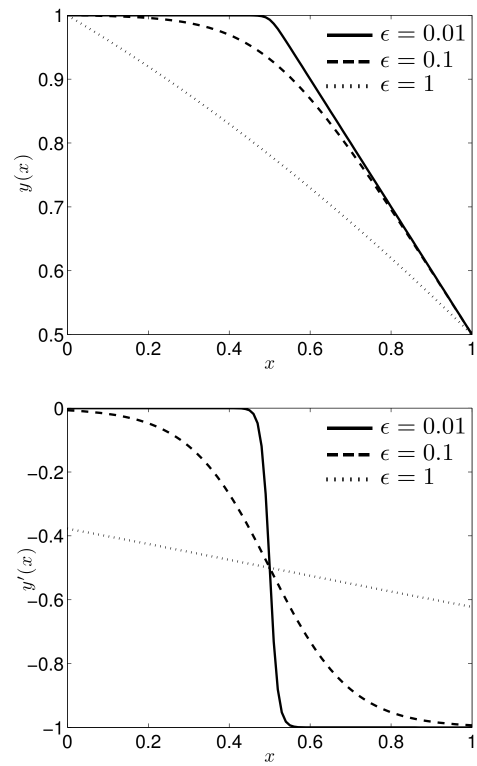

Fig. 1. The equilibrium profile (7) (top) and its derivative (bottom), for three different values of the viscosity coefficient ϵ. As $\epsilon \to 0$, the equilibrium profile becomes non-differentiable.

III. Stability Properties of the Open-Loop System

We shift the equilibrium of system (1)–(3) to the origin. Defining $\tilde{u} = u - y$ we get that $\tilde{u}$ satisfies

$$\begin{aligned} \tilde{u}_t(x,t) &= \epsilon \tilde{u}_{xx}(x,t) - \tilde{u}_x(x,t)\left(1 + \tilde{u}_x(x,t)\right) \\ &\quad -2y'(x)\tilde{u}_x(x,t) && (8) \\ \tilde{u}_x(0,t) &= \tilde{U}_0(t) && (9) \\ \tilde{u}_x(1,t) &= \tilde{U}_1(t), && (10) \end{aligned}$$

where

$$\begin{aligned} \tilde{U}_0(t) &= U_0(t) - y'(0) && (11) \\ \tilde{U}_1(t) &= U_1(t) - y'(1). && (12) \end{aligned}$$

One can observe from (8)–(10) that any constant could be an equilibrium, and hence, the zero solution of (8)–(10) is not asymptotically stable. Therefore, a control design is needed, which asymptotically stabilizes system (8)–(10) to the origin. In fact, we show next that the linearized system has one eigenvalue at zero independently of the values of $y(0)$, ϵ, and σ. For $\sigma \geq 0$ we also show that all the rest of the eigenvalues are negative. The same tools can be applied for studying the behavior of the nonzero eigenvalues for $0 > \sigma > -e^{-\frac{1}{2\epsilon}}$ or $\sigma < -e^{\frac{1}{2\epsilon}}$. Linearizing system (8)–(10) around zero we get

$$\begin{aligned} \theta_t(x,t) &= \epsilon \theta_{xx}(x,t) - \left(1 + 2y'(x)\right)\theta_x(x,t) && (13) \\ \theta_x(0,t) &= 0 && (14) \\ \theta_x(1,t) &= 0. && (15) \end{aligned}$$

Eliminating the advection term with the change of variables $\zeta=\theta e^{-\frac{1}{2\epsilon}\int_0^x\left(1+2y'(s)\right)ds}$ we get that

$$\begin{aligned}\zeta_t(x,t) &= \epsilon\zeta_{xx}(x,t)-\frac{1}{4\epsilon}\zeta(x,t) && (16)\\ \zeta_x(0,t) &= r_1\zeta(0,t) && (17)\\ \zeta_x(1,t) &= r_2\zeta(1,t), && (18)\end{aligned}$$

where

$$\begin{aligned}r_1 &= -\frac{1+2y'(0)}{2\epsilon}\\ &= \frac{1-\sigma e^{\frac{1}{2\epsilon}}}{2\epsilon\left(1+\sigma e^{\frac{1}{2\epsilon}}\right)} && (19)\\ r_2 &= -\frac{1+2y'(1)}{2\epsilon}\\ &= \frac{1-\sigma e^{-\frac{1}{2\epsilon}}}{2\epsilon\left(1+\sigma e^{-\frac{1}{2\epsilon}}\right)}, && (20)\end{aligned}$$

with $\epsilon>0$, $\sigma\geq 0$. With system (16)–(18) we associate the following Sturm-Liouville system (assuming a solution for ζ as $\zeta(x,t)=e^{-\lambda t}\phi(x)$) which is well-known to have only real and simple eigenvalues [22]

$$\begin{aligned}\phi''(x) &= \left(\frac{1}{4\epsilon^2}-\frac{\lambda}{\epsilon}\right)\phi(x) && (21)\\ \phi'(0) &= r_1\phi(0) && (22)\\ \phi'(1) &= r_2\phi(1). && (23)\end{aligned}$$

If $\lambda\leq 0$ is an eigenvalue of (21)–(23), system (21)–(23) must have a nontrivial solution of the form

$$\phi(x) = c_1e^{\mu x}+c_2e^{-\mu x}, \quad (24)$$

where $\mu=\sqrt{\frac{1}{4\epsilon^2}-\frac{\lambda}{\epsilon}}$, with $\mu\geq\frac{1}{2\epsilon}$. Substituting (24) into the boundary conditions (22), (23) one can conclude that the equation $(\mu+r_1)(\mu-r_2)=e^{-2\mu}(\mu-r_1)(\mu+r_2)$ for $\mu\geq\frac{1}{2\epsilon}$, must hold. For $\mu=\frac{1}{2\epsilon}$ this equation is satisfied, and hence, $\lambda=0$ is an eigenvalue of (16)–(18). We show next that $\lambda<0$ can not be an eigenvalue of (16)–(18) for $\sigma\geq 0$. It holds that $h'(\mu)=\frac{2(r_2-r_1)\left(\mu^2+r_1r_2\right)}{(\mu-r_1)^2(\mu+r_2)^2}+2e^{-2\mu}$, where $h(\mu)=\frac{(\mu+r_1)(\mu-r_2)}{(\mu-r_1)(\mu+r_2)}-e^{-2\mu}$, and hence, since $r_2-r_1=\frac{\sigma}{\epsilon\left(1+\sigma e^{-\frac{1}{2\epsilon}}\right)\left(1+\sigma e^{\frac{1}{2\epsilon}}\right)}\left(e^{\frac{1}{2\epsilon}}-e^{-\frac{1}{2\epsilon}}\right)\geq 0$ and $\mu^2+r_1r_2>\frac{1}{4\epsilon^2}+r_1r_2=\frac{1+\sigma^2}{2\epsilon^2\left(1+\sigma e^{-\frac{1}{2\epsilon}}\right)\left(1+\sigma e^{\frac{1}{2\epsilon}}\right)}>0$, for all $\mu>\frac{1}{2\epsilon}$ (and also $\mu-r_1>\frac{1}{2\epsilon}-r_1=\frac{\sigma e^{\frac{1}{2\epsilon}}}{\epsilon\left(1+\sigma e^{\frac{1}{2\epsilon}}\right)}\geq 0$, $\mu+r_2>\frac{1}{2\epsilon}+r_2=\frac{2}{\epsilon\left(1+\sigma e^{-\frac{1}{2\epsilon}}\right)}>0$), one can conclude that h is a strictly increasing function of μ, for all $\mu>\frac{1}{2\epsilon}$, i.e., for all $\lambda<0$. Therefore, $\mu=\frac{1}{2\epsilon}$, i.e., $\lambda=0$ is the unique solution to $\frac{(\mu+r_1)(\mu-r_2)}{(\mu-r_1)(\mu+r_2)}=e^{-2\mu}$, for all $\mu\geq\frac{1}{2\epsilon}$.

IV. Controller Design

A. Feedback linearizing transformation

We design in this section the controllers $\tilde{U}_0$, $\tilde{U}_1$ in order to asymptotically stabilize the nonlinear system (8)–(10). We linearize system (8)–(10) by introducing the following locally invertible transformation

$$\tilde{v}(x,t)=e^{-\frac{1}{\epsilon}\tilde{u}(x,t)}-1, \quad (25)$$

and choosing the control laws as

$$\begin{aligned}\tilde{U}_0(t) &= -\epsilon e^{\frac{1}{\epsilon}\tilde{u}(0,t)}\tilde{V}_0(t) && (26)\\ \tilde{U}_1(t) &= -\epsilon e^{\frac{1}{\epsilon}\tilde{u}(1,t)}\tilde{V}_1(t), && (27)\end{aligned}$$

where $\tilde{V}_0$, $\tilde{V}_1$ are the new control variables yet to be chosen. Transformation (25) and the control laws (26), (27) transform system (8)–(10) to

$$\begin{aligned}\tilde{v}_t(x,t) &= \epsilon\tilde{v}_{xx}(x,t)-\left(1+2y'(x)\right)\tilde{v}_x(x,t) && (28)\\ \tilde{v}_x(0,t) &= \tilde{V}_0(t) && (29)\\ \tilde{v}_x(1,t) &= \tilde{V}_1(t), && (30)\end{aligned}$$

Note that transformation (25) is inspired from the Hopf-Cole transformation [18], [20] and the fact that the variable $h=2u_x$ satisfies $h_t=\epsilon h_{xx}-\left(\frac{h^2}{2}+h\right)_x$. A feedback linearizing transformation for the case of the viscous Burgers equation introduced in [25] and further used in [26].

The inverse transformation of (25) is given by

$$\tilde{u}(x,t)=-\epsilon\log\left(\tilde{v}(x,t)+1\right), \quad (31)$$

which is well-defined whenever the initial condition and the solutions of the system satisfy the following condition

$$\sup_{x\in[0,1]}|\tilde{v}(x,t)|<c, \quad \text{for all } t\geq 0, \quad (32)$$

for some $c\in(0,1]$.

B. Full-state feedback controller

Our next step is to choose the control variables $\tilde{V}_0$ and $\tilde{V}_1$ in order to achieve stabilization of the linear diffusion-advection PDE (28)–(30) with an arbitrary decay rate of convergence. We first define the transformation

$$v(x,t)=\tilde{v}(x,t)e^{-\frac{1}{2\epsilon}\int_0^x\left(1+2y'(s)\right)ds}, \quad (33)$$

in order to eliminate the advection term in (28), and we choose the control variables $\tilde{V}_0$, $\tilde{V}_1$ as

$$\begin{aligned}\tilde{V}_0(t) &= -r_1\tilde{v}(0,t) && (34)\\ \tilde{V}_1(t) &= e^{\frac{1}{2\epsilon}\int_0^1\left(1+2y'(x)\right)dx}V_1(t)-r_2\tilde{v}(1,t), && (35)\end{aligned}$$

where r_1, r_2 are given in (19) and (20) respectively, and V_1 is a new control variable yet to be designed, in order to get

$$\begin{aligned}v_t(x,t) &= \epsilon v_{xx}(x,t)-\frac{1}{4\epsilon}v(x,t) && (36)\\ v_x(0,t) &= 0 && (37)\\ v_x(1,t) &= V_1(t). && (38)\end{aligned}$$

We employ next backstepping for stabilization of system (36)–(38) [32]. Introducing the following backstepping transformation

$$w(x,t)=v(x,t)-\int_0^x k(x,y)v(y,t)dy, \quad (39)$$

we transform system (36)–(38) to the "target system"

$$w_t(x,t) = \epsilon w_{xx}(x,t) - \left(\frac{1}{4\epsilon} + c_1\right) w(x,t) \quad (40)$$
$$w_x(0,t) = 0 \quad (41)$$
$$w_x(1,t) = 0, \quad (42)$$

where $c_1 > 0$ is arbitrary, when the gain kernel k satisfies

$$k_{xx}(x,y) - k_{yy}(x,y) = \frac{c_1}{\epsilon} k(x,y) \quad (43)$$
$$\frac{dk(x,x)}{dx} = -\frac{c_1}{2\epsilon} \quad (44)$$
$$k_y(x,0) = 0, \quad (45)$$

with $k(0,0) = 0$, such that (41) is satisfied given (37), and the control law V_1 is chosen as

$$V_1(t) = k(1,1)v(1,t) + \int_0^1 k_x(1,y)v(y,t)dy. \quad (46)$$

It is shown in [32] that $k \in C^2(E)$, where $E = \{(x,y) : 0 \leq y \leq x \leq 1\}$. In fact, (43)–(45) can be solved explicitly as [32]

$$k(x,y) = -\frac{c_1}{\epsilon} x \frac{I_1\left(\sqrt{\frac{c_1}{\epsilon}\left(x^2 - y^2\right)}\right)}{\sqrt{\frac{c_1}{\epsilon}\left(x^2 - y^2\right)}}, \quad (47)$$

where I_1 is a modified Bessel function of order one. Combining relations (25), (33)–(35), (46), the control laws (26), (27) are expressed in terms of the original variable $\tilde{u}$

$$\tilde{U}_0(t) = -\epsilon r_1 \left(e^{\frac{1}{\epsilon}\tilde{u}(0,t)} - 1\right) \quad (48)$$
$$\tilde{U}_1(t) = \epsilon\left(-r_2 + k(1,1)\right)\left(e^{\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right) + \frac{\epsilon e^{\frac{1}{\epsilon}\tilde{u}(1,t)}}{1 + \sigma e^{-\frac{1}{2\epsilon}}} \int_0^1 k_x(1,y)\left(e^{\frac{y-1}{2\epsilon}} + \sigma e^{-\frac{y}{2\epsilon}}\right) \times \left(1 - e^{-\frac{1}{\epsilon}\tilde{u}(y,t)}\right) dy. \quad (49)$$

where r_1 and r_2 are given in (19) and (20) respectively. The inverse transformation of (39) is well-defined and is given by [32]

$$v(x,t) = w(x,t) + \int_0^x l(x,y)w(y,t)dy, \quad (50)$$

where l satisfies a well-posed hyperbolic linear partial differential equation like (43)–(45) and $l \in C^2(E)$, where $E = \{(x,y) : 0 \leq y \leq x \leq 1\}$. In Fig. 2 we show the interconnections between the variables $\tilde{u}$, $\tilde{v}$, v, and w involved in transformations (25), (31), (33), (39), and (50).

C. Static collocated output-feedback controller

Note that if one does not intent to achieve an arbitrary decay rate c_1 for the closed-loop system then one could set $k \equiv 0$ obtaining the following simpler control laws

$$\tilde{U}_0(t) = -\epsilon r_1 \left(e^{\frac{1}{\epsilon}\tilde{u}(0,t)} - 1\right) \quad (51)$$
$$\tilde{U}_1(t) = -\epsilon r_2 \left(e^{\frac{1}{\epsilon}\tilde{u}(1,t)} - 1\right). \quad (52)$$

In this case the closed-loop system v satisfies (36)–(38) with $V_1 = 0$ which is exponentially stable. An interesting feature

$$\begin{tikzcd}
\tilde{u} \arrow[r, bend left, "e^{-\frac{1}{\epsilon}(\cdot)}-1"] & \tilde{v} \arrow[l, bend left, "-\epsilon\log(\cdot+1)"] \arrow[r, bend left, "e^{-\frac{1}{2\epsilon}\int_0^x\left(1+2y'(s)\right)ds}"] & v \arrow[l, bend left, "e^{\frac{1}{2\epsilon}\int_0^x\left(1+2y'(s)\right)ds}"] \arrow[r, bend left, "\mathscr{K}\{\cdot\}"] & w \arrow[l, bend left, "\mathscr{L}\{\cdot\}"]
\end{tikzcd}$$

Fig. 2. The interconnections between the variables $\tilde{u}$, $\tilde{v}$, v, and w involved in transformations (25), (31), (33), (39), and (50). The operators $\mathscr{K}\{\cdot\}$ and $\mathscr{L}\{\cdot\}$ are defined as $\mathscr{K}\{v\}(x) = v(x) - \int_0^x k(x,y)v(y)dy$ and $\mathscr{L}\{w\}(x) = w(x) + \int_0^x l(x,y)w(y)dy$ respectively.

of the control laws (51)–(52) is that the controllers $\tilde{U}_0$ and $\tilde{U}_1$ are given only in terms of the boundary values of $\tilde{u}$ at $x = 0$ and $x = 1$. This enables one to design an output-feedback control law assuming that the only available measurements are the boundary values $\tilde{u}(0,t)$ and $\tilde{u}(1,t)$, in contrast to the control law (49) which requires measurement of the full state $\tilde{u}(x,t)$, for all $x \in [0,1]$.

V. Stability Analysis

Theorem 1: Consider system (8)–(10) together with the control laws (48), (49), (47). There exist positive constants R and μ_1 such that for all initial conditions $\tilde{u}(\cdot,0) \in H^2(0,1)$ which are compatible with the feedback laws (48), (49) and satisfy

$$\|\tilde{u}(0)\|_{H^1} < R, \quad (53)$$

the following holds for all $t \geq 0$,

$$\|\tilde{u}(t)\|_{H^1} \leq \alpha\left(\|\tilde{u}(0)\|_{H^1}\right) e^{-\left(c_1 + \frac{1}{4\epsilon}\right)t}, \quad (54)$$

where $c_1 \geq 0$ is arbitrary, and

$$\alpha(s) = \frac{3\mu_1}{1-c} e^{\frac{2s}{\epsilon}} s, \quad (55)$$

with $0 < c < 1$. Moreover, the closed-loop system has a unique solution $\tilde{u} \in H^{2,1}\left((0,1)\times(0,\infty)\right)$.

The proof of Theorem 1 is based on a series of technical lemmas which are presented next.

Lemma 1: If $\tilde{u} \in H^1(0,1)$ then $\tilde{v} \in H^1(0,1)$ and the following holds

$$\|\tilde{v}(t)\|_{H^1} \leq \alpha_1\left(\|\tilde{u}(t)\|_{H^1}\right), \quad (56)$$

where the class $\mathcal{K}_\infty$ function α_1 is given by

$$\alpha_1(s) = \frac{3s}{\epsilon} e^{\frac{2s}{\epsilon}}. \quad (57)$$

Moreover, if $\tilde{u} \in H^2(0,1)$ then $\tilde{v} \in H^2(0,1)$.

Proof: For the function $f(r) = e^{-\frac{1}{\epsilon}r} - 1$ the following holds, $|f(r)| \leq \frac{|r|}{\epsilon} e^{\frac{|r|}{\epsilon}}$, for all $r \in \mathbb{R}$. Hence, using (25) one can conclude that

$$|\tilde{v}(x,t)| \leq \hat{\alpha}\left(|\tilde{u}(x,t)|\right), \quad (58)$$

where the class $\mathcal{K}_\infty$ function $\hat{\alpha}$ is defined as $\hat{\alpha}(s) = \frac{s}{\epsilon} e^{\frac{s}{\epsilon}}$, and hence,

$$|\tilde{v}(x,t)| \leq \hat{\alpha}\left(\sup_{x\in[0,1]} |\tilde{u}(x,t)|\right), \quad \text{for all } x \in [0,1]. \quad (59)$$

For any $u \in H^1(0,1)$ the following holds, $u(x,t) = u(0,t)+\int_0^x u_y(y,t)dy$, and hence, using Cauchy-Schwartz's inequality we obtain

$$|u(x,t)| \leq |u(0,t)| + \sqrt{\int_0^1 u_x(x,t)^2 dx}, \quad x \in [0,1]. \quad (60)$$

Since $u(0,t) = u(x,t) - \int_0^x u_y(y,t)dy$ we get with the Cauchy-Schwartz inequality that

$$|u(0,t)| \leq |u(x,t)| + \sqrt{\int_0^1 u_x(x,t)^2 dx}. \quad (61)$$

Therefore, by integrating (61) and using the Cauchy-Schwartz inequality we get

$$|u(0,t)| \leq \|u(t)\|_{H^1}. \quad (62)$$

Combining (60) and (62) we get that

$$\sup_{x\in[0,1]} |\tilde{u}(x,t)| \leq 2\|\tilde{u}(t)\|_{H^1}. \quad (63)$$

Using (59) we get that $\sup_{x\in[0,1]} |\tilde{v}(x,t)| \leq \hat{\alpha}\left(2\|\tilde{u}(t)\|_{H^1}\right)$. Hence, it also holds

$$\|\tilde{v}(t)\|_{L^2} \leq \hat{\alpha}\left(2\|\tilde{u}(t)\|_{H^1}\right). \quad (64)$$

Using (25) we obtain

$$\tilde{v}_x(x,t) = -\frac{1}{\epsilon}\tilde{u}_x(x,t)e^{-\frac{1}{\epsilon}\tilde{u}(x,t)}. \quad (65)$$

Hence, using (63) we obtain by integrating (65)

$$\sqrt{\int_0^1 \tilde{v}_x(x,t)^2 dx} \leq \frac{1}{\epsilon}e^{\frac{2}{\epsilon}\|\tilde{u}(t)\|_{H^1}}\sqrt{\int_0^1 \tilde{u}_x(x,t)^2 dx}. \quad (66)$$

Combining (64) and (66) we arrive at

$$\|\tilde{v}(t)\|_{H^1} \leq \hat{\alpha}\left(2\|\tilde{u}(t)\|_{H^1}\right) + \frac{1}{\epsilon}e^{\frac{2}{\epsilon}\|\tilde{u}(t)\|_{H^1}}\|\tilde{u}(t)\|_{H^1}. \quad (67)$$

Which gives (56) with α_1 defined in (57). Analogously, using the fact that $\tilde{v}_{xx}(x,t) = -\frac{1}{\epsilon}\tilde{u}_{xx}(x,t)e^{-\frac{1}{\epsilon}\tilde{u}(x,t)} + \frac{1}{\epsilon^2}\tilde{u}_x(x,t)^2 e^{-\frac{1}{\epsilon}\tilde{u}(x,t)}$ and relations (60)–(62) for $u = \tilde{u}_x \in H^1(0,1)$ one can prove that when $\tilde{u} \in H^2(0,1)$ then

$$\|\tilde{v}(t)\|_{H^2} \leq \alpha_2\left(\|\tilde{u}(t)\|_{H^2}\right), \quad (68)$$

where the class $\alpha_2 \in \mathcal{K}_\infty$ is $\alpha_2(s) = \frac{\sqrt{2}}{\epsilon}e^{\frac{2}{\epsilon}s}\left(\frac{2}{\epsilon}s+1\right)s$. ■

Lemma 2: For all solutions of the system that satisfy (32) for some $0 < c < 1$, if $\tilde{v} \in H^1(0,1)$ then $\tilde{u} \in H^1(0,1)$ and the following holds

$$\|\tilde{u}(t)\|_{H^1} \leq \frac{\epsilon}{1-c}\|\tilde{v}(t)\|_{H^1}. \quad (69)$$

Moreover, for all solutions of the system that satisfy (32) for some $0 < c < 1$, if $\tilde{v} \in H^2(0,1)$ then $\tilde{u} \in H^2(0,1)$.

Proof: Using (31) and (32) one can conclude from the simple fact that $|\log(r+1)| \leq \frac{1}{1-c}|r|$, for all $|r| < c$ and some $0 < c < 1$, that $|\tilde{u}(x,t)| \leq \frac{\epsilon}{1-c}|\tilde{v}(x,t)|$, and hence,

$$\|\tilde{u}(t)\|_{L^2} \leq \frac{\epsilon}{1-c}\|\tilde{v}(t)\|_{L^2}. \quad (70)$$

Using (31) it follows that $\tilde{u}_x(x,t) = -\epsilon\frac{\tilde{v}_x(x,t)}{\tilde{v}(x,t)+1}$. Therefore, with (32) we get that

$$\sqrt{\int_0^1 \tilde{u}_x(x,t)^2 dx} \leq \frac{\epsilon}{1-c}\sqrt{\int_0^1 \tilde{v}_x(x,t)^2 dx}, \quad (71)$$

and hence, combining (70) and (71) we arrive at

$$\|\tilde{u}(t)\|_{H^1} \leq \frac{\epsilon}{1-c}\|\tilde{v}(t)\|_{H^1}. \quad (72)$$

Analogously, using the fact that $\tilde{u}_{xx}(x,t) = -\epsilon\frac{\tilde{v}_{xx}(x,t)(\tilde{v}(x,t)+1)-\tilde{v}_x(x,t)^2}{(\tilde{v}(x,t)+1)^2}$, relation (32), and relations (60)–(62) for $u = \tilde{v}_x \in H^1(0,1)$ we obtain

$$\sqrt{\int_0^1 \tilde{u}_{xx}(x,t)^2 dx} \leq \frac{\sqrt{2}\epsilon}{1-c}\sqrt{\int_0^1 \tilde{v}_{xx}(x,t)^2 dx} + \frac{2\sqrt{2}\epsilon}{(1-c)^2}\|\tilde{v}(t)\|_{H^1}\|\tilde{v}(t)\|_{H^2}, \quad (73)$$

and hence, combining (70), (72), and (73) we arrive at

$$\|\tilde{u}(t)\|_{H^2} \leq \alpha_4\left(\|\tilde{v}(t)\|_{H^2}\right), \quad (74)$$

where $\alpha_4 \in \mathcal{K}_\infty$ is defined as $\alpha_4(s) = \frac{\sqrt{2}\epsilon}{1-c}s + \frac{2\sqrt{2}\epsilon}{(1-c)^2}s^2$. ■

Lemma 3: There exists a positive constant μ_1 such that the following holds for all $t \geq 0$ and any $c_1 \geq 0$,

$$\|\tilde{v}(t)\|_{H^1} \leq \mu_1\|\tilde{v}(0)\|_{H^1}e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}. \quad (75)$$

Proof: Taking the L^2-inner product of (40) with w and w_{xx}, we obtain after using integration by parts and (41), (42) that

$$\frac{d\int_0^1 w(x,t)^2 dx}{dt} = -2\epsilon\int_0^1 w_x(x,t)^2 dx - 2\left(c_1+\frac{1}{4\epsilon}\right)\int_0^1 w(x,t)^2 dx. \quad (76)$$

$$\frac{d\int_0^1 w_x(x,t)^2 dx}{dt} = -2\epsilon\int_0^1 w_{xx}(x,t)^2 dx - 2\left(c_1+\frac{1}{4\epsilon}\right)\int_0^1 w_x(x,t)^2 dx. \quad (77)$$

From (76), (77), and by using the Lyapunov functional

$$V_1(t) = \frac{1}{2}\int_0^1 w(x,t)^2 dx + \frac{1}{2}\int_0^1 w_x(x,t)^2 dx, \quad (78)$$

we get that $\dot{V}_1(t) \leq -2\left(c_1+\frac{1}{4\epsilon}\right)V_1(t)$, and hence,

$$\|w(t)\|_{H^1} \leq \sqrt{2}e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}\|w(0)\|_{H^1}. \quad (79)$$

Using the backstepping transformation (39) and its inverse (50), and the fact that $k, l \in C^2(E)$, where $E = \{(x,y) : 0 \leq y \leq x \leq 1\}$, one can conclude that [32]

$$\|w(t)\|_{H^1} \leq 2\sqrt{6}M_k\|v(t)\|_{H^1} \quad (80)$$

$$\|v(t)\|_{H^1} \leq 2\sqrt{6}M_l\|w(t)\|_{H^1}, \quad (81)$$

where $M_k = 1 + \sup_{0\leq y\leq x\leq 1}|k(x,y)| + \sup_{0\leq y\leq x\leq 1}|k_x(x,y)|$, $M_l = 1 + \sup_{0\leq y\leq x\leq 1}|l(x,y)| +$

$\sup_{0\le y\le x\le 1}|l_x(x,y)|$. With relation (33) we obtain that

$$\|v(t)\|_{H^1} \le m e^{\frac{1+2\sup_{x\in[0,1]}|y'(x)|}{2\epsilon}}\|\tilde v(t)\|_{H^1} \quad (82)$$

$$\|\tilde v(t)\|_{H^1} \le m e^{\frac{1+2\sup_{x\in[0,1]}|y'(x)|}{2\epsilon}}\|v(t)\|_{H^1} \quad (83)$$

$$m = 1+\sqrt{2}+\sqrt{2}\frac{1+2\sup_{x\in[0,1]}|y'(x)|}{2\epsilon}. \quad (84)$$

Estimate (75) then follows. ■

Proof of Theorem 1: Using relations (56), (75) we get

$$\|\tilde v(t)\|_{H^1} \le \mu_1\alpha_1\left(\|\tilde u(0)\|_{H^1}\right)e^{-\left(c_1+\frac{1}{4\epsilon}\right)t}. \quad (85)$$

Combining relations (60)–(62) for $u=\tilde v\in H^1(0,1)$ we get that $\sup_{x\in[0,1]}|\tilde v(x,t)|\le 2\|\tilde v(t)\|_{H^1}$, and hence, choosing R in (53) as $R=\alpha_1^{-1}\left(\frac{c}{2\mu_1}\right)$ one can conclude that relation (32) is satisfied. Therefore, using relations (69) and (57) we arrive at (54), (55). Existence and uniqueness of a solution $\tilde u\in H^{2,1}\left((0,1)\times(0,\infty)\right)$ follows from the target system (40)–(42), transformations (39), (50) and Lemmas 1, 2, by using almost identical arguments to [25] (Section VII).

VI. Conclusions

We develop a full-state nonlinear control design for a Hamilton-Jacobi PDE with Greenshields Hamiltonian. We prove local exponential stability of the closed-loop system using a Lyapunov functional and provide an estimate of the region of attraction.

Although in this article we consider the case of a Greenshields Hamiltonian, the same tools can be applied to the stabilization of the PDE given by $u_t=\epsilon u_{xx}-\frac{u_x^2}{2}$.

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