Motion Planning of Water Tank

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Outline

- ❑Problem Definition
- ❑Mathematical Derivation
- ❑MATLAB Simulation
- ❑Conclusion
- ❑Future Perfection

Problem

- ❑Sloshing when moving tanks containing liquids from one place to another

Goal

- ❑Simulate tank dynamics
- ❑Minimize sloshing

Approaches

- ❑Calculate flatness
- ❑Select proper output function

Assumptions

- ❑Irrotational potential flow
- ❑Shallow water
- ❑Flat Bottom
- ❑1-D motion of tank

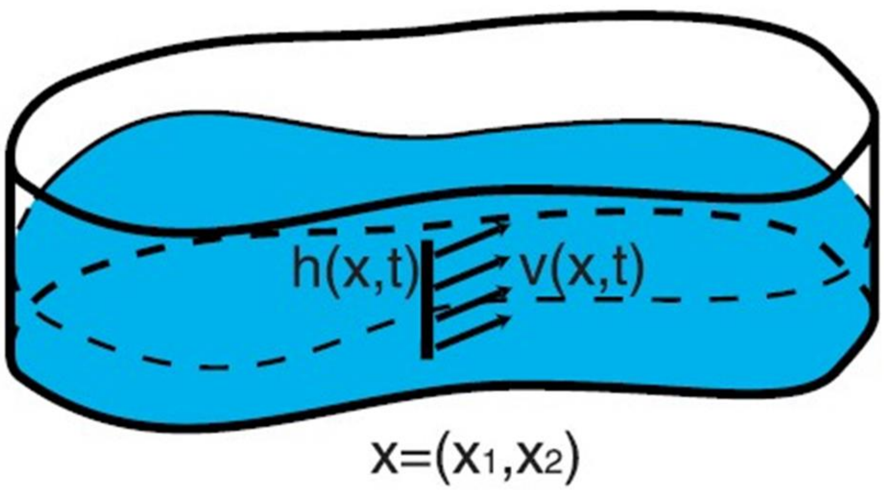


Fig. 3. The two-dimensional tank.

Petit, 2002

Notations

h(x,y,t) = Actual water depth

H(x,y,t) = Actual depth minus average depth $h = \bar{h} + H$

D = Location of the tank's bottom center

v(x,y,t) = Velocity of water

Basic Equations

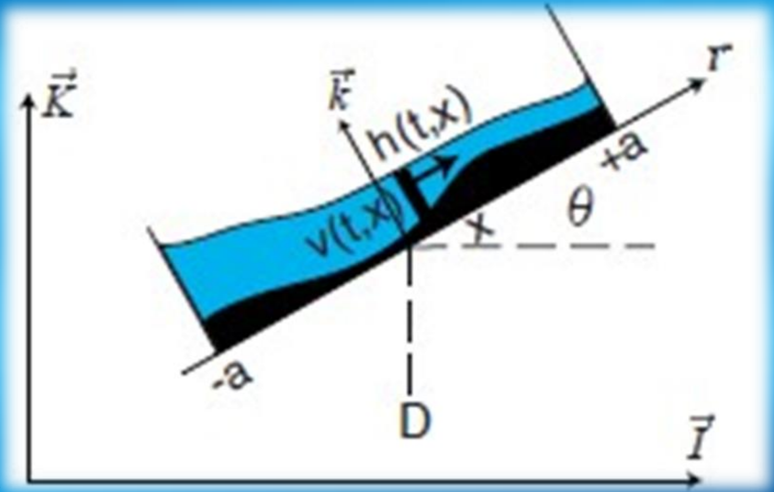
Continuity

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\vec{v}) = 0$$

Momentum

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -g \nabla h$$

Petit, 2002

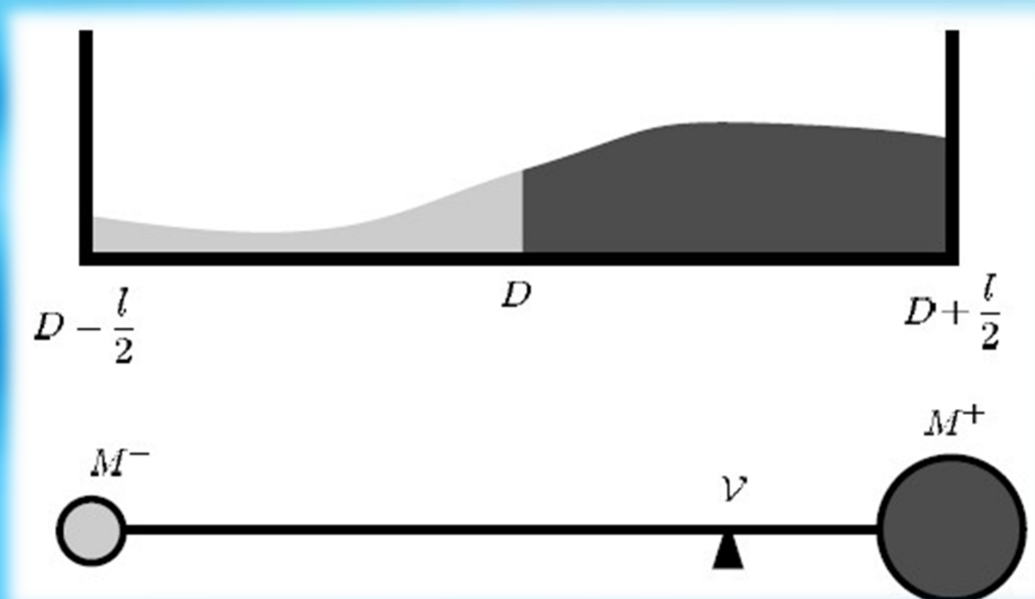


Define Input & Output

Input D(t) – Position of tank center on the bottom

Output y(t) – Center of gravity of tank's first half mass and second half mass

$$y(t) = D(t) + \frac{1}{2h}\left(\int_0^a H(x,t)\,dx - \int_{-a}^0 H(x,t)\,dx\right)$$



Dubois, 1999

System dynamics for 1D model

$$\begin{cases} \dfrac{\partial^2 H}{\partial t^2} = \bar{h}g\dfrac{\partial^2 H}{\partial x^2} \\ \dfrac{\partial H}{\partial x}(a,t) = \dfrac{\partial H}{\partial x}(-a,t) = -\dfrac{u}{g} \\ \ddot{D} = u \end{cases}$$

Recall d'Alembert's formula

$$\frac{\partial^2 H}{\partial t^2} = \bar{h}g\frac{\partial^2 H}{\partial x^2} \Rightarrow H(x,t) = \varphi(t+\frac{x}{c}) + \psi(t-\frac{x}{c})$$

Plug into the BC's to solve for D(t)

System dynamics for 1D model

$$\begin{cases} D(t) = \frac{1}{2}\left(y(t+\frac{a}{c}) + y(t-\frac{a}{c})\right) \\ H(x,t) = \frac{c}{2g}\left(\dot{y}(t+\frac{x}{c}) - \dot{y}(t-\frac{x}{c})\right) \end{cases}$$

$$y(t) = \begin{cases} p & \text{if } t \le a/c \\ \text{arbitrary} & \text{if } a/c < t < T - a/c \\ q & \text{if } t \ge T - a/c \end{cases}$$

This approach can be extended to 2D case

System dynamics for 2D rectangular tank

$$\begin{cases} \ddot{H} = g\bar{h}\Delta H \\ g\nabla H \cdot \vec{n} = -u.\vec{n} \quad \text{on } \partial\Omega \\ \ddot{D} = u \end{cases}$$

$$\begin{cases} D_1(t) = \frac{1}{2}\left(y_1(t+\frac{a_1}{c}) + y_1(t-\frac{a_1}{c})\right) \\ D_2(t) = \frac{1}{2}\left(y_2(t+\frac{a_2}{c}) + y_2(t-\frac{a_2}{c})\right) \\ H(x_1,x_2,t) = \frac{c}{2g}\left(\dot{y}_1(t+\frac{x_1}{c}) - \dot{y}_1(t-\frac{x_1}{c}) + \dot{y}_2(t+\frac{x_2}{c}) - \dot{y}_2(t-\frac{x_2}{c})\right) \end{cases}$$

Recall:
H = Fluctuation of surface
D = Location of tank bottom center
K, I = Global coordinate system
k, i = Local system
Ω = Tank boundary

2D model can be decoupled to get 1D model

System dynamics for circular tank

Laplace transform

$$\ddot{H} = g\bar{h}\Delta H \quad \rightarrow \quad \Delta\hat{H}(x,y,s) - \frac{s^2}{g\bar{h}}\hat{H}(x,y,s) = 0$$

SOV in cylindrical coordinates

$$\bar{H}(r,\theta) = R(r)\Theta(\theta) \quad \rightarrow \quad \frac{1}{R}\left(\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr}\right) + \frac{1}{r^2\Theta}\frac{d^2\Theta}{d\theta^2} - \frac{s^2}{g\bar{h}} = 0$$

Can solve for H

System dynamics for circular tank

$$
\begin{cases}
D_1(t) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y_1 \left(t - \dfrac{l \cos\varphi}{\sqrt{g\bar{h}}} \right) \cos^2\varphi \, d\varphi \\
D_2(t) = \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} y_2 \left(t - \dfrac{l \cos\varphi}{\sqrt{g\bar{h}}} \right) \cos^2\varphi \, d\varphi \\
H(r, \theta, t) = \dfrac{\cos\theta}{\pi} \sqrt{\dfrac{\bar{h}}{g}} \displaystyle\int_0^{2\pi} \dot{y}_1 \left(t - \dfrac{r}{\sqrt{g\bar{h}}} \cos\varphi \right) \cos\varphi \, d\varphi \\
\qquad + \dfrac{\sin\theta}{\pi} \sqrt{\dfrac{\bar{h}}{g}} \displaystyle\int_0^{2\pi} \dot{y}_2 \left(t - \dfrac{r}{\sqrt{g\bar{h}}} \cos\varphi \right) \cos\varphi \, d\varphi
\end{cases}
$$

Selecting y(t) function

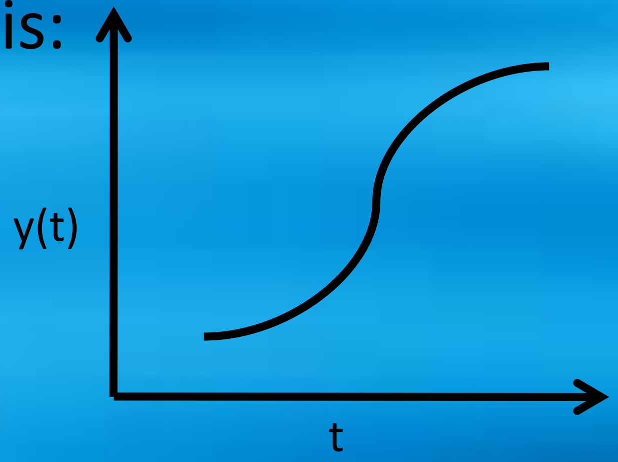
Steady state at start and end points – Controllable

Tank moves from 0 to d in time T – D(t) from 0 to d

Got four BC’s to restrict y(t):

$$y(0) = 0,\ y(T) = d,\ y'(0) = 0,\ y'(T) = 0$$

y(t) should roughly look like this:



Four y(t) functions are tested

Cosine Function

$$y(t) = \frac{d}{2}\cos\left(\frac{\pi t}{T} + \pi\right) + \frac{d}{2}$$

3rd Order Polynomial

$$y(t) = -\frac{2d}{T^3}t^3 + \frac{3d}{T^2}t^2$$

Error Function

$$y(t) = \frac{d}{2}\operatorname{erf}\left(\frac{4t - 2T}{T}\right) + \frac{d}{2}$$

Arc + Line + Arc

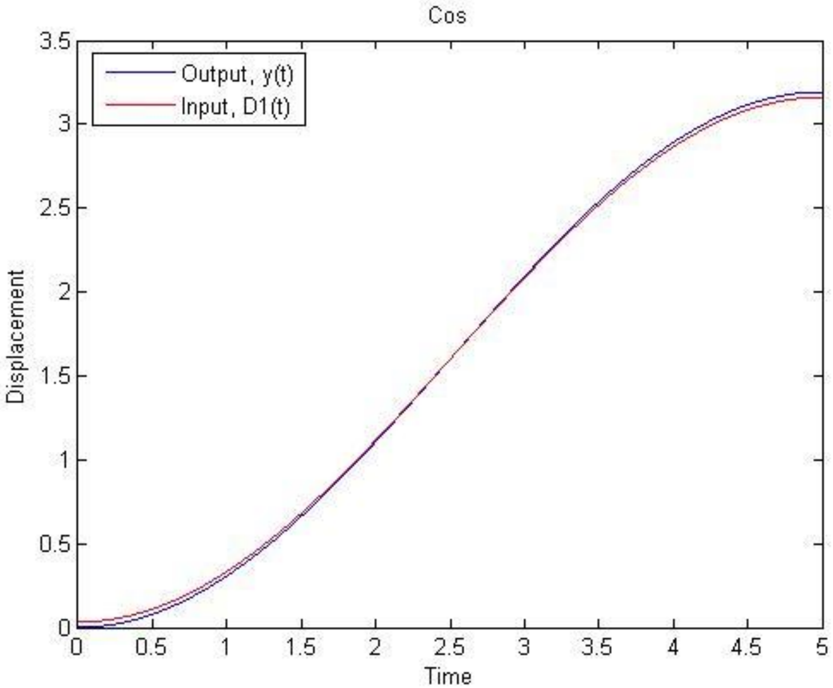
$$y(t) = \begin{cases} r - \sqrt{r^2 - t^2}, & t < r \cdot \sin\theta_3 \\ t \cdot \tan\theta_3 + r(1 - \cos\theta_3 - \tan\theta_3 \sin\theta_3), & r \cdot \sin\theta_3 < t < T - r \cdot \sin\theta_3 \\ \sqrt{r^2 - (t - T)^2} + d - r, & t > T - r \cdot \sin\theta_3 \end{cases}$$

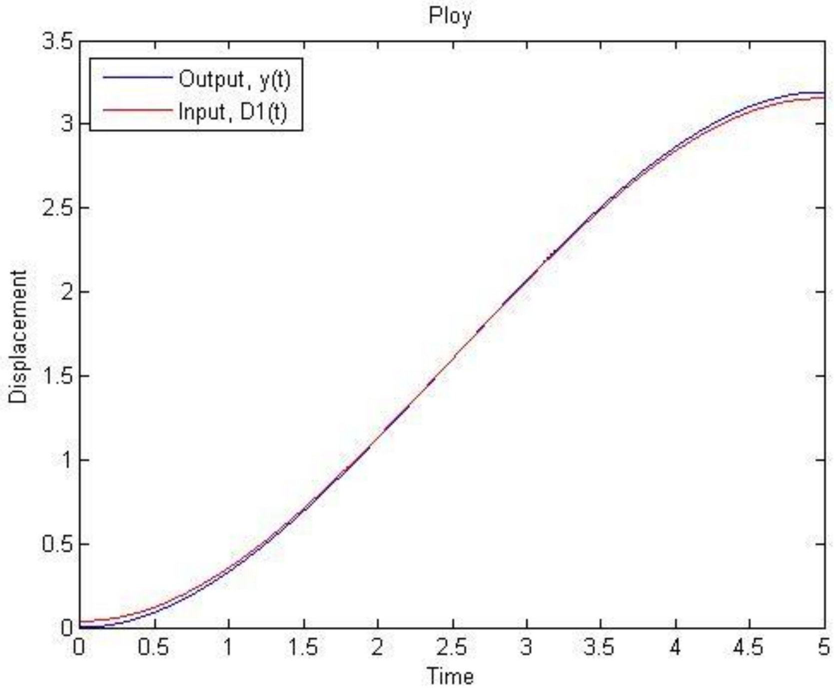
$$\theta_3 = \pi - \theta_1 - \theta_2$$

$$\theta_1 = \arctan\left(\left|\frac{T}{d - 2r}\right|\right), \quad \theta_2 = \arccos\left(\left|\frac{2r}{\sqrt{T^2 + (d - 2r)^2}}\right|\right)$$

D(t) – y(t) Plots

- 2 x 2 Rectangular tank
- Water depth h = 1
- Travel time T = 5
- Time step dt = 0.001
- Travel distance d = 3.19





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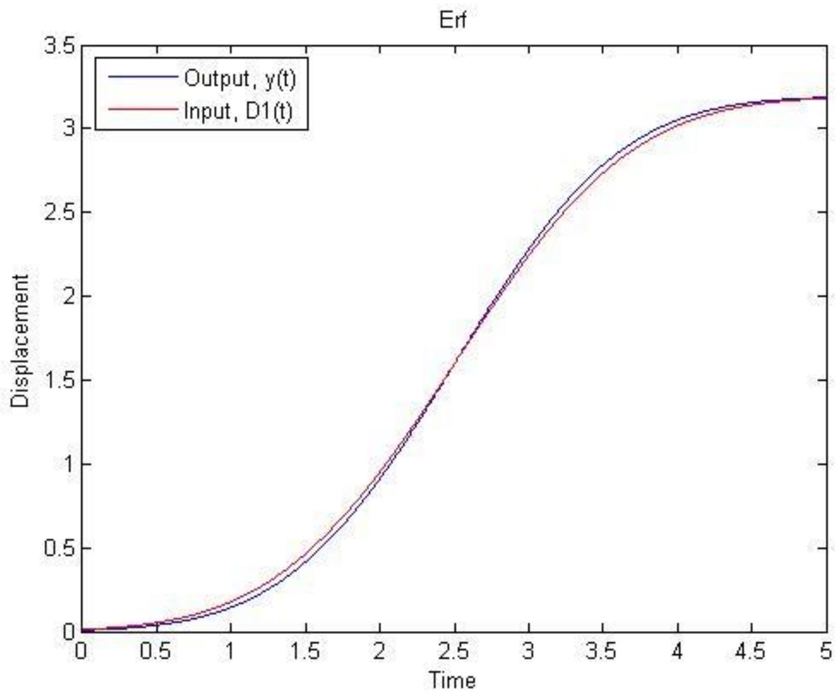

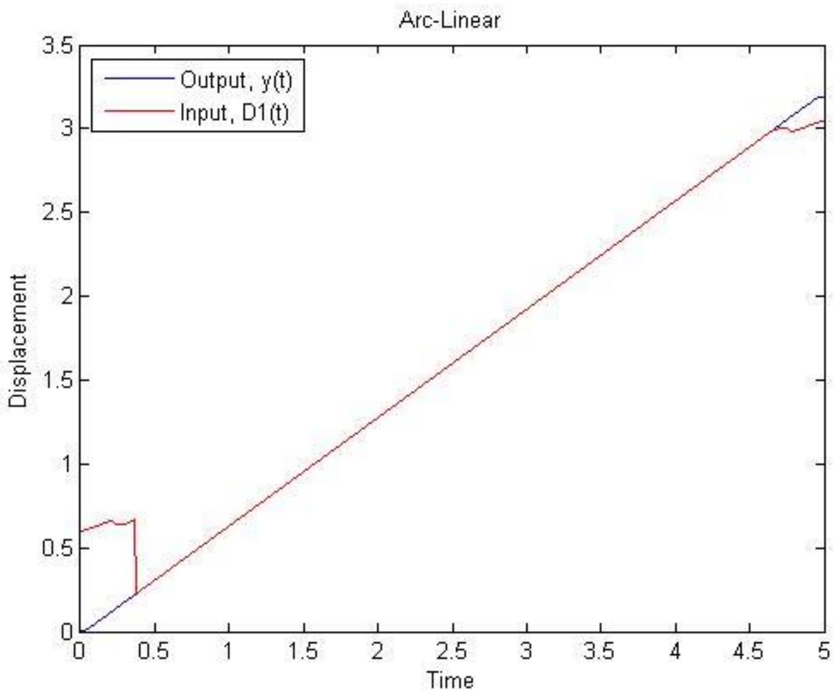

Mid-presentation Review

Two systems: Rectangular Tank and Circular Tank

$$\begin{cases} D_1(t) = \frac{1}{2}\left(y_1(t+\frac{a_1}{c}) + y_1(t-\frac{a_1}{c})\right) \\ D_2(t) = \frac{1}{2}\left(y_2(t+\frac{a_2}{c}) + y_2(t-\frac{a_2}{c})\right) \\ H(x_1,x_2,t) = \frac{c}{2g}\left(\dot{y}_1(t+\frac{x_1}{c}) - \dot{y}_1(t-\frac{x_1}{c}) + \dot{y}_2(t+\frac{x_2}{c}) - \dot{y}_2(t-\frac{x_2}{c})\right) \end{cases}$$

$$\begin{cases} D_1(t) = \frac{1}{\pi}\int_0^{2\pi} y_1\left(t - \frac{l\cos\varphi}{\sqrt{g\bar{h}}}\right)\cos^2\varphi\, d\varphi \\ D_2(t) = \frac{1}{\pi}\int_0^{2\pi} y_2\left(t - \frac{l\cos\varphi}{\sqrt{g\bar{h}}}\right)\cos^2\varphi\, d\varphi \\ H(r,\theta,t) = \frac{\cos\theta}{\pi}\sqrt{\frac{\bar{h}}{g}}\int_0^{2\pi} \dot{y}_1\left(t - \frac{r}{\sqrt{g\bar{h}}}\cos\varphi\right)\cos\varphi\, d\varphi \\ \qquad + \frac{\sin\theta}{\pi}\sqrt{\frac{\bar{h}}{g}}\int_0^{2\pi} \dot{y}_2\left(t - \frac{r}{\sqrt{g\bar{h}}}\cos\varphi\right)\cos\varphi\, d\varphi \end{cases}$$

Three output functions: Cosine, Polynomial and Erf

$$y(t) = \frac{d}{2}\cos\left(\frac{\pi t}{T} + \pi\right) + \frac{d}{2}$$

$$y(t) = -\frac{2d}{T^3}t^3 + \frac{3d}{T^2}t^2$$

$$y(t) = \frac{d}{2}\mathrm{erf}\left(\frac{4t - 2T}{T}\right) + \frac{d}{2}$$

Goal: Minimize H

Simulation Video

Analysis

The standard deviation of the water surface can be used to evaluate sloshing

	Rectangular Tank	Circular Tank
Cosine Function	0.0528	0.0323
Polynomial	0.0584	0.0310
Error Function	0.0830	0.0464

Conclusion

- ❑Cosine and 3rd order polynomial outputs have better performance
- ❑Circular tank produces less sloshing than rectangular tank

Future Perfection

- ❑Simulate non-linear moving trajectories
- ❑Extend to more complicated geometries
- ❑Find proper initial and final values of output
- ❑Test more output functions

$$y(t) = \begin{cases} p & \text{if } t \leq a/c \\ \text{arbitrary} & \text{if } a/c < t < T - a/c \\ q & \text{if } t \geq T - a/c \end{cases}$$

References

[1] N. Petit, P. Rouchon,"Dynamics and solutions to some control problems for water-tank systems", IEEE Trans. Contr. Syst. Technol., vol. 47, pp.594 -609 2002

[2] F. Dubois, N. Petit and P. Rouchon, "Motion planning and nonlinear simulation for a tank containing a fluid", European Control Conference, 1999

[3] D. Ho, B. Grunloh and N. Jenson, "Motion planning of a water tank with differential flatness and optimal control", CE291 Final Project, 2012

Thanks!