Motion Planning of a Water Tank with Differential Flatness and Optimal Control

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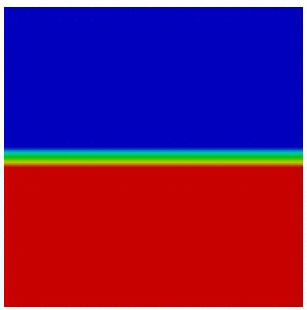
Problem Definition

- Project Goal
  - Move a tank containing a fluid to another location within a given time with minimal sloshing
- Potential Project Applications
  - Industrial process control moving tanks of liquid from station to station

Example of Constant Velocity

- Container traveling 1.5 meters in 2 seconds



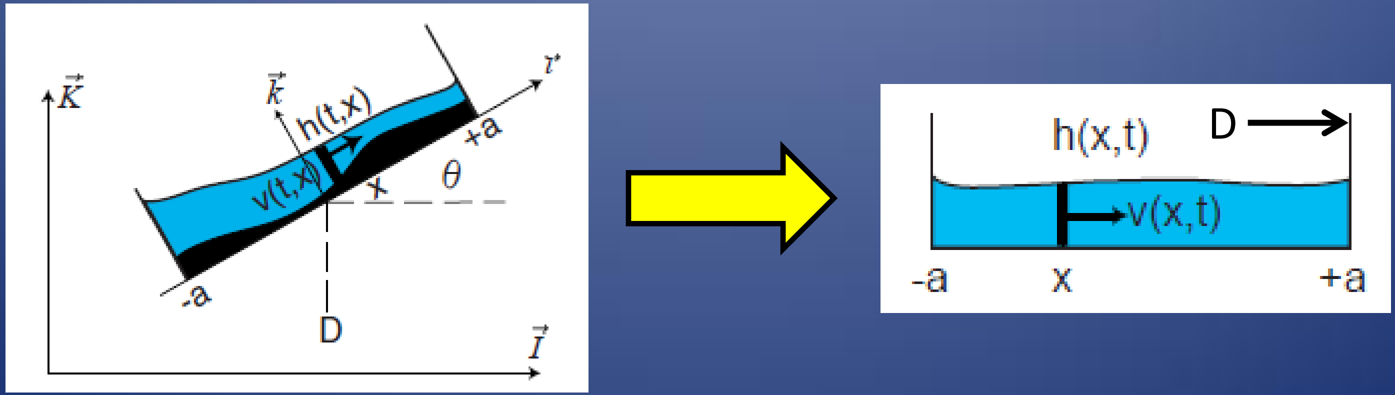
- All simulations run with 0.3 x 0.3 meter container that is half filled

Proposed Solution Methods

- Differential Flatness
  - Based on Conservation of Mass and Momentum Equations
  - Reduction to form of 1-D wave equation
- Optimal Control of Equivalent ODE Approach
  - Pendulum Approximation
  - Double Pendulum Approximation

Differential Flatness: Assumptions

- Saint Venant Equations (Shallow Water)
  - Implies vertical velocities are much smaller than horizontal velocities and are therefore negligible.
- 1-Dimensional Motion
  - Restricted to horizontal translation (non-rotational)
- Neglecting Coriolis, frictional, and viscous forces
- Flat bottom, rectangular fluid container
- Motion begins and ends at steady-state



Differential Flatness: Governing Equations

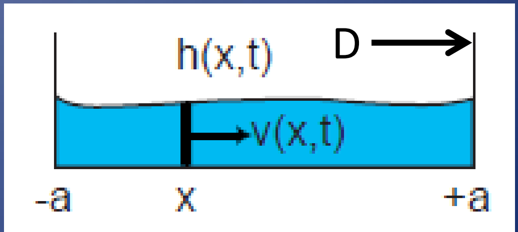
- General Results from the Conservation of Mass and Momentum Equations

$$\frac{\partial^2 H}{\partial t^2} = \bar{h} g \frac{\partial^2 H}{\partial x^2}$$

$$\frac{\partial H}{\partial x}(a,t) = \frac{\partial H}{\partial x}(-a,t) = -\frac{u}{g}$$

$$\ddot{D} = u$$

Where:
$\mathrm{H} = difference\ between\ average\ height\ and\ current\ fluid\ height$
$\bar{h} = average\ height\ of\ the\ fluid\ (constant)$
$v = velocity\ of\ the\ fluid\ at\ a\ given\ position$
$g = gravity$
$\mathrm{D} = Distance\ traveled\ by\ the\ container$
$u = feed\ forward\ input\ (acceleration)$
$t = time$
$a = distance\ from\ center\ of\ tank\ to\ edge\ of\ tank$



Differential Flatness: Results

$$D(t) = \frac{1}{2}\left(y(t+\frac{a}{c}) + y(t-\frac{a}{c})\right)$$

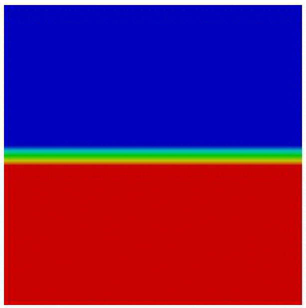
$$y(t) = \begin{cases} p & \text{if } t \leq a/c \\ \text{arbitrary} & \text{if } a/c < t < T - a/c \\ q & \text{if } t \geq T - a/c \end{cases}$$

- y(t) becomes arbitrary function between initial and final states
  - For minimal sloshing at steady state condition, must choose y(t) to have first and second derivatives equal zero at boundaries

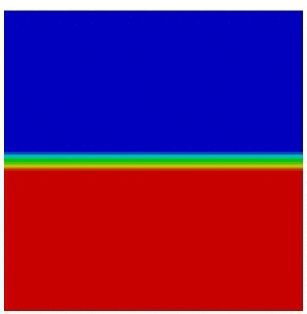
- System is steady state controllable if initial state is zero
  - Can be steered from a steady-state position to any other steady position

Simulations – Differential Flatness

- Select y(t) to be a linear function



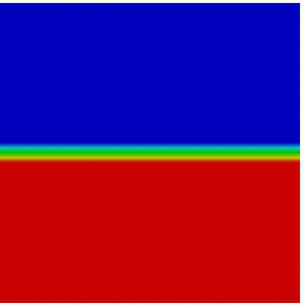
- Constant velocity model under equivalent time and distance



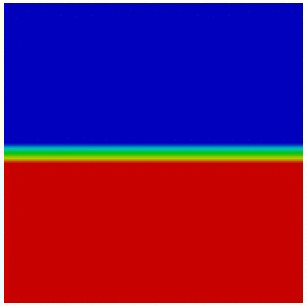
Model simulated to travel 1.5 meters over 2.0 seconds

Simulations – Differential Flatness

- Select y(t) to be a modified cosine function



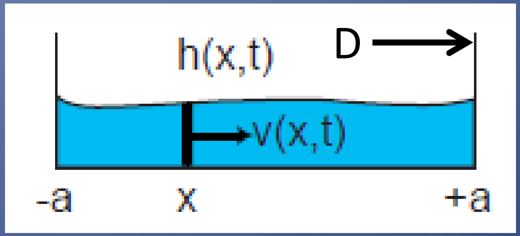
- Constant velocity model under equivalent time and distance



Model simulated to travel 1.5 meters over 2.0 seconds

Equivalent ODE

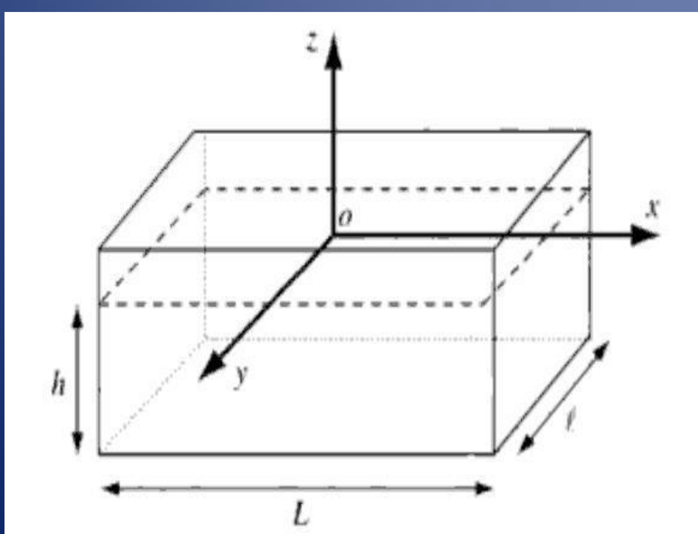
- Approach:
  - Represent the water tank as a pendulum ODE system
  - Calculating steering, D(t), with Optimal Control using the approximate ODE model
- Assumptions:
  - Inviscid fluid
  - Irrotational fluid
  - Incompressible fluid
  - No surface tension



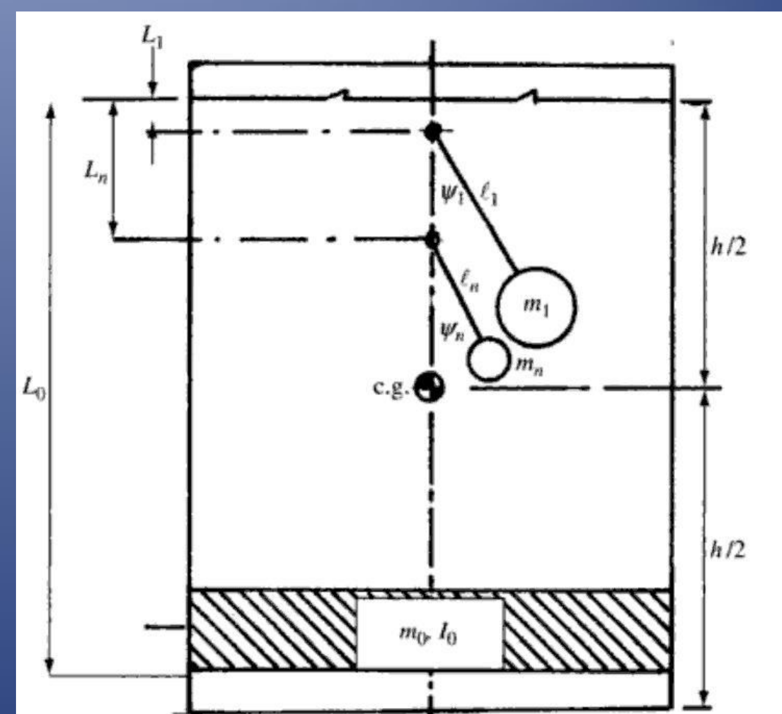
Equivalent ODE Approach: Modeling of the system

Simplification of the PDE to an ODE model consisting of N-pendulums, representing the first N eigenmodes of the water tank

$$\omega_n^2 = \frac{gn\pi}{L}\tanh\left(\frac{n\pi h}{L}\right)$$

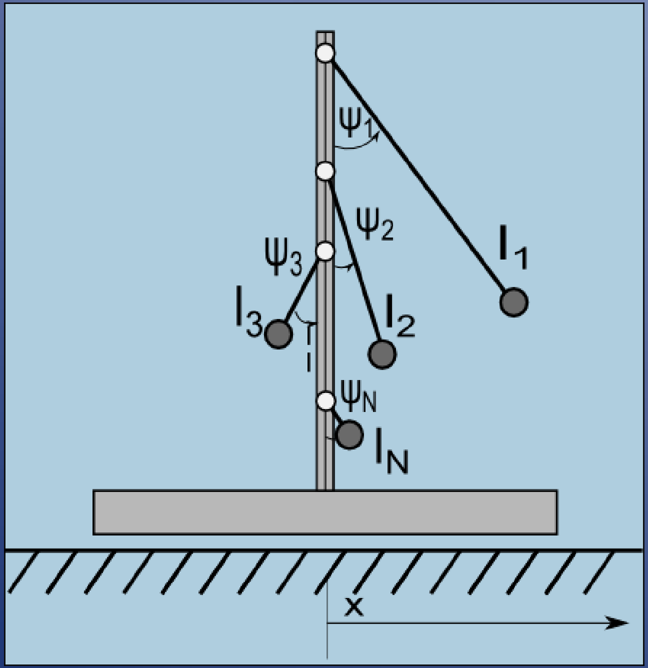


$$\ell_n = \frac{g}{\omega_n^2} = \frac{L}{n\pi}\coth\left(\frac{n\pi h}{L}\right)$$



Equivalent ODE Approach: Optimal Control

Formulating the minimum sloshing motion planning problem as an Optimal Control Problem using the equivalent pendulum model



Optimal Control Problem

$$\min_{\psi_i(t),\,u(t)} \int_0^{t_f} \left(\sum_{i=1}^{N} \omega_i \psi_i^2 + u^2 \right) dt$$

subject to:

$$\dot{\psi}_i = -\frac{1}{\ell_i}\left(u\cos(\psi_i) + g\sin(\psi_i)\right)$$

$$\dot{x} = v$$

$$\dot{v} = u$$

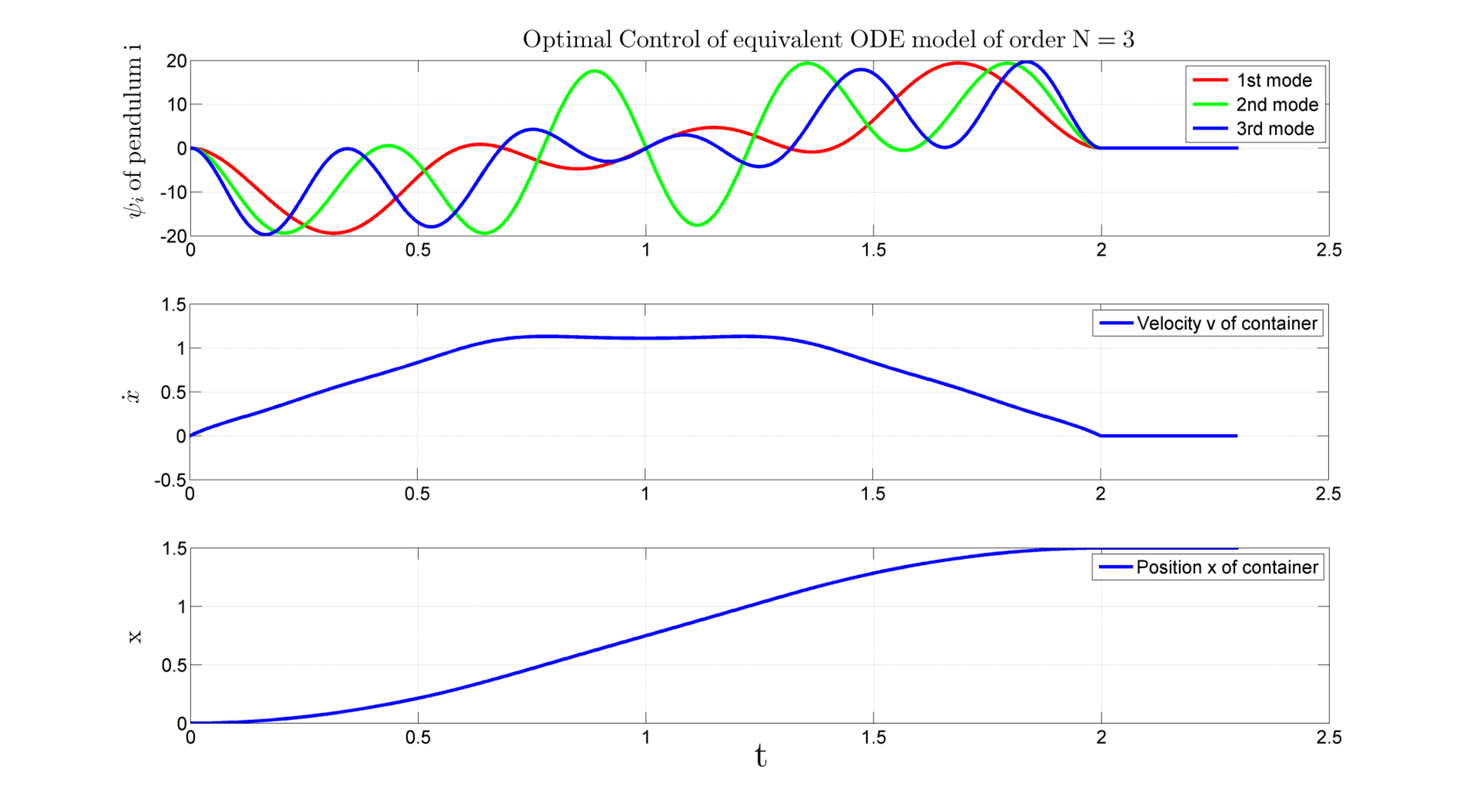
$$\psi_i(0) = v(0) = 0$$

$$\psi_i(t_f) = v(t_f) = 0$$

$$x(0) = a$$

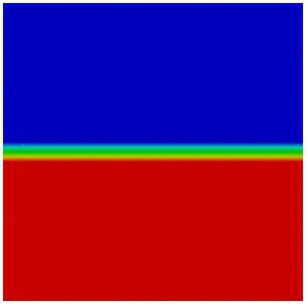
$$x(t_f) = b$$

Equivalent ODE Approach: Solution of the Optimal Control Problem

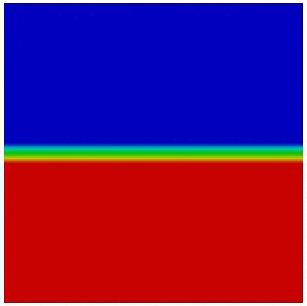


Simulations – Equivalent ODE

- Singular Pendulum Approach



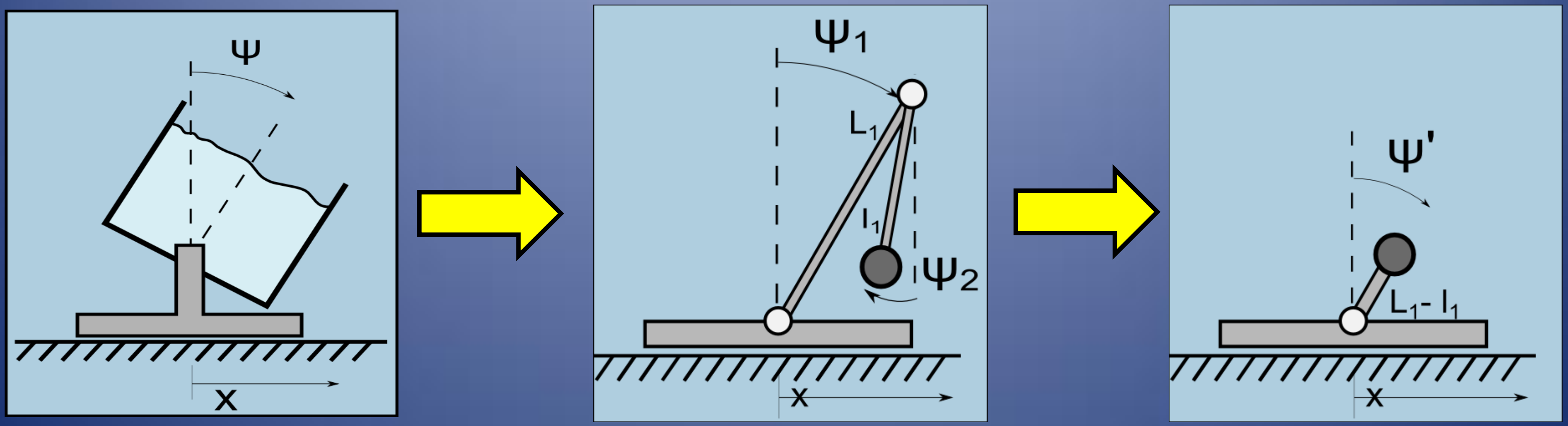
- Constant velocity model under equivalent time and distance



Model simulated to travel 1.5 meters over 2.0 seconds

Equivalent ODE Approach: Adding Tilting Motion

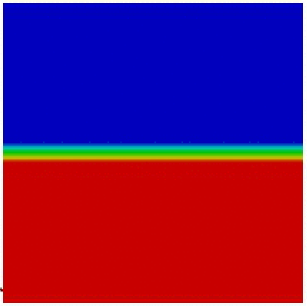
Adding a tilt-DoF enables us to model the water tank system as a double pendulum.



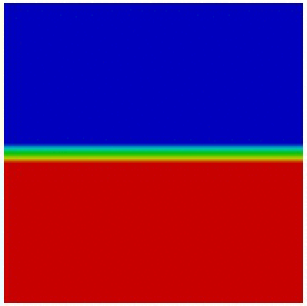
We can derive a steering strategy for φ_1 and x, by solving an Optimal Control Problem of an equivalent inverted pendulum

Simulations – Equivalent ODE

- Double Pendulum Approach with Rotation



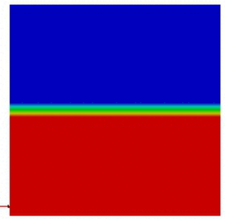
- Constant velocity model under equivalent time and distance



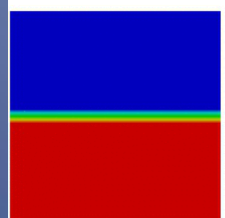
Model simulated to travel 1.5 meters over 2.0 seconds

Method Comparisons

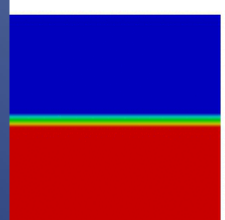
Double Pendulum



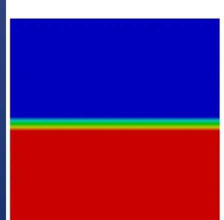
Single Pendulum



Flatness with cosine y(t)



Constant Velocity



Conclusions

- Both Differential Flatness and Optimal Equivalent ODE provide enhanced trajectories for minimizing sloshing
- Proper choice of y(t) improves trajectory
- Double Pendulum approach is optimal for reduction of fluid travel up the sides of the container due to container rotation

Future Steps

- Verification of simulation results
  - Implementation of resulting controls on linear stage system in Hesse Hall Lab