Modeling a Circular Membrane Under Pressure

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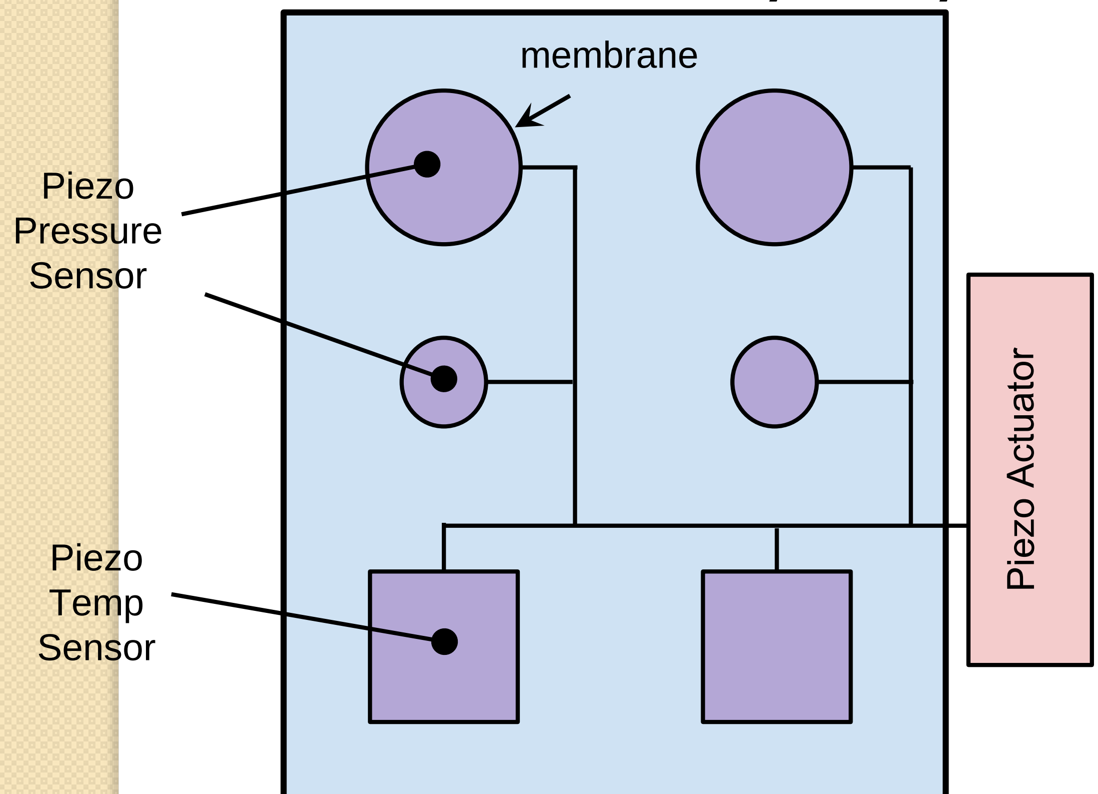
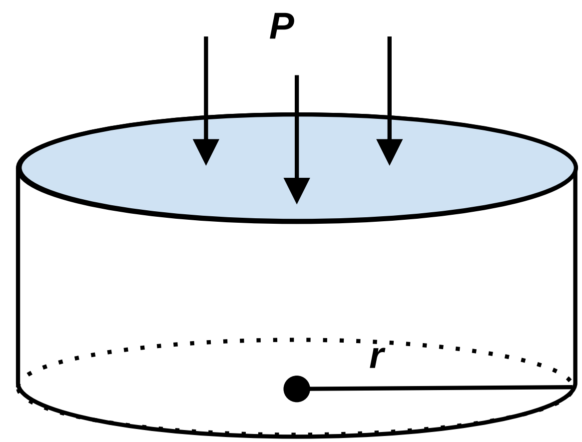
EE C291 – Spring 2013

FINAL PROJECT PRESENTATION

Application: MEMS Pressure Sensor

- Pressure increases membrane resonant freq.
- Size membranes under pressure to have same frequency range as other sensors
- **Goal**: do this analytically





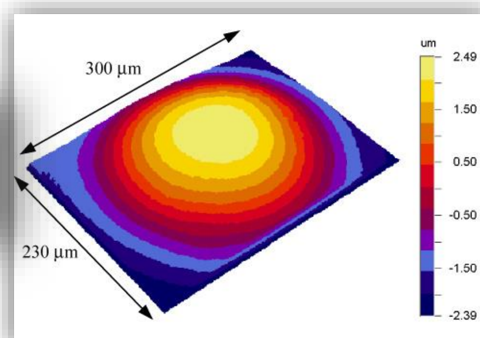
$$\omega = f(\mathrm{P}, \mathrm{r})$$

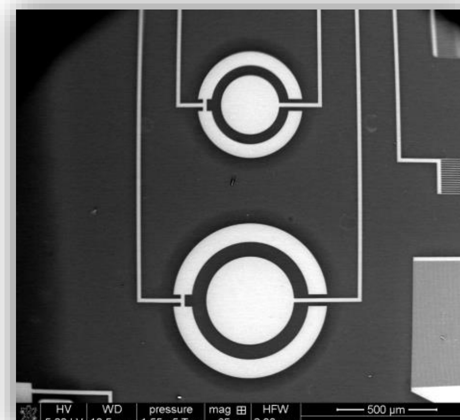
Project Approach

- Find resonant frequency of a membrane as a function of applied pressure
- Compare three methods:
  - Analytical (Hankel Transformation and SOV)
  - Numerical (Finite Element Analysis - FEA)
  - Experimental (using MEMS pressure sensor)

$$\nabla^2 u(r,t) = \frac{1}{cm^2} u_{tt}(r,t) - \frac{P(r)}{T_m}$$

$$u(r=a,t) = 0 \ (BC)$$
$$u(r,t=0) = 0 \ (IC)$$
$$u_t(r,t=0) = 0 \ (IC)$$

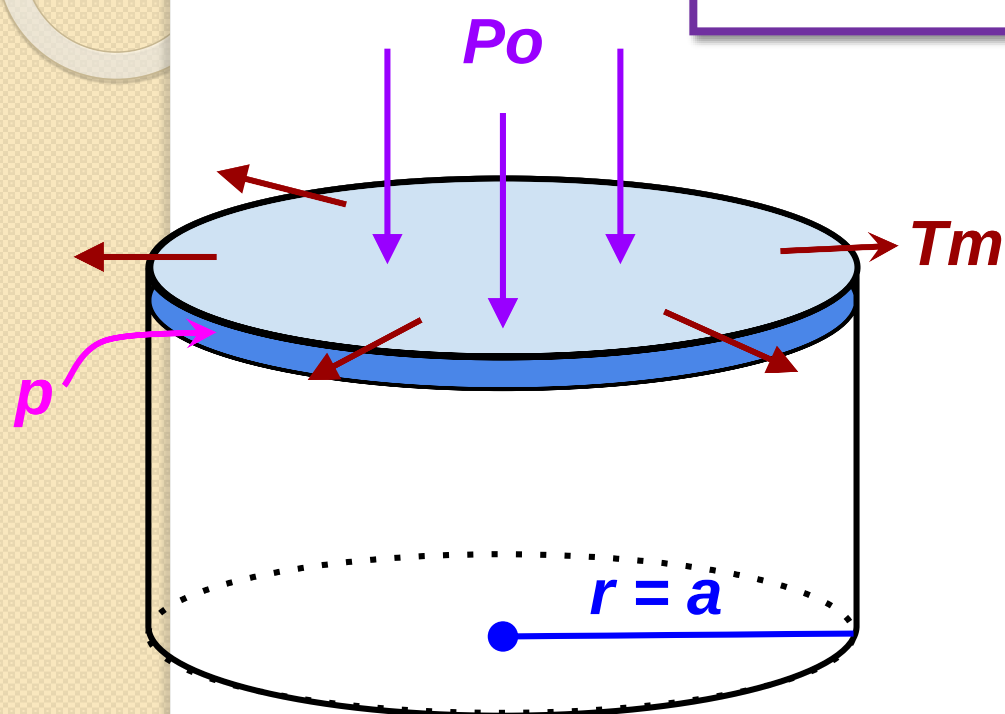




Actual MEMs Pressure Sensor on a chip

Membrane Model

$$\nabla^2 u(r,t) = \frac{1}{cm^2} u_{tt}(r,t) - \frac{P(r)}{T_m}$$



$$u(r=a,t) = 0 \ (BC)$$

$$u(r,t=0) = 0 \ (IC)$$

$$u_t(r,t=0) = 0 \ (IC)$$

$P_o =$ pressure

$T_m =$ membrane

$\rho =$ density

$r =$ radius

$u(r,t) =$ deflection

$$cm^2 = \frac{T_m}{\rho}$$

Membrane tension is calculated as a function of the pressure: $T_m = f(P_o)$

Analytical: Using Hankel Transforms

Take **Hankel** Transform
H {u(r,t)} = *U*(X ,t)

↓

Take **Laplace** Transform
L{*U*(X ,t)} = U(X ,s)

↓

Rearrange terms and use properties of **convolution**

↓

This allows you to take $\mathbf{L}^{-1}$
L^{-1}{*U*(X ,s)} = U(X ,t)

↓

Rearrange terms to take $\mathbf{H}^{-1}$
H^{-1}{*U*(X ,t)} = u(r,t)

↓

Apply **IC and BC** and simplify

$$\nabla^2 u(r,t) = \frac{1}{cm^2} u_{tt}(r,t) - \frac{P(r)}{T_m}$$

What's a Hankel Transform?

Laplace Transform **:** Hankel Transform
:: Sinusoids **:** Bessel Functions

$$u(r,t) = \frac{P_o}{T_m}\left[\frac{(a^2 - r^2)}{4} - 2a^2 \sum_{i=1,2}^{\infty} \frac{J_o(\frac{\lambda_i}{a} r)}{\lambda_i^3 J_1(\lambda_i)} cos\left(\frac{\lambda_i cmt}{a}\right)\right]$$

Analytical - Using SOV:

$$\nabla^2 u(r,t) = \frac{1}{cm^2} u_{tt}(r,t) - \frac{P(r)}{T_m}$$

Assume: $$u(r,t) = \psi(r) + \phi(r,t)$$

Particular

$$\frac{1}{r}\psi_r + \psi_{rr} = \frac{P_o}{T_m}$$

ODE - Solve!

$$\psi(r) = -\frac{P_o}{4T_m}(a^2 - r^2)$$

Homogeneous

$$\frac{1}{r}\phi_r + \phi_{rr} = \frac{1}{cm^2}\phi_{tt}$$

Solve with SOV

$$\phi(r,t) = AJ_o\left(\frac{\lambda_i}{a}\right) cos\left(\frac{cm\lambda_i}{a}t\right)$$

$$\oplus$$

$$u(r,t) = \frac{P_o}{T_m}\left[\frac{(a^2 - r^2)}{4} - 2a^2 \sum_{i=1,2}^{\infty} \frac{J_o(\frac{\lambda_i}{a}r)}{\lambda_i^3 J_1(\lambda_i)} cos\left(\frac{\lambda_i cmt}{a}\right)\right]$$

Analytic Conclusions

- Hankel and SOV produce the same expression

$$u(r,t) = \frac{P_o}{T_m}\left[\frac{(a^2 - r^2)}{4} - 2a^2 \sum_{i=1,2}^{\infty} \frac{J_o(\frac{\lambda_i}{a} r)}{\lambda_i^3 J_1(\lambda_i)} cos\left(\frac{\lambda_i cmt}{a}\right)\right]$$

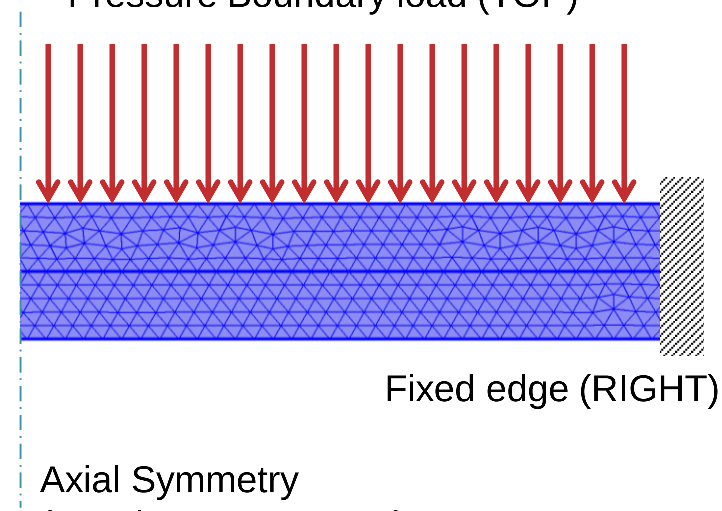
$$u(r,t,\theta) = T(t)\Theta(\theta)J_{\alpha,k}(z_{\alpha,k} r/r_o)$$

$$\omega = f(T_m(P_o))$$

Finite Element Analysis - Approach

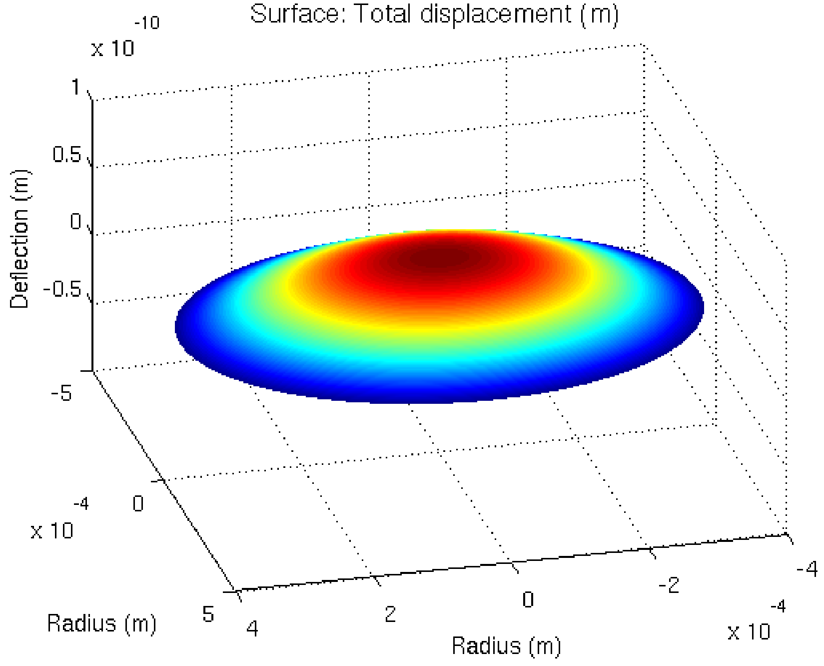
COMSOL Multiphysics was used to implement numerical solution

- Model
  - Axisymmetric 2D model
  - Nonlinear, Large deformations
- Boundary Conditions
  - Clamped edge
  - Symmetry
- Load
  - Pressure
- Analysis
  - Static
  - Prestressed Eigenfrequency

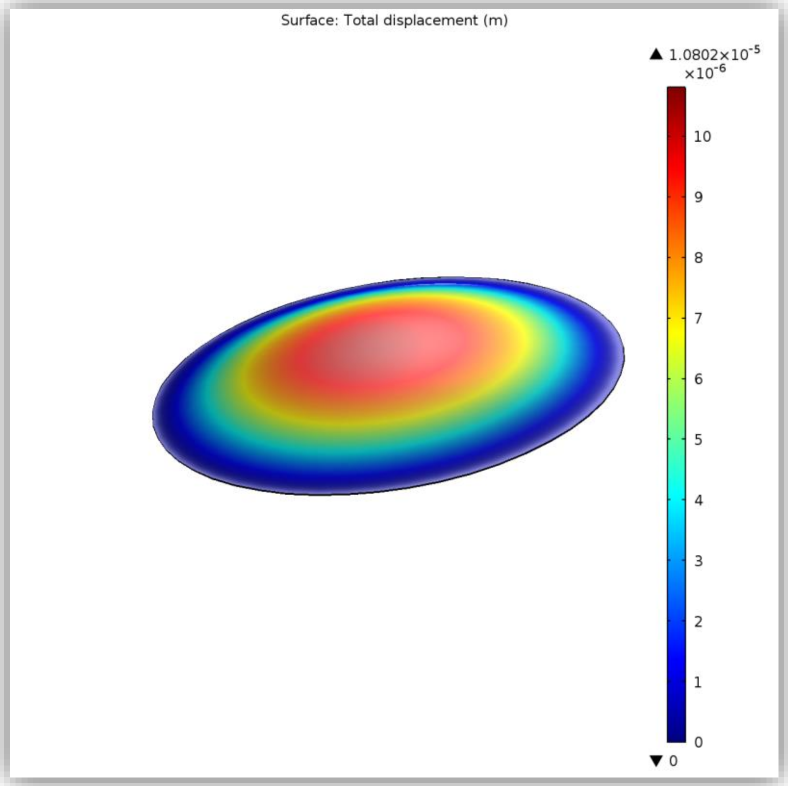


Material Property	Aluminum Nitride
Density (kg/m^3)	3260
Young's Modulus (Pa)	340×10^9
Poisson's Ratio	0.3

Results: Mode Shape

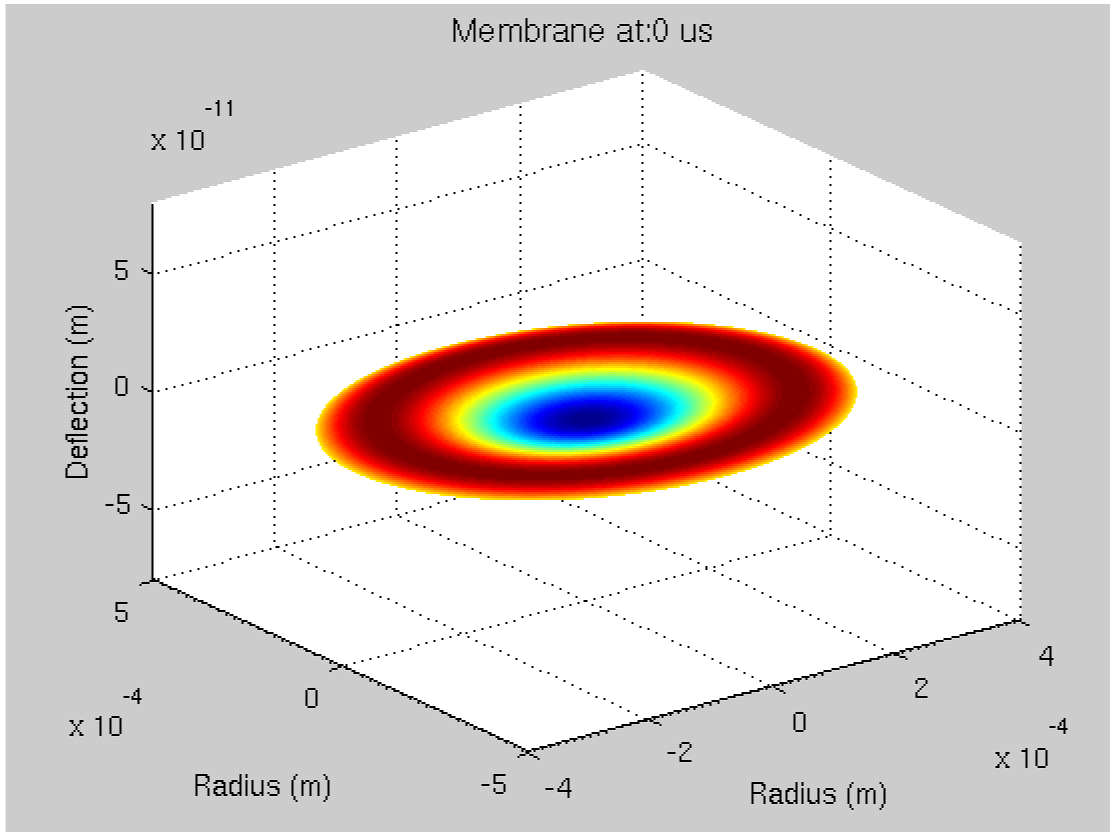


Analytical Result



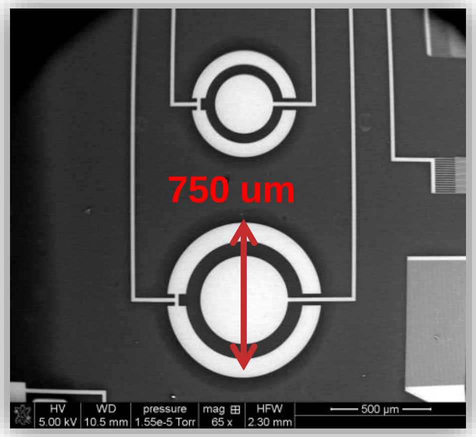
FEA Result

Results: Video

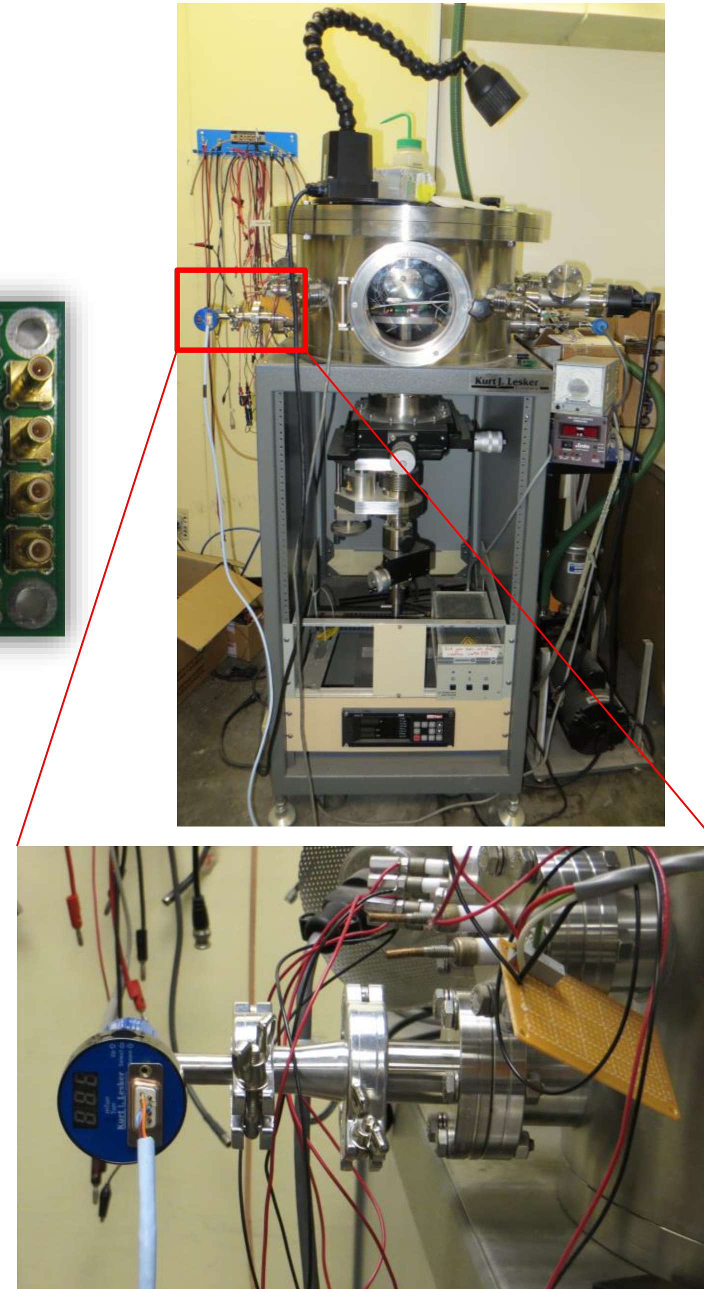


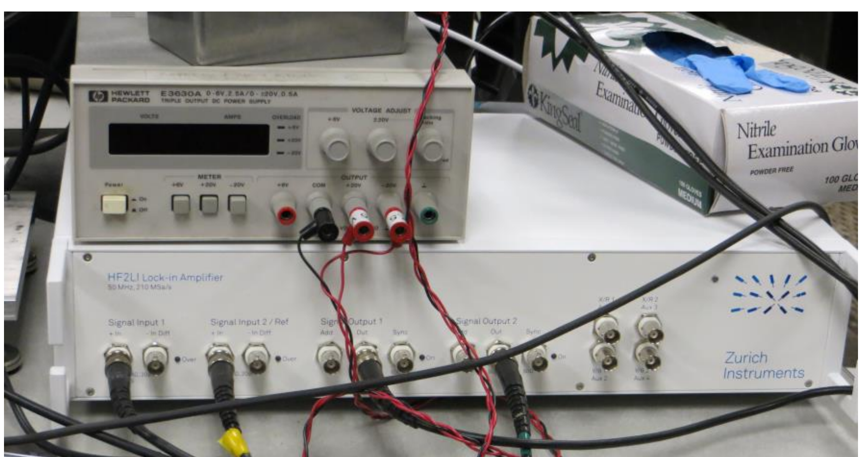
Analytical Result

Experimental Setup & MEMs Chip

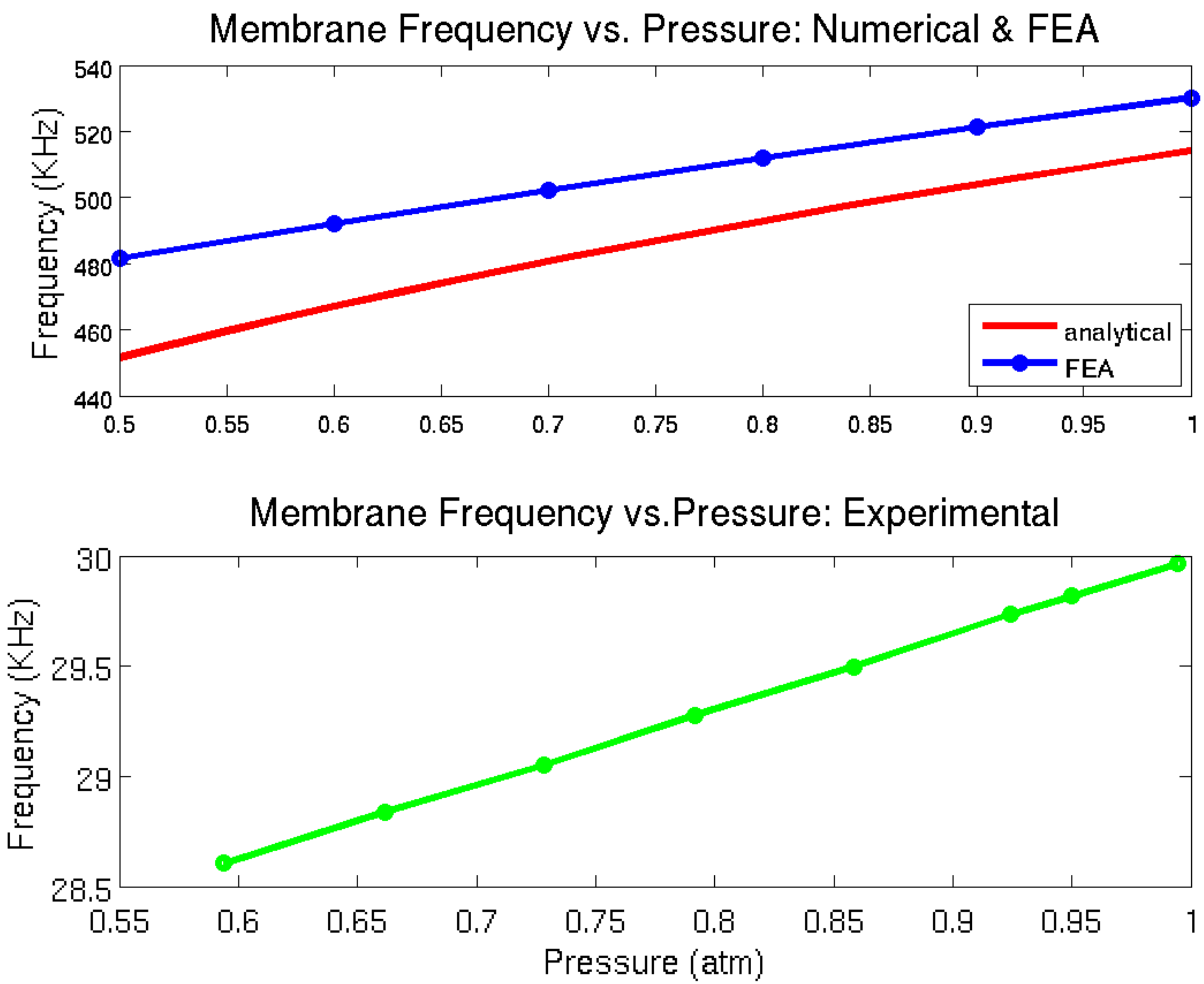


Actual MEMs Pressure Sensor on a chip



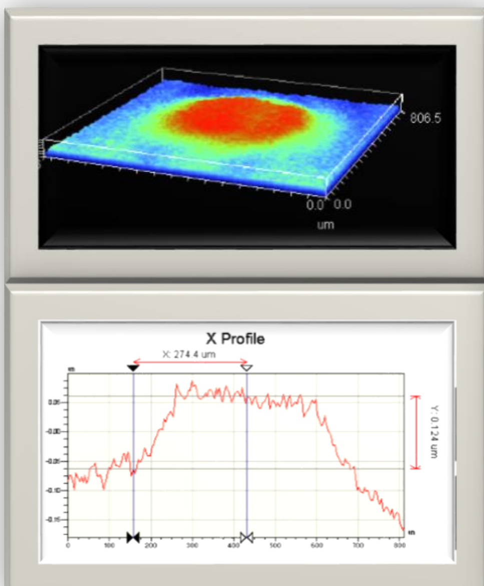


Results: Frequency vs. Pressure

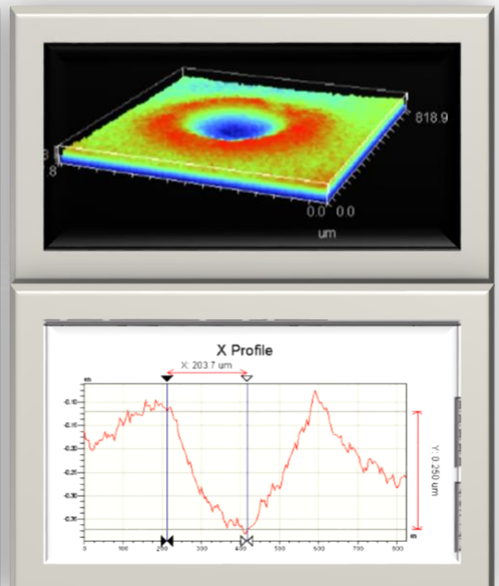


Sources of error → Future Work

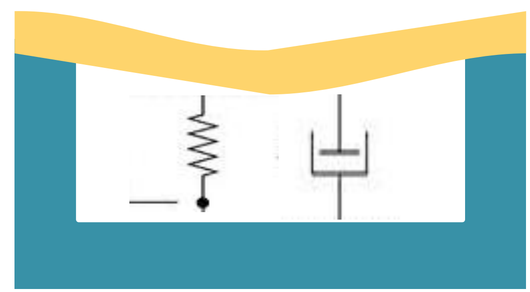
- Damping effects – squeeze film damping
- Substrate interaction
- Initial stress estimation of released film



600 um membrane defelction, NO PRESSURE



600 um membrane defelction, PRESSURE ON



Spring-Damper model for Squeeze film damping

Acknowledgements

- Fabian Goericke, GSR who helped take experimental measurements

Key References

- Selvadurai A., "Partial Differential Equations in Mechanics"
- Reynolds, "Introduction to PDEs Class Notes"

Thank You.
Any Questions?