EE291 Project Presentation

Mean Field Games

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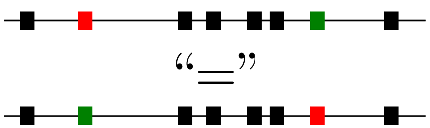
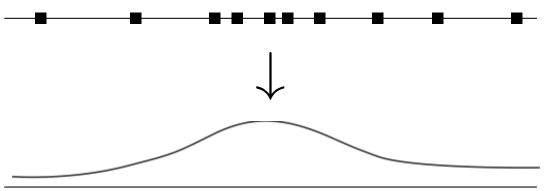
Mean Field Games – basic assumptions

- Players are rational
- Players are indistinguishable
  - identical cost functions and dynamics
  - outcome of the game is invariant under permutation of players



- Continuum of players
  - player density $m(t, x)$ over the state space $\mathcal{X}$



$\int dm_t = 1, \ \forall\, t$

- Decisions (control) based on limited Information
  - agent state $x(t)$ (local information)
  - distribution $m(t, x)$ of all other agents (aggregate global information)

J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. I – Le cas stationnaire. C. R. Acad. Sci. Paris, 2006
J.-M. Lasry and P.-L. Lions. Jeux à champ moyen. II – Horizon fini et contrôle optimal. C. R. Acad. Sci. Paris, 2006

Agent dynamics and cost functions

- Agent dynamics described by an Itō SDE

$$dX_t = f(t, X_t, \alpha)\, dt + \Sigma(t, X_t, \alpha)\, dW_t + dN_t(X_t)$$

- Cost function (finite horizon case)

$$J(x_0, \alpha) = \mathbb{E}\left[\int_0^T l(t, X_t^\alpha, \alpha_t, m_t) dt + g(X_T^\alpha, m_T)\right] \qquad X_0^\alpha = x_0$$

- Existence and uniqueness of Nash equilibria under reasonable assumptions (e.g. convexity, monotonicity…)

- A well studied special case:
  - Periodic state space $\mathcal{X} = \mathbb{T}^n$ identified to $[0,1]^n$ with periodicity
  - velocity control, i.i.d. noise $dX_t = \alpha_t\, dt + \sigma\, dW_t$
  - quadratic cost $l = \frac{1}{2}\|\alpha\|_2^2 + V_{[m]}(x)$

Optimal control problem of an agent

- Rationality: agents assume everyone will behave optimally;
  At Nash equilibrium: expected and actual evolution of $m(t,x)$ coincide

- The value function

$$v(t,x(t)) = \inf_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_t^T l(s, X_s^\alpha, \alpha_s, m_s)ds + g(X_T^{\alpha.x}, m_T)\right] \qquad X_t^\alpha = x(t)$$

satisfies the following Hamilton-Jacobi-Bellman (HJB) PDE

$$\partial_t v(t,x) + \frac{1}{2}\operatorname{div}\left(\Sigma^2\,\nabla_x v(t,x)\right) + H(t,x,\nabla_x v, m_t) = 0, \qquad v(T,x) = g_{m_T}(x)$$

where $H(t,x,p,m) := \inf_{a\in\mathcal{A}}\{\langle p, f(t,x,a)\rangle + l(t,x,a,m)\}$

- The optimal control is given by

$$\alpha^*(t,x,m) = \arg\min_{a\in\mathcal{A}}\{\langle\nabla_x v(t,x), f(t,x,a)\rangle + l(t,x,a,m)\}$$

Evolution of the player density

- Define the distribution $m : [0, T] \times \mathcal{X} \mapsto \mathbb{R}_+$ such that

$$\int \varphi \, dm_t = \int \mathbb{E}[\varphi(X_t^\alpha)] \, dm_0(x) \qquad \text{for all } C^2 \text{ functions} \quad \varphi : \mathcal{X} \mapsto \mathbb{R}$$

where $m_t(x) := m(t, x)$ for some fixed t

- Then the player density evolves according to the following Fokker-Planck (FPK) PDE:

$$\partial_t m(t, x) - \frac{1}{2} \operatorname{div} \left(\Sigma^2 m(t, x)\right) + \operatorname{div}(m(t, x) \, f(t, x, \alpha^*)) = 0, \qquad m|_{t=0} = m_0$$

The Mean Field Game equations

- Assumptions: $\Sigma(t,x,\alpha) = \mathrm{diag}(\sigma_1,\dots,\sigma_n)$, $dX_t = \alpha\, dt + \Sigma\, dW_t$
- Finite horizon problem: $J(x_0,\alpha) = \mathbb{E}\left[\int_0^T l(t,x,\alpha,m)\, dt + g_{m_T}(x_T)\right]$

$$\partial_t v + \frac{1}{2}\,\mathrm{div}\left(\Sigma^2 v\right) + H(t,x,\nabla v,m) = 0, \qquad v|_{t=T} = g_{m_T}$$

$$\partial_t m - \frac{1}{2}\,\mathrm{div}\left(\Sigma^2 m\right) + \mathrm{div}\left(m\,\partial_p H(t,x,\nabla_x v,m)\right) = 0, \qquad m|_{t=0} = m_0$$

$$m \geq 0, \quad \int_{\mathcal{X}} dm_t = 1$$

- Discounted infinite horizon problem: $J(x_0,\alpha) = \lim_{T\to\infty} \mathbb{E}\int_0^T l(x,\alpha,m)e^{-\gamma t}\, dt$

$$\frac{1}{2}\,\mathrm{div}\left(\Sigma^2 v\right) + H(x,\nabla v,m) - \gamma v = 0$$

$$-\frac{1}{2}\,\mathrm{div}\left(\Sigma^2 m\right) + \mathrm{div}\left(m\,\partial_p H(x,\nabla_x v,m)\right) = 0$$

A congestion model for traffic (finite horizon)

"road condition" congestion

- State space: $\mathcal{X} = [0, L]$
- No control constraints: $\mathcal{A} = \mathbb{R}$
- Agent cost function: $l(x, \alpha, m) = \frac{1}{2}(q(x) + m(x)^\beta)\alpha^2$
- Results in Hamiltonian $H(t, x, p, m) = -\frac{1}{2}\frac{p^2}{q(x) + m(x)^\beta}$
- Terminal cost: $g_m(x) = g(x, m(x))$ ← "distance" to destination
- Change of variables $t = t - T$ ("time to go") yields

$$\partial_t v - \frac{\sigma^2}{2}\partial_{xx} v + \frac{1}{2}\frac{\partial_x v^2}{q(x) + m^\beta} = 0, \qquad v|_{t=0} = g_{m(t=0)}$$

$$\partial_t m + \frac{\sigma^2}{2}\partial_{xx} m + \partial_x\left(\frac{m\,\partial_x v}{q(x) + m^\beta}\right) = s(t, x), \qquad m|_{t=T} = m_0$$

source term (in-/outflow of drivers)

A congestion model for traffic (infinite horizon)

"distance" to destination

- Cost functional:

$$J(x_0, \alpha) = \lim_{T \to \infty} \mathbb{E} \int_0^T \left(\frac{1}{2} (q(x) + m(x)^\beta) \alpha^2 + k(x) \right) e^{-\gamma t} \, dt$$

- Stationary mean field game equations:

$$-\frac{\sigma^2}{2} \partial_{xx} v + \frac{1}{2} \frac{(\partial_x v)^2}{q(x) + m(x)^\beta} + \gamma v = k(x)$$

source term (in-/outflow of drivers)

$$\frac{\sigma^2}{2} \partial_{xx} m + \partial_x \left(\frac{m \, \partial_x v}{q(x) + m(x)^\beta} \right) = s(x)$$

Summary and outlook

Project work:

- Studied recent work on mean field games
- Did formal derivation of the mean field games equations
- Formulated simple MFG-based traffic congestion model (dynamic & stationary)
- Started implementing a simple numerical scheme to solve the system of PDEs

Further research questions:

- Traffic model
  - Survey theoretical results on the traffic model (existence, uniqueness, regularity …)
  - Investigate advanced numerical schemes for solving the resulting system of PDEs
  - Compare simulation results with those of conventional traffic models
- General Mean Field Games
  - Application of MFG to problems in energy (e.g. power markets, demand response)
  - Extension to the case with multiple populations and major players (synchronous or Stackelberg games)