Control of 2×2 Linear Hyperbolic Systems: Backstepping-Based Trajectory Generation and PI-Based Tracking

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***Abstract*— We consider the problems of trajectory generation and tracking for general 2×2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the trajectory generation problem via backstepping. The reference input, which generates the desired output, incorporates integral operators acting on advanced and delayed versions of the reference output with kernels which were derived by Vazquez, Krstic, and Coron for the backstepping stabilization of 2×2 linear hyperbolic systems. For tracking the desired trajectory we employ a PI control law on the tracking error of the output. We prove exponential stability of the closed-loop system, under the proposed PI control law, when the parameters of the plant and the controller satisfy certain conditions, by constructing a novel "non-diagonal" Lyapunov functional.**

I. Introduction

Control of 2×2 systems of first-order hyperbolic PDEs is an active area of research since numerous processes can be modeled with this class of PDE systems. Among various applications, 2×2 systems model the dynamics of traffic [15], [17], hydraulic [2], [8], [12], [13], as well as gas pipeline networks [18], and the dynamics of transmission lines [7].

Several articles are dedicated to the control and analysis of 2×2 linear [2], [9], [12], [22] [29], [30], [31] and nonlinear [4], [5], [6], [19], [25], [26] systems. Results for the control of $n \times n$ systems also exist [10], [11], [21]. Algorithms for disturbance rejection in 2×2 systems are recently developed [1], [28]. The motion planning problem is solved in [14], [23], for a class of 2×2 systems and in [16], [24] for a class of wave PDEs. Perhaps the most relevant results to the present article are the results in [12], dealing with the Lyapunov-based output-feedback control of 2×2 linear systems, the results in [30], dealing with the backstepping stabilization of 2×2 linear systems, and the results in [23], dealing with the motion planning for a class of 2×2 systems.

In this paper, we are concerned with the trajectory generation and tracking problems for general 2×2 systems of first-order linear hyperbolic PDEs with anti-collocated boundary input and output. We solve the motion planning problem for this class of systems employing backstepping (Section II). Specifically, we start from a simple transformed system, namely, a cascade of two first-order hyperbolic PDEs coupled only at the boundary, for which the motion planning problem can be trivially solved. We then apply an inverse backstepping transformation to derive the reference trajectory and reference input for the original system. The reference output is assumed to be continuously differentiable and uniformly bounded. Our approach is different than the one in [30], in that we use backstepping for trajectory generation rather than stabilization, and the one in [23], in that we employ a different conceptual idea to a different class of systems. Although the idea of the backstepping-based trajectory generation for PDEs was conceived in [20], and applied to a beam PDE [27] and the Navier-Stokes equations [3], this approach has neither been systematized nor been applied to the class of systems considered in the present article.

We then employ a PI control law for the stabilization of the error system, namely, the system whose state is defined as the difference between the state of the plant and the reference trajectory. We prove exponential stability in the L_2 norm of the closed-loop system by constructing a Lyapunov functional which incorporates cross-terms between the PDE states of the system and the ODE state of the controller, when the parameters of the system and the controller satisfy certain conditions (Section III). Our result differs than the result in [12] in that we employ PI control on an output of the system in the Riemann coordinates and we construct a non-diagonal Lyapunov functional for proving closed-loop stability.

II. Trajectory Generation for 2×2 Linear Hyperbolic Systems Using Backstepping

We consider the following system

$$\begin{aligned} z_t^1 + \varepsilon_1(x) z_x^1 &= c_1(x) z^1 + c_2(x) z^2 && (1)\\ z_t^2 - \varepsilon_2(x) z_x^2 &= c_3(x) z^1 + c_4(x) z^2 \,, && (2) \end{aligned}$$

under the boundary conditions

$$\begin{aligned} z^1(0,t) &= q z^2(0,t) && (3)\\ z^2(1,t) &= S(t) && (4)\\ z^2(0,t) &= y(t) \,, && (5) \end{aligned}$$

where $t \in [0,+\infty)$ is the time variable, $x \in [0,1]$ is the spatial variable, y is the output of the system, $q \neq 0$ is a constant parameter, and V is the control input. The functions ε_1, ε_2 belong to $C^2([0,1])$ and satisfy $\varepsilon_1(x), \varepsilon_2(x) > 0$, for all $x \in [0,1]$, and the functions c_i, $i = 1,2,3,4$ belong to $C^1([0,1])$. Defining the change of variables (see, for

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example, [2])

$$\begin{aligned} \chi_1(x) &= \exp\left(-\int_0^x \frac{c_1(s)}{\varepsilon_1(s)} ds\right) && (6)\\ \chi_2(x) &= \exp\left(\int_0^x \frac{c_4(s)}{\varepsilon_2(s)} ds\right) && (7)\\ \chi(x) &= \frac{\chi_1(x)}{\chi_2(x)}\,, && (8) \end{aligned}$$

and the new coordinates

$$\begin{aligned} u &= \chi_1(x) z^1 && (9)\\ v &= \chi_2(x) z^2\,, && (10) \end{aligned}$$

system (1)–(5) is transformed into the following system

$$\begin{aligned} u_t + \varepsilon_1(x) u_x &= \gamma_1(x) v && (11)\\ v_t - \varepsilon_2(x) v_x &= \gamma_2(x) u\,, && (12) \end{aligned}$$

with

$$\begin{aligned} \gamma_1(x) &= \chi(x) c_2(x) && (13)\\ \gamma_2(x) &= \chi^{-1}(x) c_3(x)\,. && (14) \end{aligned}$$

The boundary conditions become

$$\begin{aligned} u(0,t) &= q v(0,t) && (15)\\ v(1,t) &= U(t) && (16)\\ v(0,t) &= y(t)\,, && (17) \end{aligned}$$

where the original control variable satisfies

$$U = \chi_2(1) V\,. \tag{18}$$

Theorem 1: Let $y^r \in C^1(\mathbb{R})$ be uniformly bounded. The functions

$$\begin{aligned} u^r(x,t) &= q y^r(t - \Phi_1(x)) \\ &\quad + q\int_0^x L^{\alpha\alpha}(x,\xi) y^r(t-\Phi_1(\xi)) d\xi \\ &\quad + \int_0^x L^{\alpha\beta}(x,\xi) y^r(t+\Phi_2(\xi)) d\xi && (19)\\ v^r(x,t) &= y^r(t+\Phi_2(x)) \\ &\quad + q\int_0^x L^{\beta\alpha}(x,\xi) y^r(t-\Phi_1(\xi)) d\xi \\ &\quad + \int_0^x L^{\beta\beta}(x,\xi) y^r(t+\Phi_2(\xi)) d\xi && (20)\\ U^r(t) &= y^r(t+\Phi_2(1)) \\ &\quad + q\int_0^1 L^{\beta\alpha}(1,\xi) y^r(t-\Phi_1(\xi)) d\xi \\ &\quad + \int_0^1 L^{\beta\beta}(1,\xi) y^r(t+\Phi_2(\xi)) d\xi\,, && (21) \end{aligned}$$

where

$$\begin{aligned} \Phi_1(x) &= \int_0^x \frac{1}{\varepsilon_1(\xi)} d\xi && (22)\\ \Phi_2(x) &= \int_0^x \frac{1}{\varepsilon_2(\xi)} d\xi\,, && (23) \end{aligned}$$

and $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta}$ are the solutions of the following equations

$$\begin{aligned} \varepsilon_2(x) L_x^{\beta\alpha} - \varepsilon_1(\xi) L_\xi^{\beta\alpha} &= \varepsilon_1'(\xi) L^{\beta\alpha} - \gamma_2(x) L^{\alpha\alpha} && (24)\\ \varepsilon_2(x) L_x^{\beta\beta} + \varepsilon_2(\xi) L_\xi^{\beta\beta} &= -\varepsilon_2'(\xi) L^{\beta\beta} - \gamma_2(x) L^{\alpha\beta} && (25)\\ \varepsilon_1(x) L_x^{\alpha\alpha} + \varepsilon_1(\xi) L_\xi^{\alpha\alpha} &= -\varepsilon_1'(\xi) L^{\alpha\alpha} + \gamma_1(x) L^{\beta\alpha} && (26)\\ \varepsilon_1(x) L_x^{\alpha\beta} - \varepsilon_2(\xi) L_\xi^{\alpha\beta} &= \varepsilon_2'(\xi) L^{\alpha\beta} + \gamma_1(x) L^{\beta\beta}\,, && (27) \end{aligned}$$

with the boundary conditions

$$\begin{aligned} L^{\beta\alpha}(x,x) &= -\frac{\gamma_2(x)}{\varepsilon_1(x)+\varepsilon_2(x)} && (28)\\ L^{\beta\alpha}(x,0) &= \frac{\varepsilon_2(0)}{q\varepsilon_1(0)} L^{\beta\beta}(x,0) && (29)\\ L^{\alpha\beta}(x,0) &= \frac{q\varepsilon_1(0)}{\varepsilon_2(0)} L^{\alpha\alpha}(x,0) && (30)\\ L^{\alpha\beta}(x,x) &= \frac{\gamma_1(x)}{\varepsilon_1(x)+\varepsilon_2(x)}\,, && (31) \end{aligned}$$

are uniformly bounded and solve the boundary value problem (11), (12), (15), (16). In particular, $v^r(0,t) = y^r(t)$.

Proof: First note that since ε_1, $\varepsilon_2 \in C^2([0,1])$ with $\varepsilon_1(x)$, $\varepsilon_2(x) > 0$, for all $x \in [0,1]$ and γ_1, $\gamma_2 \in C^1([0,1])$, system (24)–(31) has a unique solution with $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, $L^{\beta\beta} \in C^1(\mathcal{T})$ where $\mathcal{T} = \{(x,\xi) : 0 \le \xi \le x \le 1\}$ [6]. Hence, from (19)–(21) and the uniform boundness of y^r it follows that u^r, v^r, and U^r are bounded for all $t \ge 0$ and $x \in [0,1]$. Taking the time and space derivatives of u^r we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = q&\int_0^x L^{\alpha\alpha}(x,\xi) {y^r}'(t-\Phi_1(\xi)) d\xi \\ &+ \int_0^x L^{\alpha\beta}(x,\xi) {y^r}'(t+\Phi_2(\xi)) d\xi \\ &+ \varepsilon_1(x) \int_0^x L_x^{\alpha\beta}(x,\xi) y^r\left(t+\Phi_2(\xi)\right) d\xi \\ &+ q\varepsilon_1(x) \int_0^x L_x^{\alpha\alpha}(x,\xi) y^r\left(t-\Phi_1(\xi)\right) d\xi \\ &+ \varepsilon_1(x) L^{\alpha\beta}(x,x) y^r\left(t+\Phi_2(x)\right) \\ &+ q\varepsilon_1(x) L^{\alpha\alpha}(x,x) y^r\left(t-\Phi_1(x)\right)\,. \qquad (32) \end{aligned}$$

Integrating by parts the first two integrals we get

$$\begin{aligned} u_t^r + \varepsilon_1(x) u_x^r = q&\int_0^x \big(\varepsilon_1(x) L_x^{\alpha\alpha}(x,\xi) + \varepsilon_1(\xi) L_\xi^{\alpha\alpha}(x,\xi) \\ &+ \varepsilon_1'(\xi) L^{\alpha\alpha}(x,\xi)\big) y^r(t-\Phi_1(\xi)) d\xi \\ &+ \int_0^x \Big(\varepsilon_1(x) L_x^{\alpha\beta}(x,\xi) - \varepsilon_2(\xi) L_\xi^{\alpha\beta}(x,\xi) \\ &- \varepsilon_2'(\xi) L^{\alpha\beta}(x,\xi)\Big) y^r(t+\Phi_2(\xi)) d\xi \\ &+ \big(q\varepsilon_1(0) L^{\alpha\alpha}(x,0) \\ &- \varepsilon_2(0) L^{\alpha\beta}(x,0)\big) y^r(t) \\ &+ \big(\varepsilon_1(x) L^{\alpha\beta}(x,x) \\ &+ \varepsilon_2(x) L^{\alpha\beta}(x,x)\big) y^r(t+\Phi_2(x))\,. \qquad (33) \end{aligned}$$

Due to the fact that $L^{\alpha\beta}$ and $L^{\alpha\alpha}$ are the solutions of (26) and (27) with the boundary conditions (30) and (31) one gets, by using (20), that u^r satisfies (11). The proof that v^r satisfies (12) follows analogously. Setting $x=0$ in (19), (20) and using (22), (23), we get that u^r and v^r satisfy (15). Setting $x=1$ in (20) it follows that (21) satisfies (16). Setting in (20) $x=0$ and using (23) we get $v^r(0,t)=y^r(t)$. ■

Remark 1: The approach for the trajectory generation introduced here is inspired from backstepping. Consider the following system

$$\alpha_t + \varepsilon_1(x)\alpha_x = 0 \tag{34}$$

$$\beta_t - \varepsilon_2(x)\beta_x = 0\,, \tag{35}$$

with boundary conditions

$$\alpha(0,t) = q\beta(0,t) \tag{36}$$

$$\beta(1,t) = y^r\left(t+\Phi_2(1)\right)\,. \tag{37}$$

It is shown that the functions

$$\alpha(x,t) = qy^r(t-\Phi_1(x)) \tag{38}$$

$$\beta(x,t) = y^r(t+\Phi_2(x))\,, \tag{39}$$

where Φ_1 and Φ_2 are defined in (22) and (23) respectively, satisfy (34)–(37) and, in particular, $\beta(0,t)=y^r(t)$. Using the inverse backstepping transformation introduced in [30]

$$\begin{aligned} u^r(x,t) &= \alpha(t,x)+\int_0^x L^{\alpha\alpha}(x,\xi)\alpha(\xi,t)d\xi \\ &\quad +\int_0^x L^{\alpha\beta}(x,\xi)\beta(\xi,t)d\xi \end{aligned} \tag{40}$$

$$\begin{aligned} v^r(x,t) &= \beta(t,x)+\int_0^x L^{\beta\alpha}(x,\xi)\alpha(\xi,t)d\xi \\ &\quad +\int_0^x L^{\beta\beta}(x,\xi)\beta(\xi,t)d\xi\,, \end{aligned} \tag{41}$$

relations (38), (39) and the fact that the functions $L^{\alpha\alpha}$, $L^{\alpha\beta}$, $L^{\beta\alpha}$, and $L^{\beta\beta}$ satisfy (24)–(31), one can conclude that the functions u^r, v^r, and $U^r=v^r(1)$ solve the trajectory generation problem for system (11), (12), (15)–(17).

Example 1: We consider the following system

$$z^1_t+\varepsilon_1 z^1_x = -\frac{1}{\tau}z^1 \tag{42}$$

$$z^2_t-\varepsilon_2 z^2_x = -\frac{1}{\tau}z^1\,, \tag{43}$$

with boundary conditions

$$z^1(0,t) = qz^2(0,t) \tag{44}$$

$$z^2(1,t) = V(t)\,, \tag{45}$$

where τ is a positive parameter. Among various systems that can be modeled by (42)–(45) (for instance, the Saint-Venant equations, see [12], [8]), system (42)–(45) can be viewed as a linearized version of the Aw-Rascle-Zhang (ARZ) macroscopic model of traffic flow in the Riemann coordinates

$$z^1 = w-V'(s^*)s \tag{46}$$

$$z^2 = w\,, \tag{47}$$

where w and s correspond to the velocity and density of the vehicles at time t and location x, respectively. The variable $V(s^*)$ is the nominal velocity of the cars and s^* is the nominal density. The opposite transport velocities in (42), (43) correspond to traffic flow in a congested mode. The parameter $\frac{1}{\tau}$ is an indicator of the convergence rate of the velocity w of the cars to the nominal velocity $V(s)$. For more details the reader is referred to [15]. The boundary condition (44) in the original variables is written as

$$w=\frac{V'(s^*)s}{1-q}\,. \tag{48}$$

Hence, the boundary condition (44) dictates that there is a static relation, at the entrance of the road, between the density and the velocity similarly to the static relation between the nominal velocity $V(s)$ and the density of the cars in the road. The change of variables (9), (10), (13), and (14) transform system (42)–(45) to

$$u_t+\varepsilon_1 u_x = 0 \tag{49}$$

$$v_t-\varepsilon_2 v_x = -\frac{1}{\tau}\exp\left(-\frac{1}{\tau\varepsilon_1}x\right)u \tag{50}$$

$$u(0,t) = qv(0,t) \tag{51}$$

$$v(1,t) = U(t)\,, \tag{52}$$

where $U(t)$ is given by (18). Observing that $\gamma_1=0$, relations (24)–(31) can be solved explicitly as

$$L^{\alpha\alpha}(x,\xi) = 0 \tag{53}$$

$$L^{\alpha\beta}(x,\xi) = 0 \tag{54}$$

$$L^{\beta\alpha}(x,\xi) = \frac{\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1x+\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)\right)}{\tau\left(\varepsilon_1+\varepsilon_2\right)} \tag{55}$$

$$L^{\beta\beta}(x,\xi) = \frac{q\varepsilon_1\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1x-\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)\right)}{\tau\varepsilon_2\left(\varepsilon_1+\varepsilon_2\right)}\,. \tag{56}$$

Therefore, for system (42)–(45), the reference input which generates the desired output $y^r(t)$ is

$$\begin{aligned} V^r(t) &= y^r\left(t+\frac{1}{\varepsilon_2}\right)+\frac{q}{\tau\left(\varepsilon_1+\varepsilon_2\right)} \\ &\quad\times\int_0^1\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1+\varepsilon_2\xi}{\varepsilon_1+\varepsilon_2}\right)\right)y^r\left(t-\frac{\xi}{\varepsilon_1}\right)d\xi \\ &\quad+\frac{q\varepsilon_1}{\tau\varepsilon_2\left(\varepsilon_1+\varepsilon_2\right)}\int_0^1\exp\left(-\frac{1}{\tau\varepsilon_1}\left(\frac{\varepsilon_1-\varepsilon_1\xi}{\varepsilon_1+\varepsilon_2}\right)\right) \\ &\quad\times y^r\left(t+\frac{\xi}{\varepsilon_2}\right)d\xi\,. \end{aligned} \tag{57}$$

III. Trajectory Tracking Using PI Control

For stabilizing the system around the desired trajectory for any initial condition $(u(x,0),v(x,0))$, rather than only for $(u(x,0),v(x,0))=(u^r(x,0),v^r(x,0))$, we employ a PI-feedback control law. We first write the dynamics of the tracking errors $\tilde{u}(x,t)=u(x,t)-u^r(x,t)$ and $\tilde{v}(x,t)=v(x,t)-v^r(x,t)$ as

$$M_1 = \begin{bmatrix} -q^2-\beta\left(k_P^2e^{\mu}-1\right)-\frac{\kappa\gamma}{2} & -\beta k_Pk_Ie^{\mu}+\frac{\gamma}{2}\left(e^{\nu}k_P+1\right)-\frac{\rho}{2} \\ -\beta k_Pk_Ie^{\mu}+\frac{\gamma}{2}\left(e^{\nu}k_P+1\right)-\frac{\rho}{2} & -\beta k_I^2e^{\mu}+\gamma e^{\nu}k_I-\frac{\gamma}{2} \end{bmatrix} \tag{58}$$

$$M_2(x) = \begin{bmatrix} \left(\mu-\frac{\theta}{\varepsilon_1(x)}\right)e^{-\mu x}+\frac{\gamma^2}{2(\theta\rho-\gamma)}\frac{\gamma_2^2(x)}{\varepsilon_2^2(x)}e^{2\nu x} & -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}-\frac{\gamma^2}{2(\theta\rho-\gamma)}\frac{\gamma_2(x)}{\varepsilon_2(x)}\left(\nu-\frac{\theta}{\varepsilon_2(x)}\right)e^{2\nu x} \\ -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}-\frac{\gamma^2}{2(\theta\rho-\gamma)}\frac{\gamma_2(x)}{\varepsilon_2(x)}\left(\nu-\frac{\theta}{\varepsilon_2(x)}\right)e^{2\nu x} & \beta\left(\mu-\frac{\theta}{\varepsilon_2(x)}\right)e^{\mu x}-\frac{\gamma}{2\kappa}\frac{e^{2\nu x}}{\varepsilon_2^2(x)}+\frac{\gamma^2}{2(\theta\rho-\gamma)}\left(\nu-\frac{\theta}{\varepsilon_2(x)}\right)^2e^{2\nu x} \end{bmatrix} \tag{59}$$

$$\tilde{u}_t+\varepsilon_1(x)\tilde{u}_x = \gamma_1(x)\tilde{v} \tag{60}$$
$$\tilde{v}_t-\varepsilon_2(x)\tilde{v}_x = \gamma_2(x)\tilde{u} \tag{61}$$
$$\tilde{u}(0,t) = q\tilde{v}(0,t) \tag{62}$$
$$\tilde{v}(1,t) = \tilde{U}(t)\,, \tag{63}$$

where $\tilde{U}=U-U^r$ and U^r is the reference input generating the desired reference trajectory. We employ the controller

$$\tilde{U}(t)=-k_P\tilde{v}(0,t)-k_I\tilde{\eta}(t)\,, \tag{64}$$

with

$$\dot{\tilde{\eta}}(t)=\tilde{v}(0,t)\,. \tag{65}$$

Theorem 2: Consider system (60)–(63) together with the control law (64), (65). Let the positive constants μ, β, ρ, γ, ν, κ, and θ be such that the matrices (58), (59), shown at the top of the next page, are positive semi-definite for all $x\in[0,1]$, and the inequalities

$$\beta\rho > \frac{\gamma^2e^{(2\nu-\mu)x}}{2\varepsilon_2(x)}\,,\quad \forall x\in[0,1] \tag{66}$$
$$\gamma > \theta\rho\,, \tag{67}$$

hold. Then, there exist positive constants λ and κ such that the following holds for all $t\geq 0$

$$\Omega(t)\leq\kappa e^{-\lambda t}\Omega(0)\,, \tag{68}$$

where

$$\Omega(t)=\int_0^1\left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t)\,. \tag{69}$$

Proof: In order to analyze the stability of system (60)–(65) we propose the following Lyapunov functional

$$\begin{aligned} V(t) &= \int_0^1\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}^{\top}P(x)\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}dx \\ &= R_1(t)+R_2(t)+R_3(t)+R_4(t)\,, \end{aligned} \tag{70}$$

with

$$P(x)=\begin{bmatrix}\frac{e^{-\mu x}}{\varepsilon_1(x)} & 0 & 0\\ 0 & \beta\frac{e^{\mu x}}{\varepsilon_2(x)} & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)}\\ 0 & \frac{\gamma e^{\nu x}}{2\varepsilon_2(x)} & \frac{\rho}{2}\end{bmatrix}, \tag{71}$$

and

$$R_1(t) = \int_0^1\tilde{u}^2(x,t)\frac{e^{-\mu x}}{\varepsilon_1(x)}dx \tag{72}$$
$$R_2(t) = \beta\int_0^1\tilde{v}^2(x,t)\frac{e^{\mu x}}{\varepsilon_2(x)}dx \tag{73}$$
$$R_3(t) = \gamma\tilde{\eta}(t)\int_0^1\tilde{v}(x,t)\frac{e^{\nu x}}{\varepsilon_2(x)}dx \tag{74}$$
$$R_4(t) = \frac{\rho}{2}\tilde{\eta}^2(t)\,. \tag{75}$$

Let us introduce the constants

$$\underline{\lambda} = \min_{x\in[0,1]}\lambda_{\min}(P(x)) \tag{76}$$
$$\overline{\lambda} = \max_{x\in[0,1]}\lambda_{\max}(P(x))\,. \tag{77}$$

Inequality (66) ensures that $P(x)$ is positive definite and symmetric for all $x\in[0,1]$, and hence, using the fact that $\varepsilon_1,\,\varepsilon_2\in C^2\left([0,1]\right)$ with $\varepsilon_1(x)$, $\varepsilon_2(x)>0$, for all $x\in[0,1]$, one can conclude that, $\overline{\lambda}$, $\underline{\lambda}>0$. Therefore,

$$\underline{\lambda}\left(\int_0^1\left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t)\right)\leq V(t)\,, \tag{78}$$

and

$$V(t)\leq\overline{\lambda}\left(\int_0^1\left(\tilde{u}^2(x,t)+\tilde{v}^2(x,t)\right)dx+\tilde{\eta}^2(t)\right)\,. \tag{79}$$

Using (72)–(75) we get along the solutions of system (60)–(65) that

$$\begin{aligned} \dot{R}_1(t) &= -2\int_0^1\tilde{u}(x,t)\tilde{u}_x(x,t)e^{-\mu x}dx \\ &\quad +2\int_0^1\tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx \\ &= \left(q^2\tilde{v}^2(0,t)-e^{-\mu}\tilde{u}^2(1,t)\right)-\mu\int_0^1\tilde{u}^2(x,t)e^{-\mu x}dx \\ &\quad +2\int_0^1\tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}dx \end{aligned} \tag{80}$$

$$\begin{aligned} \dot{R}_2(t) &= 2\beta\int_0^1\tilde{v}(x,t)\tilde{v}_x(x,t)e^{\mu x}dx \\ &\quad +2\beta\int_0^1\tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx \\ &= \beta\left(k_P^2e^{\mu}\tilde{v}^2(0,t)+2k_Pk_Ie^{\mu}\tilde{v}(0,t)\tilde{\eta}(t)\right. \\ &\quad \left.+k_I^2e^{\mu}\tilde{\eta}^2(t)-\tilde{v}^2(0,t)\right)-\mu\beta\int_0^1\tilde{v}^2(x,t)e^{\mu x}dx \\ &\quad +2\beta\int_0^1\tilde{u}(x,t)\tilde{v}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x}dx \end{aligned} \tag{81}$$

$$\begin{aligned} \dot{R}_3(t) &= \gamma\tilde{\eta}(t)\int_0^1\tilde{v}_x(x,t)e^{\nu x}dx \\ &\quad +\gamma\tilde{v}(0,t)\int_0^1\tilde{v}(x,t)\frac{e^{\nu x}}{\varepsilon_2(x)}dx \\ &\quad +\gamma\tilde{\eta}(t)\int_0^1\tilde{u}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx \end{aligned}$$

$$\leq \gamma\tilde{\eta}(t)\left(e^{\nu}\left(-k_P\tilde{v}(0,t)-k_I\tilde{\eta}(t)\right)-\tilde{v}(0,t)\right)$$
$$-\nu\gamma\tilde{\eta}(t)\int_0^1 \tilde{v}(x,t)e^{\nu x}dx+\frac{\kappa\gamma}{2}\tilde{v}^2(0,t)$$
$$+\frac{\gamma}{2\kappa}\int_0^1 \tilde{v}^2(x,t)\frac{e^{2\nu x}}{\varepsilon_2^2(x)}dx$$
$$+\gamma\tilde{\eta}(t)\int_0^1 \tilde{u}(x,t)\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x}dx \quad (82)$$
$$\dot{R}_4(t)=\rho\tilde{v}(0,t)\tilde{\eta}(t)\,, \quad (83)$$

where we used integration by parts in the first terms of (80)–(82) and Young's inequality in the second term of (82). Using (70), (80)–(83) we get

$$\dot{V}(t)\leq -\begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}^\top M_1\begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}$$
$$-\int_0^1\begin{bmatrix}\tilde{u}(x,t) & \tilde{v}(x,t) & \tilde{\eta}(t)\end{bmatrix}M(x)\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\\ \tilde{\eta}(t)\end{bmatrix}dx$$
$$-e^{-\mu}\tilde{u}^2(1,t)-\theta V(t)\,, \quad (84)$$

where M_1 is given by (58) and

$$M(x)=\begin{bmatrix}A(x) & B^\top(x)\\ B(x) & C\end{bmatrix}, \quad (85)$$

with

$$A(x)=\begin{bmatrix}A_1(x) & A_2(x)\\ A_3(x) & A_4(x)\end{bmatrix}, \quad (86)$$

where

$$A_1(x) = \left(\mu-\frac{\theta}{\varepsilon_1(x)}\right)e^{-\mu x} \quad (87)$$
$$A_2(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \quad (88)$$
$$A_3(x) = -\frac{\gamma_1(x)}{\varepsilon_1(x)}e^{-\mu x}-\beta\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\mu x} \quad (89)$$
$$A_4(x) = \beta\left(\mu-\frac{\theta}{\varepsilon_2(x)}\right)e^{\mu x}-\frac{\gamma}{2\kappa}\frac{e^{2\nu x}}{\varepsilon_2^2(x)} \quad (90)$$
$$B(x) = \begin{bmatrix}-\frac{\gamma}{2}\frac{\gamma_2(x)}{\varepsilon_2(x)}e^{\nu x} & \frac{\gamma}{2}\left(\nu-\frac{\theta}{\varepsilon_2(x)}\right)e^{\nu x}\end{bmatrix} \quad (91)$$
$$C = \frac{\gamma-\theta\rho}{2}\,. \quad (92)$$

Using the Schur complement of C in $M(x)$ and (67), (92) one has that $M(x)\geq 0$ for all $x\in[0,1]$, if and only if

$$M_2(x)=A(x)-B^\top(x)C^{-1}B(x)\geq 0\,. \quad (93)$$

Thus, if $M_1\geq 0$ and $M_2(x)\geq 0$, for all $x\in[0,1]$, one has

$$\dot{V}(t)\leq -e^{-\mu}\tilde{u}^2(1,t)-\theta V(t)\,, \quad (94)$$

and hence, $V(t)\leq e^{-\theta t}V(0)$, for all $t\geq 0$. Combining this relation with (78), (79) the proof is complete. ■

Remark 2: A control law with an integral action is designed in [12] for 2×2 hyperbolic systems. Stability of the closed-loop system is proved using a diagonal Lyapunov functional. Here the non-diagonal term in the Lyapunov functional is needed for proving stability using a quadratic Lyapunov functional. Indeed, let us assume that the Lyapunov functional is diagonal. We can write it as

$$V(t) = \int_0^1\left(q_1(x)\tilde{u}^2(x,t)+q_2(x)\tilde{v}^2(x,t)\right)dx$$
$$+\frac{\rho}{2}\tilde{\eta}^2(t)\,, \quad (95)$$

where the functions q_1 and q_2 belong to $C^1([0,1])$ with $q_1(x)$, $q_2(x)>0$, for all $x\in[0,1]$. The time derivative of V along the solutions of system (60), (61) with boundary conditions (62)–(65) is given by

$$\dot{V}(t) = \begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}^\top\begin{bmatrix}D_1 & D_2\\ D_3 & D_4\end{bmatrix}\begin{bmatrix}\tilde{v}(0,t)\\ \tilde{\eta}(t)\end{bmatrix}$$
$$+\int_0^1\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\end{bmatrix}^\top E(x)\begin{bmatrix}\tilde{u}(x,t)\\ \tilde{v}(x,t)\end{bmatrix}dx$$
$$-q_1(1)\varepsilon_1(1)\tilde{u}^2(1,t)\,, \quad (96)$$

where

$$D_1=q_1(0)\varepsilon_1(0)q^2-q_2(0)\varepsilon_2(0)+q_2(1)\varepsilon_2(1)k_P^2 \quad (97)$$
$$D_2=\frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_Pk_I+\rho\right) \quad (98)$$
$$D_3=\frac{1}{2}\left(q_2(1)\varepsilon_2(1)k_Pk_I+\rho\right) \quad (99)$$
$$D_4=q_2(1)\varepsilon_2(1)k_I^2 \quad (100)$$
$$E(x)=\begin{bmatrix}(q_1(x)\varepsilon_1(x))_x & q_1(x)\gamma_1(x)+q_2(x)\gamma_2(x)\\ q_1(x)\gamma_1(x)+q_2(x)\gamma_2(x) & -(q_2(x)\varepsilon_2(x))_x\end{bmatrix}. \quad (101)$$

Using (96) and (100) one can conclude that when $k_I\neq 0$ the inequality $\dot{V}\leq 0$ can not be satisfied for any $\begin{bmatrix}\tilde{u} & \tilde{v} & \tilde{\eta}\end{bmatrix}^\top$.

As explained in Remark 2 the non-diagonal term in the Lyapunov functional is crucial for proving stability using a quadratic Lyapunov functional. However, this term adds considerable complexity in verifying analytically that the matrices (58), (59) are positive definite and that (66) holds. Next, we numerically verify the conditions of Theorem 2 for the system from Example 1.

Example 2 (Example 1 Continued): We set in (49)–(51)

$$(\varepsilon_1,\varepsilon_2,\tau,q)=(3,6,\tau,5,0.2)\,, \quad (102)$$

and choose U in (63) according to (64) with

$$k_P = 0.1 \quad (103)$$
$$k_I = 1.0583\,, \quad (104)$$

in order to stabilize the zero equilibrium of (49)–(51). We verify numerically that the conditions of Theorem 2 are satisfied with

$$(\beta,\kappa,\mu,\nu,\theta,\rho,\gamma)=(0.7,0.2,0.5,0.2,0.7,2,2)\,. \quad (105)$$

From (58) we get that

$$M_1=\begin{bmatrix}0.4485 & 0\\ 0 & 0.2926\end{bmatrix}>0\,. \quad (106)$$

The verification of the positive definiteness of matrix (59) is more delicate due to its dependence on x. Fig. 1 shows the evolution of the eigenvalues of $M_2(x)$ and the determinant of matrix (71), which remain positive for all $x\in[0,1]$.

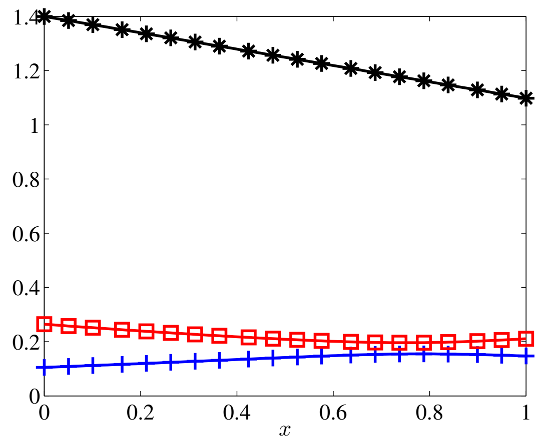

Fig. 1. Evolution of the eigenvalues of (59) as a function of x (square and cross markers), and of the determinant of $P(x)$ in (71) (star marker) for Example 2.

IV. Conclusions

We presented solutions to the trajectory generation and tracking problems for general 2×2 systems of first-order linear hyperbolic PDEs. We solved the motion planning problem with backstepping and the trajectory tracking problem with PI control. We proved exponential stability of the closed-loop system by constructing a Lyapunov functional.

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