Lecture 8: dynamic programming

- Knapsack
- Dynamic programming approach to knapsack
- A practical example for knapsack
- Dijkstra's algorithm revisited
- Dynamic programming idea behind Dijkstra's algorithm
- How to construct dynamic programming algorithms
- Landing scheduling via dynamic programming
- Travelling salesman

Knapsack problem

How to pack as much value with a weight constraint W?

$$\begin{array}{lll} \textbf{max:} & \sum_{i=1}^{p} u_i x_i & \\ \textbf{s.t.} & \sum_{i=1}^{p} w_i x_i \leq W & \\ & x_i \in \{0,1\} & \text{for all i} \end{array}$$

p	number of items
i	index of items
W	maximum weight to be carried
w_i	weight of object i
u_i	value of object i
x_i	decision variable to take ($x_i = 1$) or not to take ($x_i = 0$) object i

Dynamic programming solution of knapsack

Let us index by i the items.

Let us index by j the weight restriction.

Question (to be answered by induction)

- If I can take objects 1, 2, 3, … i,
- How much value can I take away
- Given that I am restricted to take a maximum weight of j

This question is to be answered by induction, on i AND j

Dynamic programming induction relation

Introduce a quantity d(i,j), indexed by

- i, the number of items to be taken away
  - → i = 0, 1, 2, … p
- j, the weight restriction
  - → j = 1, 2, …, W

Note that all quantities of this problem have to be integer

Introduce d(i,j), the maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

We will thus compute d(i,j) recursively, in an array of dimension p x W.

Dynamic programming induction relation

$$\underbrace{d(i,j)}_{\text{maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j}} = \max\left\{d(i-1,j), u_i + d(i-1, j-w_i)\right\}$$

maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

Dynamic programming induction relation

$$d(i,j) = \max\left\{d(i-1,j), u_i + \underbrace{d(i-1, j-w_i)}\right\}$$

value of for i-1 items, with restriction j minus weight of object i

maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

Dynamic programming induction relation

$$d(i,j) = \max\{d(i-1,j), u_i + d(i-1, j-w_i)\}$$

value of for i-1 items, with restriction j minus weight of object i
\+
value of item i

maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

Dynamic programming induction relation

$$d(i,j) = \max\{d(i-1,j), u_i + d(i-1, j-w_i)\}$$

keep i-1 items
(do nothing)

value of for i-1 items, with restriction j minus weight of object i
\+
value of item i

maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

Dynamic programming induction relation

$$d(i,j) = \max\{d(i-1,j), u_i + d(i-1,j-w_i)\}$$

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(do nothing)

value of for i-1 items, with
restriction j minus weight
of object i
+
value of item i

maximum value of the selected items, if
we are allowed to take items 1 to i, and
we have a weight restriction of j

Dynamic programming induction relation

Best option: - keep the previous selection ?
- add the new object ?

$$d(i,j) = \max\{d(i-1,j), u_i + d(i-1,j-w_i)\}$$

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maximum value of the selected items, if
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Dynamic programming induction relation

Best option: - keep the previous selection ?
- add the new object ?

$$d(i,j) = \max\left\{d(i-1,j), u_i + d(i-1,j-w_i)\right\}$$

keep i-1 items
(do nothing)

value of for i-1 items, with restriction j minus weight of object i
+
value of item i

maximum value of the selected items, if we are allowed to take items 1 to i, and we have a weight restriction of j

What do we do next?

We fill an array of size p x W

$$w_1 = 1$$
$$w_2 = 7$$
$$w_3 = 4$$
$$w_4 = 2$$

Weight = value
p=4
W=11 (arbitrary, but less than 14 obviously)

$$d(i,j) = \max\left\{d(i-1,j), u_i + d(i-1,j-w_i)\right\}$$

What do we do next?

We fill an array of size p x W

$$d(i,j) = \max\left\{d(i-1,j), u_i + d(i-1,j-w_i)\right\}$$

Weight restriction j (j=1,2, … , W)

Items picked (i=1, 2, 3, 4, …)

	1	2	3	4	5	6	7	8	9	10	11
1											
2											
3											
4											

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1	1	1	1	1	1	1	1	1	1	1	1
	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}
2	1	1	1	1	1	1	7	8	8	8	8
	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
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	{1}	{4}	{4,1}	{3}	{3,1}	{4,3}	{4,3,1}	{2,1}	{4,2}	{4,2,1}	{3,2}

We fill an array of size p x W

$$w_1 = 1,\ w_2 = 7,\ w_3 = 4,\ w_4 = 2$$

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	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
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	{1}	{4}	{4,1}	{3}	{3,1}	{4,3}	{4,3,1}	{2,1}	{4,2}	{4,2,1}	{3,2}

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	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}
2	1	1	1	1	1	1	7	8	8	8	8
	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
3	1	1	1	4	5	5	7	8	8	8	11
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	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
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	{1}	{1}	{1}	{3}	{3,1}	{3,1}	{2}	{2,1}	{2,1}	{2,1}	{3,2}
4	1	2	3	4	5	6	7	8	9	10	11
	{1}	{4}	{4,1}	{3}	{3,1}	{4,3}	{4,3,1}	{2,1}	{4,2}	{4,2,1}	{3,2}

We fill an array of size p x W

$$w_1 = 1,\ w_2 = 7,\ w_3 = 4,\ w_4 = 2$$

No uniqueness: could pick also item 2

Weight restriction j (j=1,2, … , W)

Items picked (i=1, 2, 3, 4, …)

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}
2	1	1	1	1	1	1	7	8	8	8	8
	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
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	{1}	{4}	{4,1}	{3}	{3,1}	{4,3}	{4,3,1}	{2,1}	{4,2}	{4,2,1}	{3,2}

What do we do next?

We fill an array of size p x W

$$d(i,j) = \max\{d(i-1,j),\, u_i + d(i-1,j-w_i)\}$$

Weight restriction j (j=1,2, … , W)

Items picked (i=1, 2, 3, 4, …)

	1	2	3	4	5	6	7	8	9	10	11
1											
2											
3											
4											

Final result

Weight restriction j (j=1,2, … , W)

Items picked (i=1, 2, 3, 4, …)

	1	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}	{1}
2	1	1	1	1	1	1	7	8	8	8	8
	{1}	{1}	{1}	{1}	{1}	{1}	{2}	{2,1}	{2,1}	{2,1}	{2,1}
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Dynamic programming: main idea

Main idea: when solving an integer program, in order to avoid enumeration (to expensive computationally), cut the problem in two and use induction:

At current step: assume you know the best solution at previous steps

Compute the best solution for the current step, and pair it with the solution at the previous steps

A more precise definition of dynamic programming will be given later in class.

Dijkstra's algorithm revisited

```
begin
        S:=Ø
        d(i):=+∞ for each node i
        d(s):=0 and pred(s)=0
        while |S|<n do
                begin
                        let i in S* for which d(i)=min{d(j), j in S*}
                        S = S U {i}
                        S* = S* \ {i}
                for each (i,j) in the graph do
                        if d(j)>d(i)+cij
                        then
                                d(j):=d(i)+cij
                                pred(j):=i
                        end
                end
        end
end
```

$$\boxed{d(j) := \min_{\text{connected nodes}} \{d(j), d(i) + c_{ij}\}}$$

Core of Dijkstra's algorithm

$$\underbrace{d(j) :=}_{\text{Value at next iteration}} \min_{\text{connected nodes}} \{d(j), \underbrace{d(i) + c_{ij}}\}$$

best between the previous iteration and whatever is best at this iteration

previous Iteration (other path)

for all connected node i, compute the path length to node i plus the length from i to j

This is the main idea under dynamic programming algorithms

How to construct dynamic programming algorithms

1) **View the choice of a feasible solution as a sequence of decisions occuring in stages, and so that the total cost is the sum of the costs of individual decisions.**

2) **Define the state as a summary of all relevant past decisions**

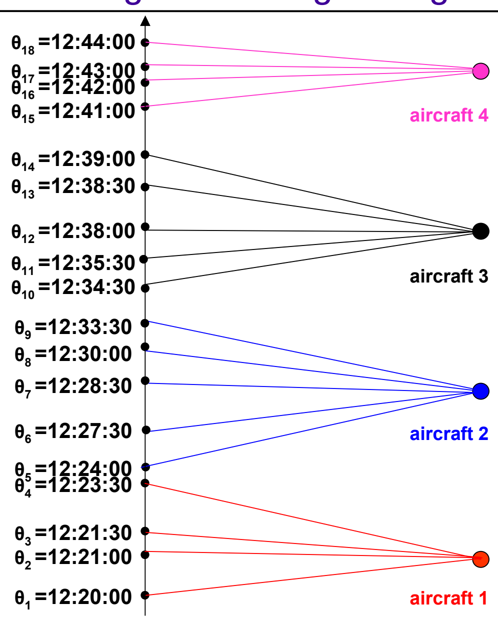
3) **Determine which state transitions are possible. Let the cost of each state transition be the cost of the corresponding decision.**

4) **Write a recursion on the optimal cost from the origin state to a destination state**

Shaded square indicates that time is meant for the algorithm (more about this in the lab).

[Introduction to linear optimization, Bertsimas, Tsitsiklis, 1997]

Landing scheduling through dynamic programming

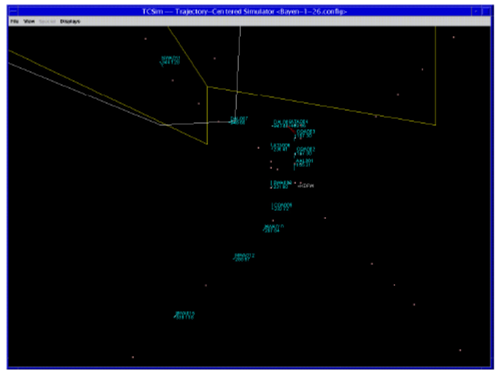


How to pick a landing time for each aircraft so that the minimum separation between any two aircraft is a large as possible?

Landing scheduling through dynamic programming

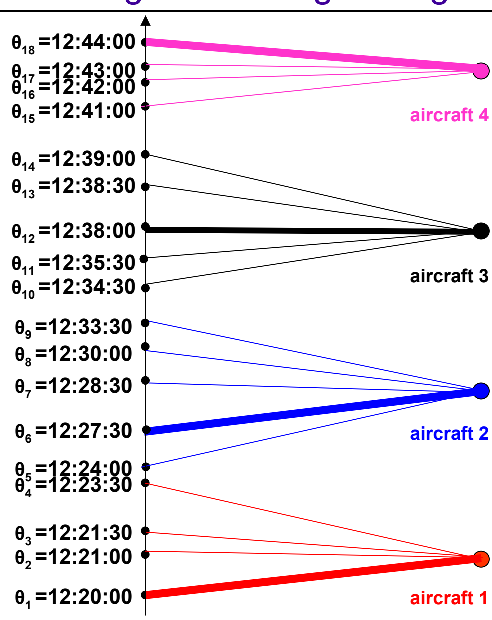
Example of data used for this type of problem:

ATA001	1295	1305	1310	1311	1316	1320	1325	1327	1335
UAL002	1413	1423	1429	1439	1447	1458	1468	1478	1489
DAL003	1522	1532	1541	1551	1557	1567	1581	1587	1601
UAL004	1606	1613	1619	1629	1638	1648	1659	1673	1682
COA005	1693	1700	1705	1710	1710	1715	1720	1725	1730
SWA006	1787	1794	1799	1799	1804	1809	1814	1819	1824



[TCSim, T. Callantine, NASA Ames]

Landing scheduling through dynamic programming



How to pick a landing time for each aircraft so that the minimum separation between any two aircraft is a large as possible?

This looks like a reasonable solution.

Landing scheduling through dynamic programming

t_{ij} j-th possible landing time of aircraft i

n_i number of possible landing times for aircraft i

$\delta(i,j)$ Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

- Aircraft 1 should obviously arrive as early as possible
- If aircraft 2 is assigned the j-th arrival time, the spacing between aircraft i and j is obviously

$$\delta(2,j) = t_{2,j} - t_{1,1}$$

Landing scheduling through dynamic programming

t_{ij} j-th possible landing time of aircraft i

n_i number of possible landing times for aircraft i

$\delta(i,j)$ Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

$$\delta(2,j) = t_{2,j} - t_{1,1}$$

Recursion

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}} \{\min\{t_{i,j} - t_{i-1,j'}\ ,\ \delta(i-1,j')\}\}$$

Landing scheduling through dynamic programming

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}}\{\min\{t_{i,j} - t_{i-1,j'} \ , \ \delta(i-1,j')\}\}$$

Recursion is done with variable i, i.e. with the number of aircraft. For a number i of aircraft, this represents the largest smallest spacing if aircraft i arrives at time j

t_{ij}	j-th possible landing time of aircraft i
n_i	number of possible landing times for aircraft i
$\delta(i,j)$	Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

$$\delta(2,j) = t_{2,j} - t_{1,1}$$

Landing scheduling through dynamic programming

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}}\{\min\{t_{i,j} - t_{i-1,j'} \ , \ \delta(i-1,j')\}\}$$

Maximal minimum separation between any two aircraft within the first i-1 aircraft, if aircraft i-1 is assigned arrival time j'

t_{ij}	j-th possible landing time of aircraft i
n_i	number of possible landing times for aircraft i
$\delta(i,j)$	Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

$$\delta(2,j) = t_{2,j} - t_{1,1}$$

Landing scheduling through dynamic programming

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}} \{ \min \{ \underbrace{t_{i,j} - t_{i-1,j'}} \ , \ \delta(i-1,j') \} \}$$

Time separation resulting from assigning aircraft i to its j-th arrival time and aircraft i-1 to its j'-th arrival time.

t_{ij} j-th possible landing time of aircraft i

n_i number of possible landing times for aircraft i

$\delta(i,j)$ Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

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Landing scheduling through dynamic programming

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}} \{ \underbrace{\min \{ t_{i,j} - t_{i-1,j'} \ , \ \delta(i-1,j') \}} \}$$

For a given j' (aircraft i-1' s arrival time), the overall resulting minimum separation is the worst between

- aircraft i and aircraft i-1

- any two other aircraft within 1, 2, ..., i-1

t_{ij} j-th possible landing time of aircraft i

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Landing scheduling through dynamic programming

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}}\{\min\{t_{i,j} - t_{i-1,j'}\ ,\ \delta(i-1,j')\}\}$$

For aircraft i, we have to compute the maximum spacing so far when trying all possible assignments for the previous i-1 aircraft

t_{ij}	j-th possible landing time of aircraft i
n_i	number of possible landing times for aircraft i
$\delta(i,j)$	Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

Initialization of the recursion:

$$\delta(2,j) = t_{2,j} - t_{1,1}$$

Landing scheduling through dynamic programming

Number of possible arrival times for aircraft i-1

$$\delta(i,j) = \max_{j'=1}^{n_{i-1}}\{\min\{t_{i,j} - t_{i-1,j'}\ ,\ \delta(i-1,j')\}\}$$

For aircraft i, we have to compute the maximum spacing so far when trying all possible assignments for the previous i-1 aircraft

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Initialization of the recursion:

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Landing scheduling through dynamic programming

$$\underbrace{\delta(i,j) = \max_{j'=1}^{n_{i-1}}\{\min\{t_{i,j} - t_{i-1,j'} \ , \ \delta(i-1,j')\}\}}$$

The goal of the dynamic algorithm is to compute the largest spacing, by trying the best among all possible assignments of aircraft i-1 (since this is the induction step between i-1 and i).

t_{ij} j-th possible landing time of aircraft i

n_i number of possible landing times for aircraft i

$\delta(i,j)$ Maximal minimum spacing between any two aircraft, for the subset of aircraft 1, 2, … , i, if the aircraft number i is assigned the arrival time t_{ij}

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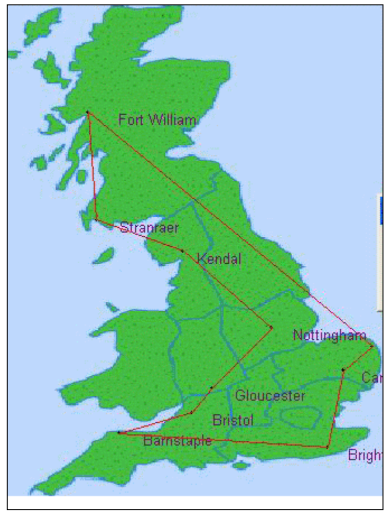
[Introduction to linear optimization, Bertsimas, Tsitsiklis, 1997]

Traveling salesman

What is the shortest path to loop through N cities?



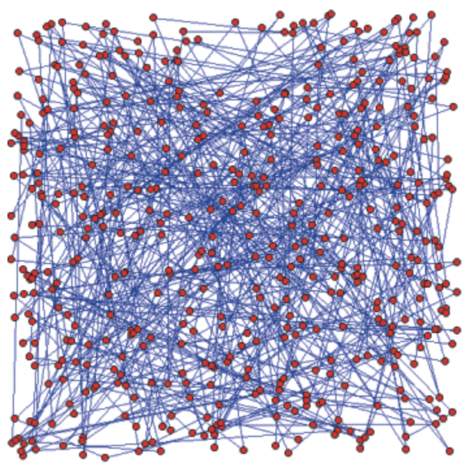
[http://www.informatik.uni-leipzig.de/~meiler]

[http://www.superbasescientific.com/]

Traveling salesman: engineering applications

What is the shortest path to loop through N cities?

500 cities, random solution!

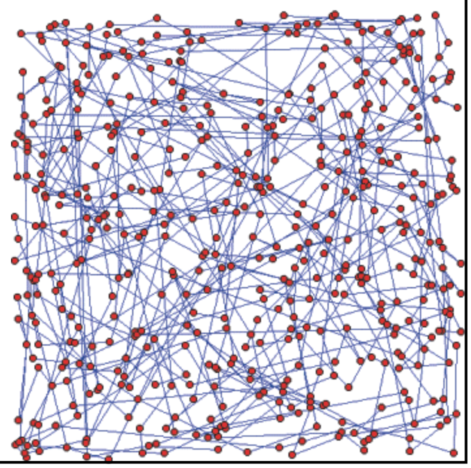


[http://www.logicalgenetics.com/]

Traveling salesman: engineering applications

What is the shortest path to loop through N cities?

500 cities, a better solution!



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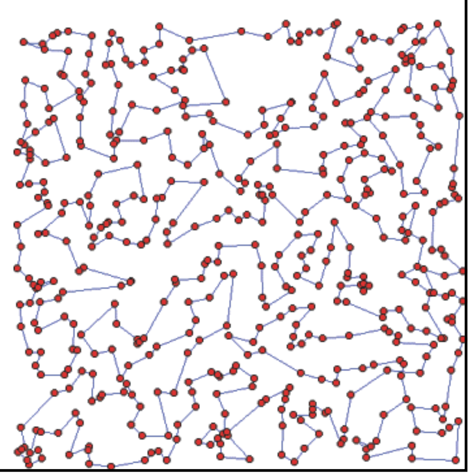
Traveling salesman: engineering applications

What is the shortest path to loop through N cities?

500 cities, a much better solution!



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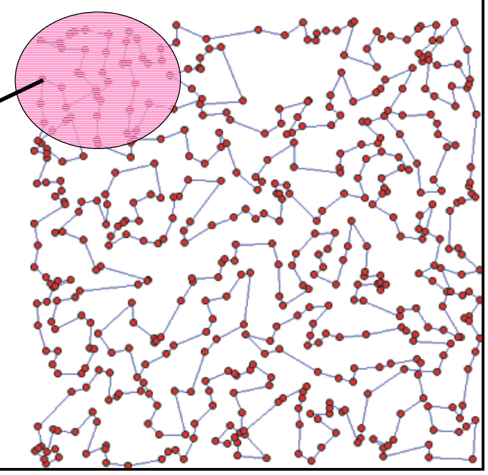
Traveling salesman: dynamic programming solution

$$C(S,k) = \min_{m \in S \backslash \{k\}} \left(C(S \backslash \{k\}, m) + c_{mk} \right)$$

S subset of cities including city 1 (departure city)

City 1



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k a city in S

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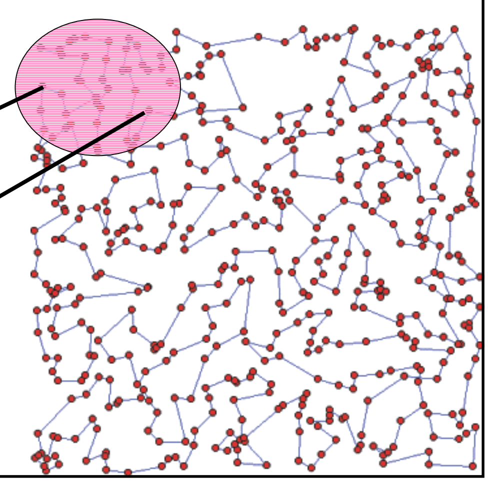
a city in S



Induction is done on S

Traveling salesman: dynamic programming solution

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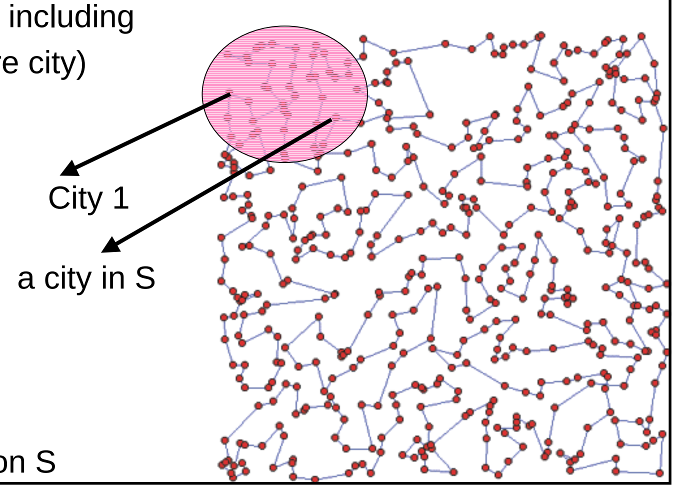
$C(S,k)$ shortest path from city 1 to city k that visits all nodes in S



Induction is done on S

Traveling salesman: dynamic programming solution

$$C(S,k) = \min_{m \in S \backslash \{k\}} \left(\underbrace{C(S\backslash\{k\}, m)}_{\text{Shortest way to go to m without going through k}} + \underbrace{c_{mk}}_{\text{Cost to go from m to k}} \right)$$

$C(S,k)$ shortest path from city 1 to city k that visits all nodes in S

S subset of cities including city 1 (departure city)

k a city in S

Induction is done on S

c_{mk} cost to go from city m to city k

Where m is a node in S, but not k

Traveling salesman: dynamic programming solution

$$C(S,k) = \underbrace{\min_{m \in S \backslash \{k\}} (C(S \backslash \{k\}, m) + c_{mk})}_{\text{Best over all possible m (not equal to k)}}$$

$C(S,k)$ shortest path from city 1 to city k that visits all nodes in S

S subset of cities including city 1 (departure city)

k a city in S

Induction is done on S

c_{mk} cost to go from city m to city k

Where m is a node in S, but not k

Traveling salesman: dynamic programming solution

$$C(S,k) = \min_{m \in S \backslash \{k\}} (C(S \backslash \{k\}, m) + c_{mk})$$

$$C(\{1\}, 1) = 0$$

Cost to go from city 1 to city 1: zero

$C(S,k)$ shortest path from city 1 to city k that visits all nodes in S

S subset of cities including city 1 (departure city)

k a city in S

Induction is done on S

c_{mk} cost to go from city m to city k

Where m is a node in S, but not k

How to construct dynamic programming algorithms

1) **View the choice of a feasible solution as a sequence of decisions occuring in stages, and so that the total cost is the sum of the costs of individual decisions.**

2) **Define the state as a summary of all relevant past decisions**

3) **Determine which state transitions are possible. Let the cost of each state transition be the cost of the corresponding decision.**

4) **Write a recursion on the optimal cost from the origin state to a destination state**

Shaded square indicates that time is meant for the algorithm (more about this in the lab).

[Introduction to linear optimization, Bertsimas, Tsitsiklis, 1997]