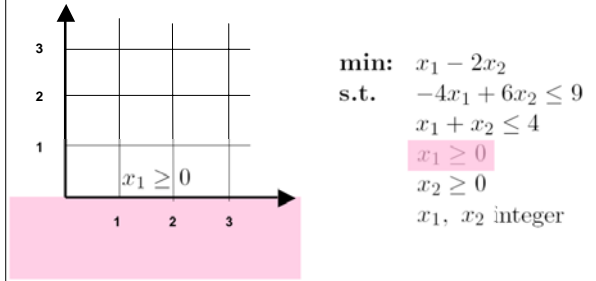


Lecture 6: branch and bound

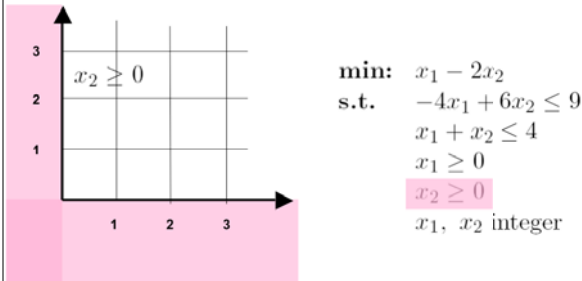
- Reminder: graphical construction of the feasible set
- An example of solution of an IP with branch and bound
- Branch and bound: summary of subproblems
- A generic branch and bound algorithm

[Bertsimas and Tsitsiklis, Introduction to Linear Optimization, Chap. 11, sec. 11.2, pp. 485-490]

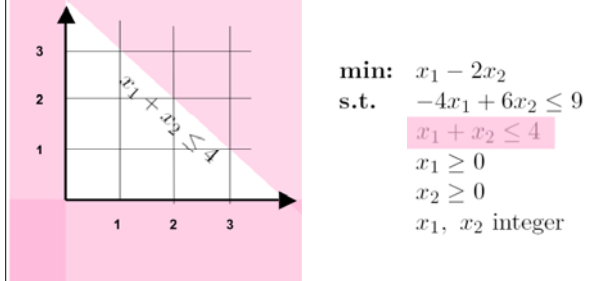
Reminder: finding the feasible set of a linear program



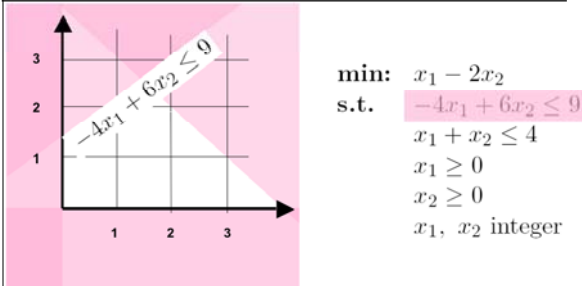
Reminder: finding the feasible set of a linear program



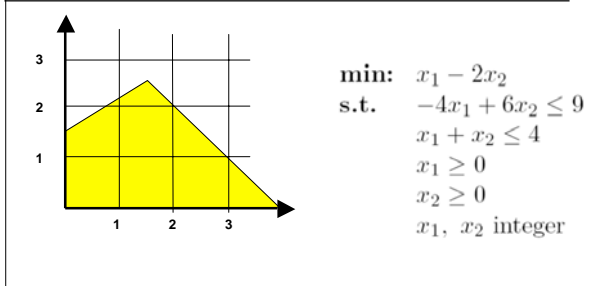
Reminder: finding the feasible set of a linear program



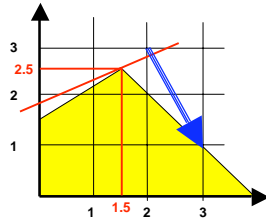
Reminder: finding the feasible set of a linear program



Reminder: finding the feasible set of a linear program

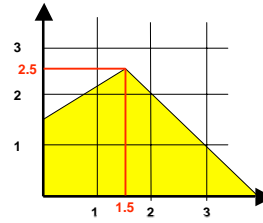


Optimum of the linear program is not integer



min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

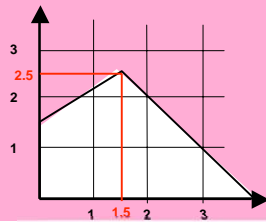
Branch and bound algorithm



Solve LP $\rightarrow (1.5, 2.5), Z^* = -3.5$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

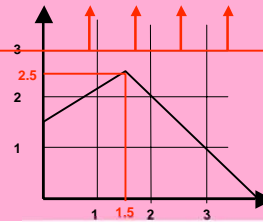
Problem P1: add constraint $x_2 \geq 3$



Solve P0 $\rightarrow (1.5, 2.5), Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

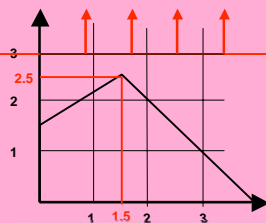
Problem P1: add constraint $x_2 \geq 3$



Solve P0 $\rightarrow (1.5, 2.5), Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

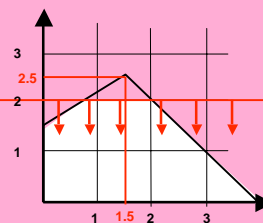
Problem P1: discard



Solve P0 $\rightarrow (1.5, 2.5), Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

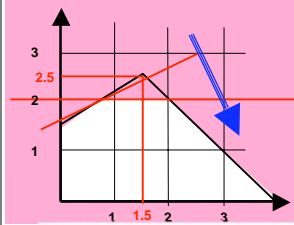
Problem P2: add constraint $x_2 \leq 2$



Solve P0 $\rightarrow (1.5, 2.5), Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

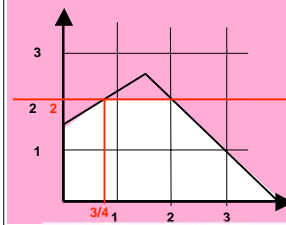
Problem P2: solve for optimal



Solve P0 $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

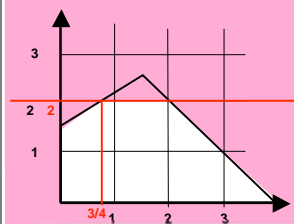
Problem P2: fractional optimum



Solve LP $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

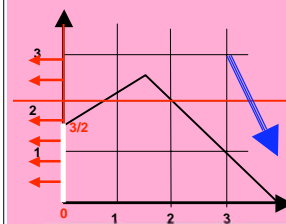
Problem P3: add constraint $x_1 \leq 0$



Solve LP $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$
 P3: P2 + constraint $x_1 \leq 0$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

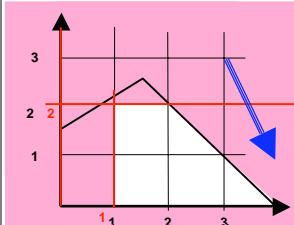
Problem P3: fractional solution



Solve LP $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$
 P3: P2 + constraint $x_1 \leq 0$
 Solve P3 $\rightarrow (0, 3/2)$, $Z^* = -3$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

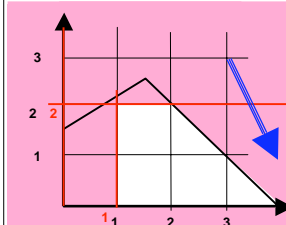
Problem P4: add constraint $x_2 \geq 1$



Solve LP $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$
 P3: P2 + constraint $x_1 \leq 0$
 Solve P3 $\rightarrow (0, 3/2)$, $Z^* = -3$
 P4: P2 + constraint $x_2 \geq 1$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

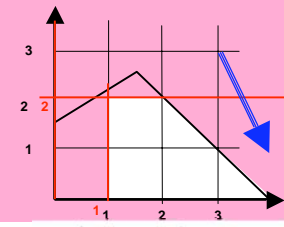
Problem P4: add constraint $x_2 \geq 1$



Solve LP $\rightarrow (1.5, 3.5)$, $Z^* = -3.5$
 P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
 P2: P0 + constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2)$, $Z^* = -3.25$
 P3: P2 + constraint $x_1 \leq 0$
 Solve P3 $\rightarrow (0, 3/2)$, $Z^* = -3$
 P4: P2 + constraint $x_2 \geq 1$
 Solve P4 $\rightarrow (1, 2)$, $Z^* = -3$

min: $x_1 - 2x_2$
 s.t. $-4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 x_1, x_2 integer

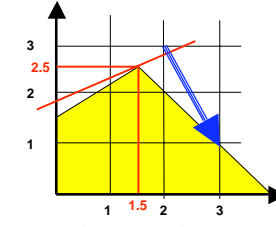
Problem P4: add constraint $x_2 \geq 1$



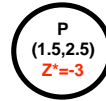
$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$

Solve LP $\rightarrow (1.5, 3.5), Z^* = -3.5$
P1: Add constraint $x_2 \geq 3$
 Problem P1 infeasible
 Discard P1
P2: Add constraint $x_2 \leq 2$
 Solve P2 $\rightarrow (3/4, 2), Z^* = -3.25$
P3: Add constraint $x_1 \leq 0$
 Solve P3 $\rightarrow (0, 3/2), Z^* = -3$
P4: Add constraint $x_2 \geq 1$
 Solve P4 $\rightarrow (1, 2), Z^* = -3$
 Optimum is (1, 2)
 Terminate

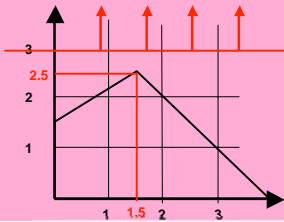
Branch and bound: summary



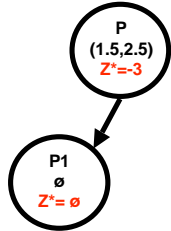
$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$



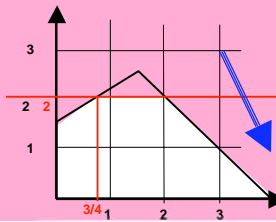
Branch and bound: summary



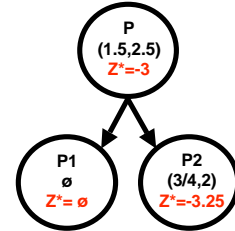
$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$



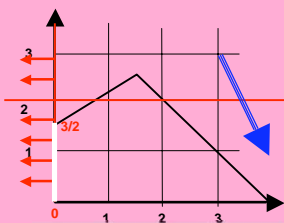
Branch and bound: summary



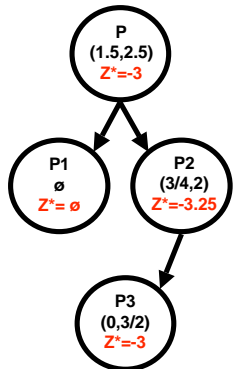
$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$



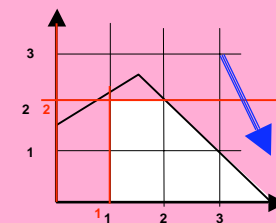
Branch and bound: summary



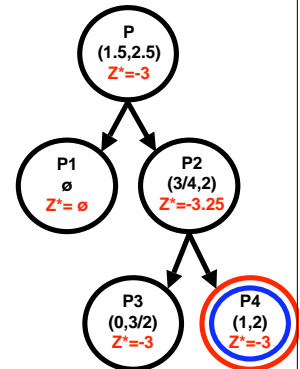
$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$



Branch and bound: summary



$\min: x_1 - 2x_2$
 $\text{s.t.} \quad -4x_1 + 6x_2 \leq 9$
 $x_1 + x_2 \leq 4$
 $x_1 \geq 0$
 $x_2 \geq 0$
 $x_1, x_2 \text{ integer}$



A generic branch and bound algorithm (min. problem)

- Get upper bound U (solving relaxed LP)
- Select an active subproblem P_i
- If subproblem infeasible, delete it, otherwise, compute optimum for this subproblem (called U_i)
- If optimum greater than U , delete subproblem
- If optimum smaller than U , obtain optimal solution to the subproblem, or break corresponding subproblem into further subproblems which are added to the list of active subproblems
- Stop when list of active problems is empty