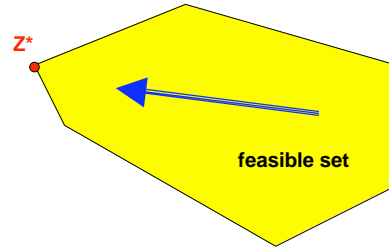## Lecture 4: Integer Programming (IP)

- Fractional solutions vs. integer solutions
- Upper/lower bounds on the optimal
- Fractional feasible set vs. integer feasible set
- Decision variables
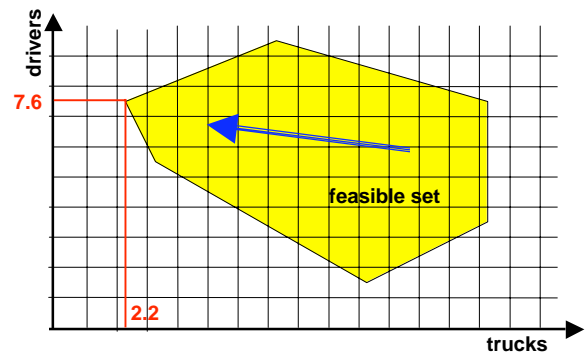- Shortest path revisited: decision variables

Optimum Z^* of a LP

Z^*

feasible set

What if the desired solution is not an integer?

What if this represents the number of trucks and drivers?

drivers

Z^*

feasible set
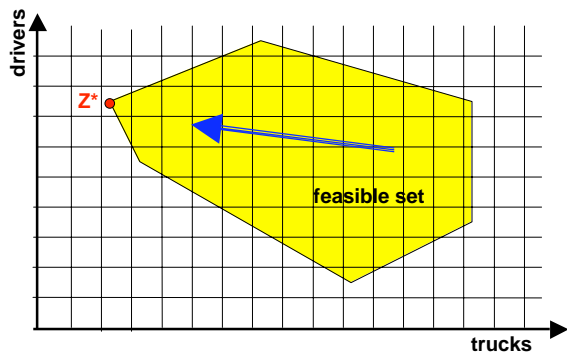
trucks

Fractional solution

What if this represents the number of trucks and drivers?

drivers

7.6

feasible set

2.2

trucks

Fractional solution

What should one do?

2 trucks
8 drivers

3 trucks
8 drivers

7.6

2 trucks
7 drivers

3 trucks
7 drivers

2.2

Fractional solution

What should one do?

Good
candidate
solution 1

Good
candidate
solution 2

1
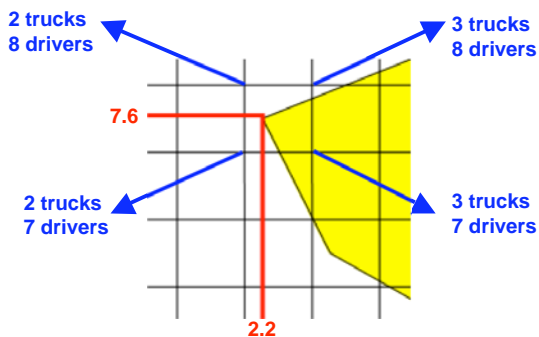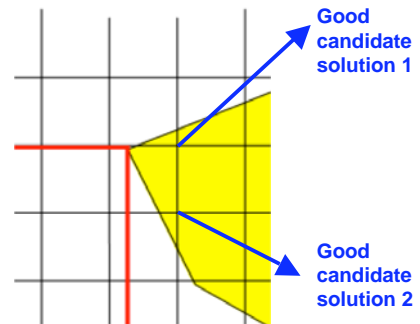
Upper or lower bound on the optimal

The fractional solution gives you an upper bound or a lower bound on the optimal.

→ **If this is a minimization problem, it gives you a lower bound on the optimal**
→ **If this is a maximization problem, it gives you an upper bound on the optimal**
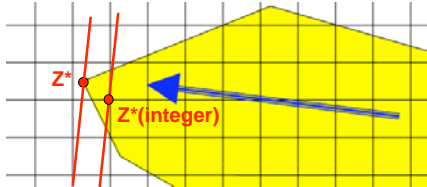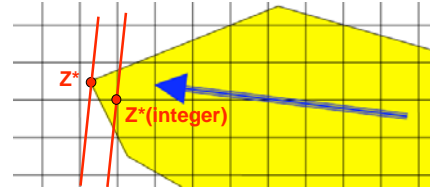

Upper or lower bound on the optimal

The fractional solution gives you an upper bound or a lower bound on the optimal.

Integer problems are sometimes very hard to solve exactly. But maybe your boss would prefer a guaranteed upper bound on the cost (quick and dirty, but correct)


Problems of fractional vs. integer solution

The feasible set for fractional solutions is larger

→ **The result is better**
If this is a maximization problem, it is larger
If it is a minimization problem it is smaller

→ **The result might not be physical**
Cut the driver in two and the truck in three

→ **Is there a way to find A / THE optimal integer solution from the fractional solution?**
This is a hard problem!!!

Optimum, value of the optimal solution

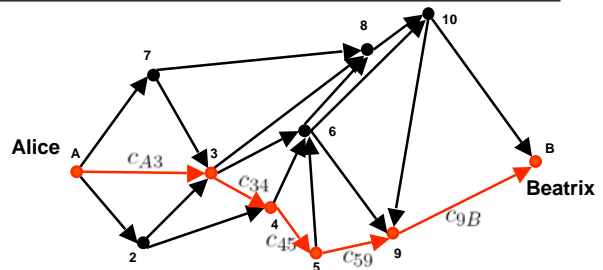**Does an integer solution to the problem exist?**
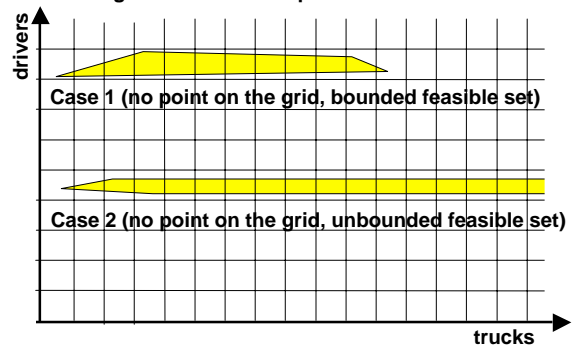
Decision variables

A possible definition for a decision variable is a variable that corresponds to a « yes/no » problem, i.e. that corresponds to a discrete choice.
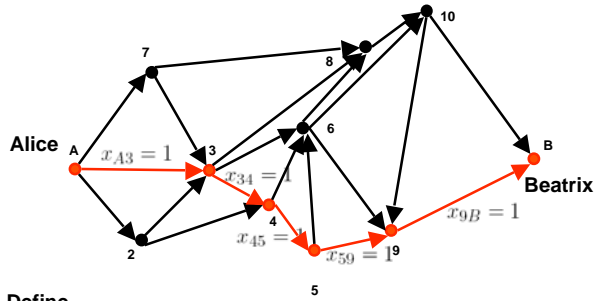
Decision variables can be modeled by integer variables

→**Example 1: do I hire this worker (d=1) or not (d=0)**
→**Example 2: how many cars are allowed in this parking lot everyday: n=0,1,2,3,4,5,6,**
→**Example 3: do I take   - the first train (d=1)**
                  - the second train (d=2)
                  - the fifth train (d=5)
→**Example 4: at this intersection, do I take**
                        - the first left
                        - the second left
                        - the first right
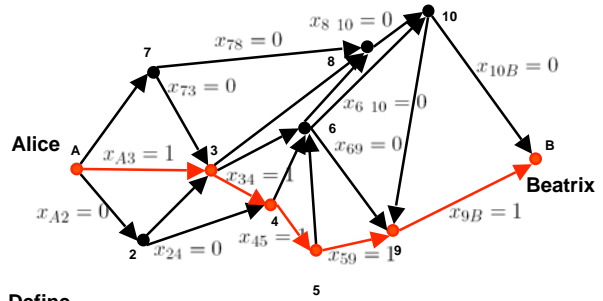
Shortest path revisited: decision variables

Shortest path revisited: decision variables

Alice — A — $x_{A3}=1$ — 3 — $x_{34}=1$ — 4 — $x_{45}=1$ — $x_{59}=1$ — 9 — $x_{9B}=1$ — B — Beatrix

Define

$x_{ij}=1$ **For every (i,j) on the shortest path**

$x_{ij}=0$ **For every (i,j) not on the shortest path**

Shortest path revisited: decision variables

$x_{8\,10}=0$, $x_{78}=0$, $x_{73}=0$, $x_{6\,10}=0$, $x_{10B}=0$, $x_{69}=0$

Alice — A — $x_{A3}=1$ — 3 — $x_{34}=1$ — Beatrix — B

$x_{A2}=0$, $x_{24}=0$, $x_{45}=1$, $x_{59}=1$, $x_{9B}=1$

Define

$x_{ij}=1$ **For every (i,j) on the shortest path**

$x_{ij}=0$ **For every (i,j) not on the shortest path**

Shortest path revisited: decision variables

Alice, Beatrix, i, x_{ij}, c_{ij}, j, c_{34}, x_{34}, 4, 3

Define

$x_{ij}=1$ **For every (i,j) on the shortest path**

$x_{ij}=0$ **For every (i,j) not on the shortest path**

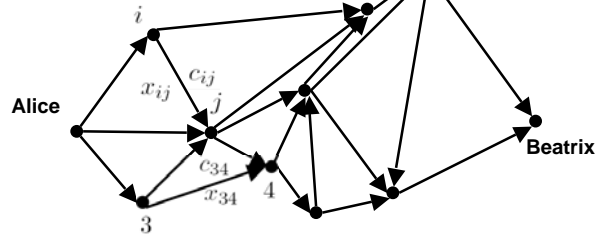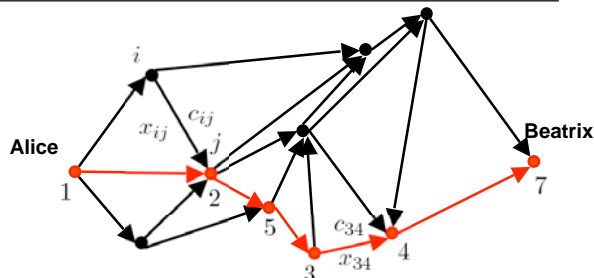## Shortest path revisited: decision variables

Define a graph (road network)

Alice, Beatrix, i, x_{ij}, c_{ij}, j, c_{34}, x_{34}, 4, 3

Take $x_{ij}=1$ **if Eva decides to go through link (i,j), zero otherwise**

For example $x_{34}=1$ **if Eva decides to use route (3,4)**

Shortest path revisited: decision variables

Alice, Beatrix, i, x_{ij}, c_{ij}, j, 1, 2, 5, c_{34}, x_{34}, 4, 3, 7

$x_{12}=x_{25}=x_{53}=x_{34}=x_{47}=1$

All other x_{ij} **are zero**

Total length of this path: $c_{12}+c_{25}+c_{53}+c_{34}+c_{47}$

Shortest path revisited: decision variables

Alice, Beatrix, i, x_{ij}, c_{ij}, j, 1, 2, 5, c_{34}, x_{34}, 4, 3, 7
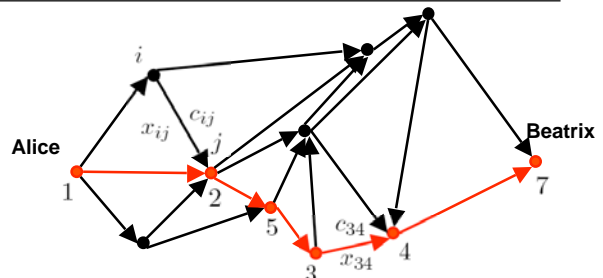
$x_{12}=x_{25}=x_{53}=x_{34}=x_{47}=1$ **All other** x_{ij} **are zero**

Decision variables x_{ij} **cannot be fractional, i.e. 0.534.**

How do we ensure that the program solving the LP provides an integer solution?

Shortest path revisited: decision variables

minimize:
$$Z = \sum_{j \in N_A} c_{Aj} x_{Aj} + \sum_{i=1}^{10} \sum_{j \in N_i} c_{ij} x_{ij} + \sum_{j \in N_B} c_{jB} x_{jB}$$

such that:
$$\sum_{j \in N_i} x_{ji} = \sum_{j \in N_i} x_{ij}, \quad i = 1, \cdots, 10$$

$$\sum_{i \in N_A} x_{Aj} = 1$$

$$\sum_{j \in N_B} x_{jB} = 1$$

$$x_{ij} \geq 0, \ x_{jB} \geq 0, \ x_{Aj} \geq 0$$

In other words, if we put this in MATLAB to solve the shortest path problem, how do we know that MATLAB will return integer decision variables?