Lecture 3: engineering solutions of LPs

- Engineering LPs
- Matrix form of a LP
- How to solve a LP in MATLAB
- A graphical illustration of the SIMPLEX method
- Engineering algorithms

In general, LPs encountered in engineering are big

It is very unlikely that your boss will give you something that simple to solve:

$$\begin{array}{ll} \textbf{min:} & Z = 140x_1 + 160x_2 \\ \textbf{Subject to:} & 2x_1 + 4x_2 \leq 28 \\ & 5x_1 + 5x_2 \leq 50 \\ & x_1 \leq 8 \\ & x_2 \leq 6 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{array}$$

In general, you get something that looks like this

Example: air traffic control

$$\min_{w_{ft}^j} \sum_{f \in \mathcal{F}} \left[c_f^g g_f + c_f^a a_f \right]$$

such that:

$$\sum_{f:P(f,1)=k} (w_{ft}^k - w_{f,t-1}^k) \leq D_k(t) \qquad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

$$\sum_{f:P(f,N_f)=k} (w_{ft}^k - w_{f,t-1}^k) \leq A_k(t) \qquad \forall k \in \mathcal{K},\ t \in \mathcal{T}$$

$$\sum_{(f,j'):P(f,i)=j, P(f,i+1)=j', i<N_f} \left(w_{ft}^j - w_{ft}^{j'} \right) \leq S_j(t) \qquad \forall j \in \mathcal{J},\ t \in \mathcal{T}$$

$$w_{f,t+l_{fj}}^{j'} - w_{ft}^j \leq 0 \qquad \forall f \in \mathcal{F},\ t \in T_f^j,\ j = P(f,i), j' = P(f,i+1),\ i < N_f$$

$$w_{ft}^k - w_{f',t-s_{f'}}^k \leq 0 \qquad \forall (f',f) \in \mathcal{C},\ t \in T_f^k, k = P(f,1) = P(f',N_f)$$

$$w_{ft}^j - w_{f,t-1}^j \geq 0 \qquad \forall f \in \mathcal{F},\ j \in P_f,\ t \in T_f^j$$

[Bertsimas, Stock-Patterson, 1997]

Reminder: general formulation of a LP

$$\begin{array}{lll}
\textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\
\textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\
& \vdots & \vdots \\
& a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M
\end{array}$$

Or equivalently,

$$\begin{array}{lll}
\textbf{min:} & \sum_{i=1}^N c_jx_j & \\
\textbf{s.t.:} & \sum_{i=1}^N a_{1,j}x_j & \leq b_1 \\
& \sum_{i=1}^N a_{2,j}x_j & \leq b_2 \\
& \vdots & \vdots \\
& \sum_{i=1}^N a_{M,j}x_j & \leq b_M
\end{array}$$

Cost function is a dot product

$$\textbf{min:} \quad c_1x_1 + c_2x_2 + \cdots + c_Nc_N$$

This cost function is in fact a dot product (scalar product between a vector x and a vector c:

$$\begin{bmatrix} c_1, & c_2, \cdots, \cdots, c_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix}$$

$$\textbf{min:} \qquad \mathbf{c^T} \cdot \mathbf{x}$$

Inequality constraints

$$\begin{array}{lll} \textbf{min:} & c_1x_1 + c_2x_2 + \cdots + c_Nc_N & \\ \textbf{s.t.:} & a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N & \leq b_1 \\ & a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N & \leq b_2 \\ & \vdots & \vdots \\ & a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N & \leq b_M \end{array}$$

Inequality constraints

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The constraints are in fact in vector form:

$$\begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,j}x_j \cdots + a_{1,N}x_N \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,j}x_j \cdots + a_{2,N}x_N \\ \vdots \\ a_{M,1}x_1 + a_{M,2}x_2 + \cdots + a_{M,j}x_j \cdots + a_{M,N}x_N \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

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The constraints are in fact in vector form:

$$\begin{bmatrix} a_{1\ 1} & a_{1\ 2} & \cdots & a_{1N} \\ a_{2\ 1} & a_{2\ 2} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

LP in matrix form

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$$\mathbf{A}\ \mathbf{x} \leq \mathbf{b}$$

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LP in matrix form

The constraints are in fact in vector form:

$$\begin{array}{ll} \textbf{min:} & \mathbf{c}^{\mathbf{T}} \cdot \mathbf{x} \\ \textbf{s.t.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \end{array}$$

Introduce objective function vector:

$$\mathbf{c}^T = \left[\ c_1,\ c_2, \cdots, \cdots, c_N\ \right]$$

The constraints are in fact in vector form:

$$\begin{bmatrix} a_{1\ 1} & a_{1\ 2} & \cdots & a_{1N} \\ a_{2\ 1} & a_{2\ 2} & \cdots & a_{2N} \\ \vdots & \vdots & \vdots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NM} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} \leq \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_M \end{bmatrix}$$

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$$\begin{array}{ll} \textbf{min:} & \mathbf{c}^{\mathbf{T}} \cdot \mathbf{x} \\ \textbf{s.t.} & \mathbf{A}\ \mathbf{x} \leq \mathbf{b} \end{array}$$

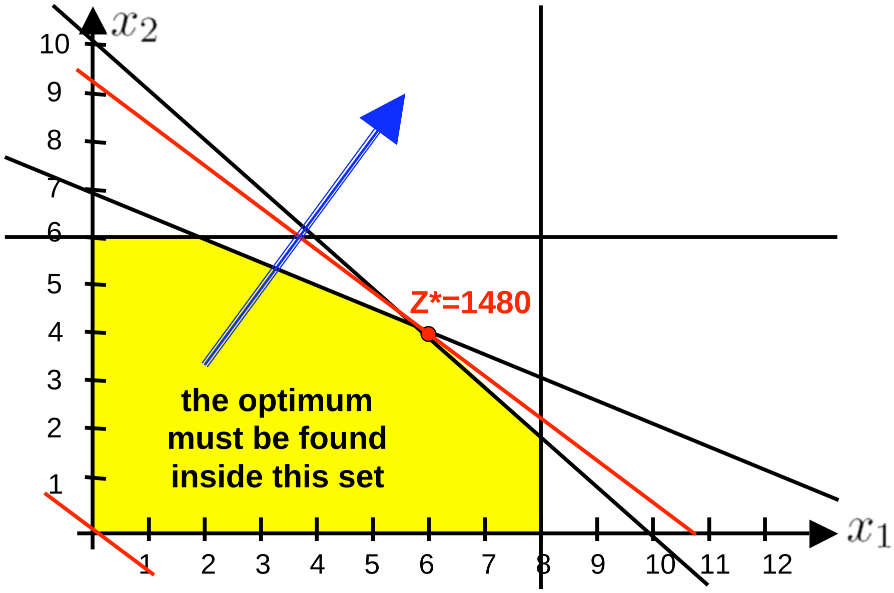
This is exactly the way you will use this in matlab:

```
x = linprog(c,A,b)
```

Several very powerful methods have been developed to solve linear programs, in particular:

- **Simplex method**
- **Interior point method**

A simplified illustration of the simplex method



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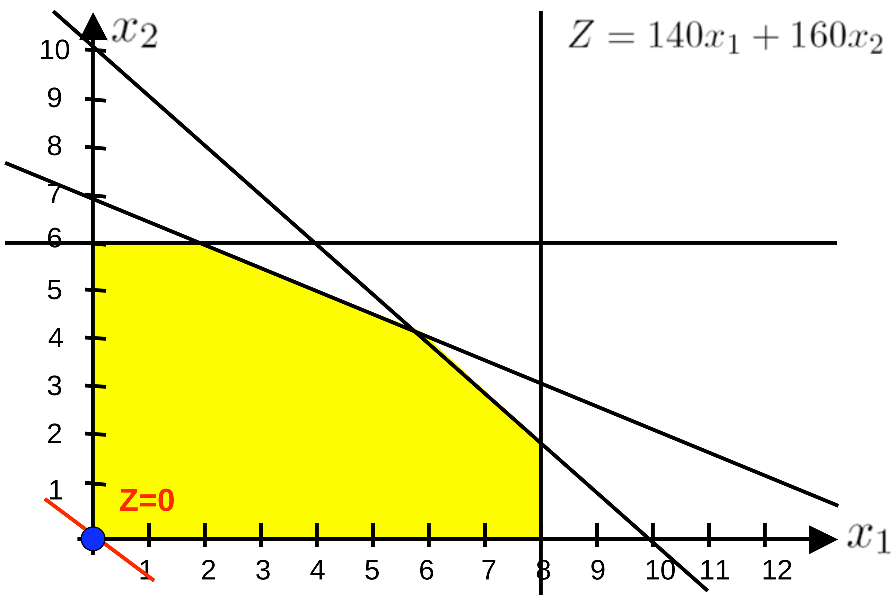

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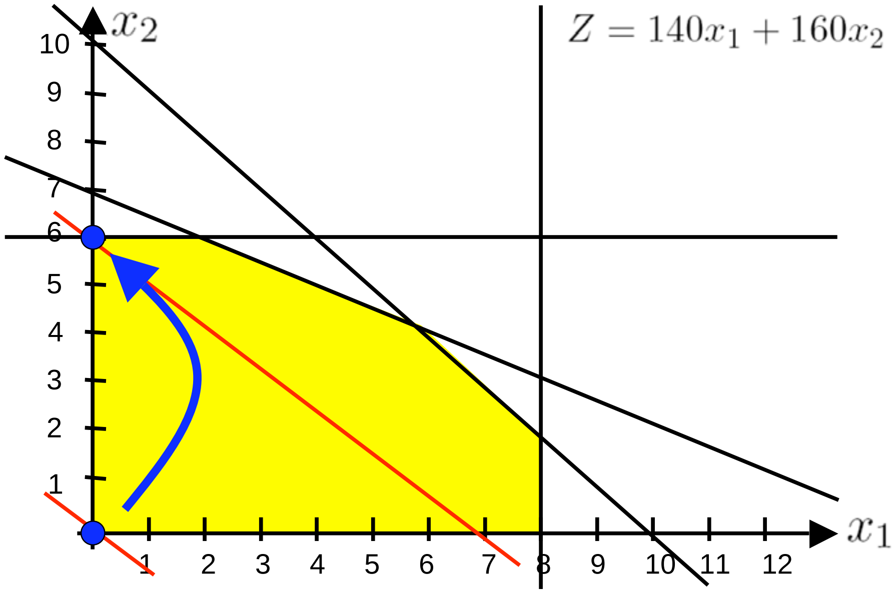

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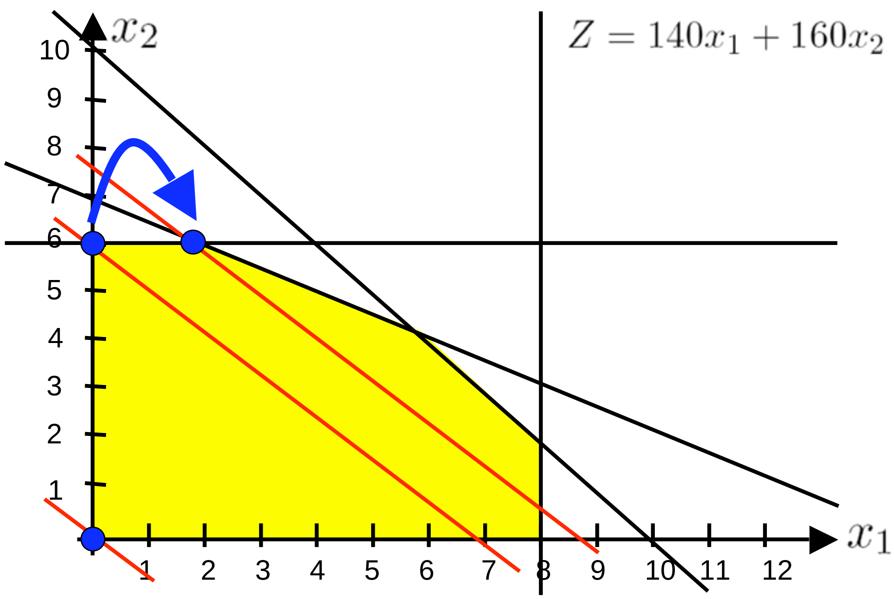

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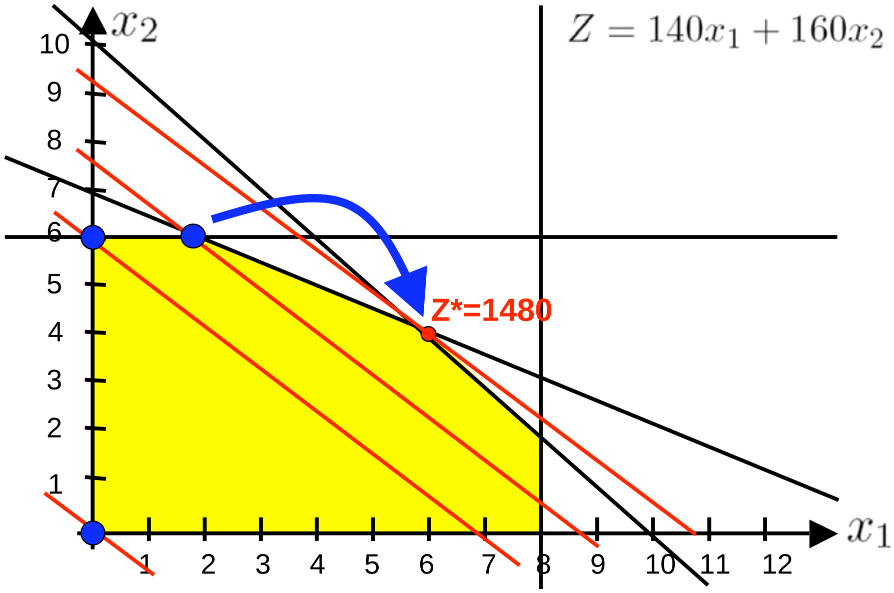

Engineering algorithms

As an engineer, you formulate a physical problem in a known format (for example a linear program). Then, you can use an algorithm to solve it (either your own, or a toolbox, for example using MATLAB).

BEGIN
  input: guess, starting point
  DO
    repeat simple task
  UNTIL
    stopping criterion is satisfied
END