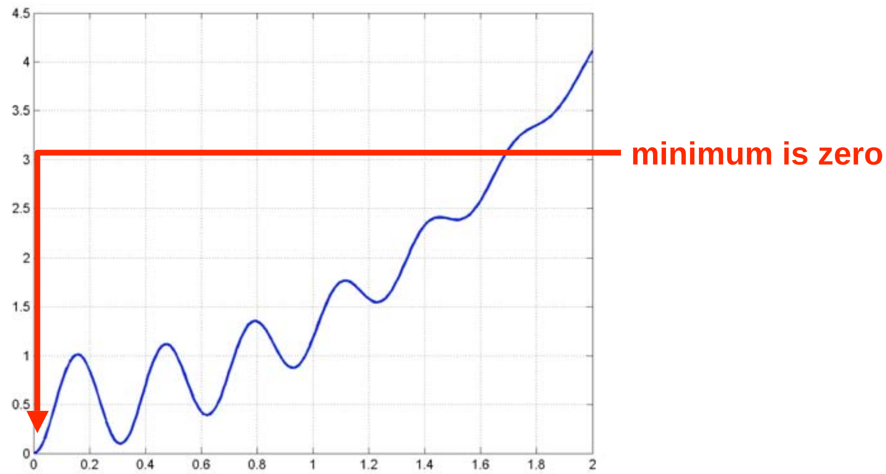
Lecture 11: constrained nonlinear optimization

- Fundamental problem: violation of the constraints
- Barrier functions, properties of the barriers
- Logarithmic barriers
- Constrained optimization algorithm
- Illustration of the algorithm
- Formal description of the algorithm
- Generalization of the algorithm to multiple dimensions

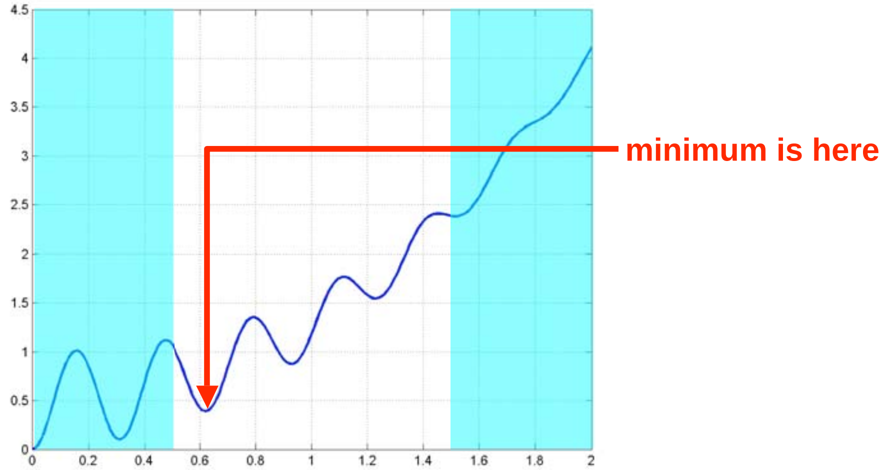
Constrained vs. unconstrained optimization

Example: find the optimum of the following function within the range [0,+∞)



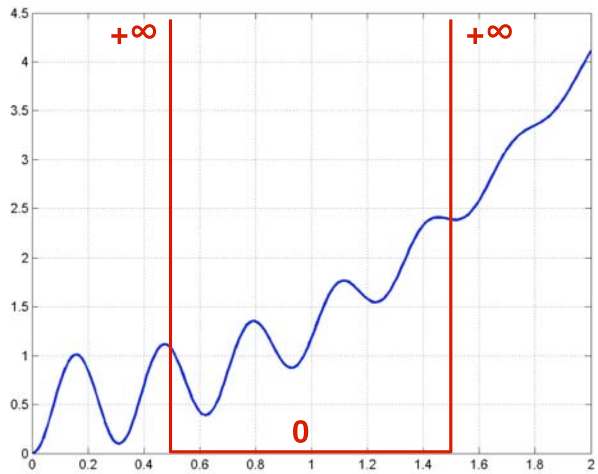
Constrained vs. unconstrained optimization

Example: find the optimum of the following function within the range [0.5 , 1.5]



Main idea of barrier methods

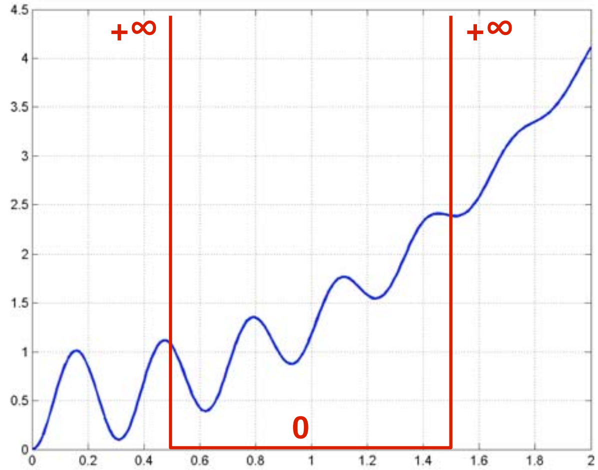
Add a barrier function which is infinite outside of the constraint domain, i.e. [a,b]



$$\begin{aligned} \textbf{min:} \quad & f(x) \\ \textbf{s.t.} \quad & x \leq b \\ & x \geq a \end{aligned}$$

Main idea of barrier methods

In practice, such functions do not exist, so they have to be approximated by acceptable functions



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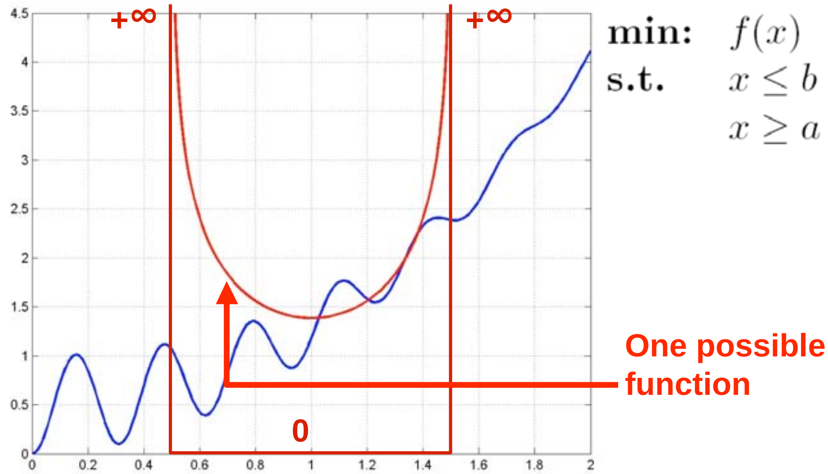

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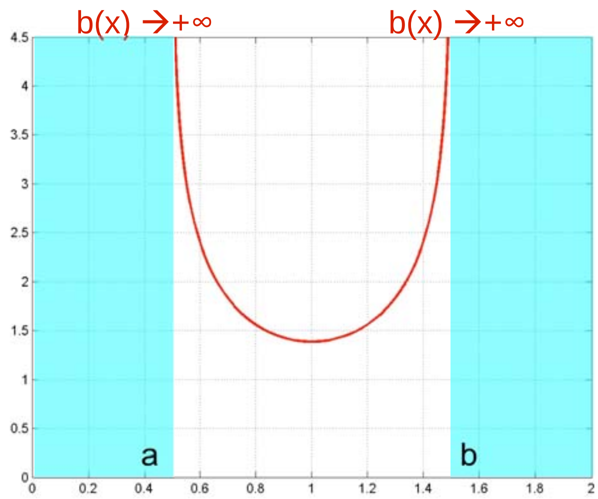
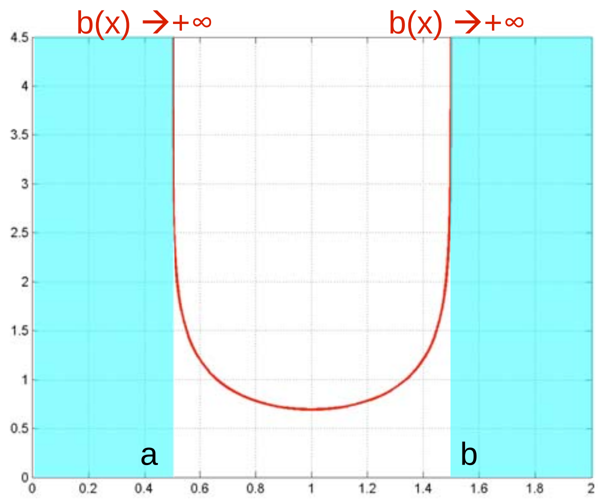
Logarithmic barrier

$b(x) = - \varepsilon \log((x-a)(b-x)),$ $\varepsilon=1$



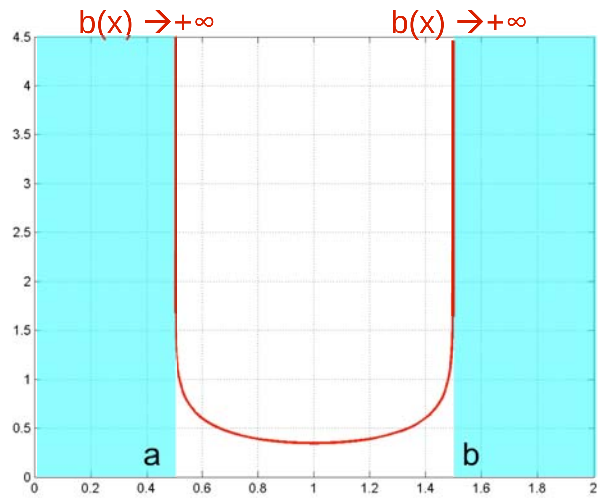
An interesting property of barriers

$b(x) = - \varepsilon \log((x-a)(b-x)),$ $\varepsilon=1/2$



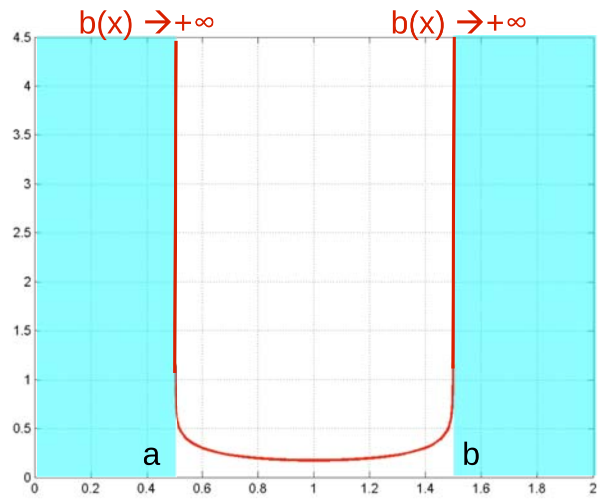
An interesting property of barriers

b(x) = - ε log((x-a)(b-x)), ε=1/4



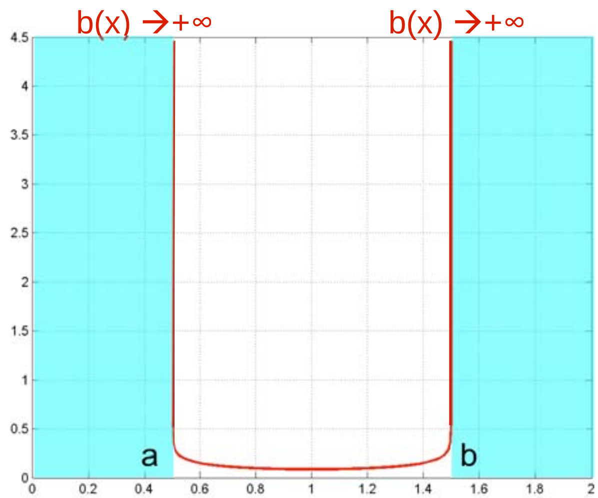
An interesting property of barriers

b(x) = - ε log((x-a)(b-x)), ε=1/8



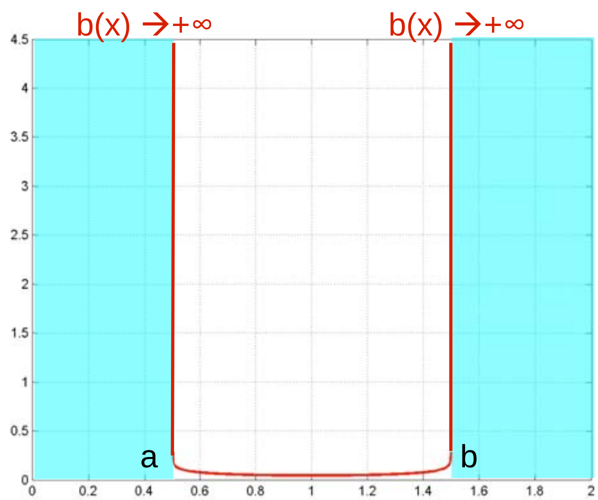
An interesting property of barriers

b(x) = - ε log((x-a)(b-x)),   ε=1/16



An interesting property of barriers

b(x) = - ε log((x-a)(b-x)),   ε=1/32



Utilization of the barrier functions

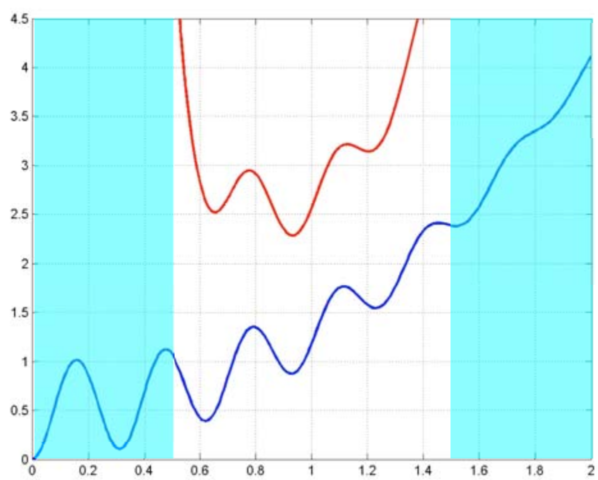
Idea of the barrier function:

- add the barrier and the function: this is called the augmented function
  1) inside the constraint set, barrier ~ 0
  2) outside the constraint set, barrier is infinite

- if the barrier is almost zero inside the constraint set, the minimum of the function and the augmented function are almost the same.

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1$

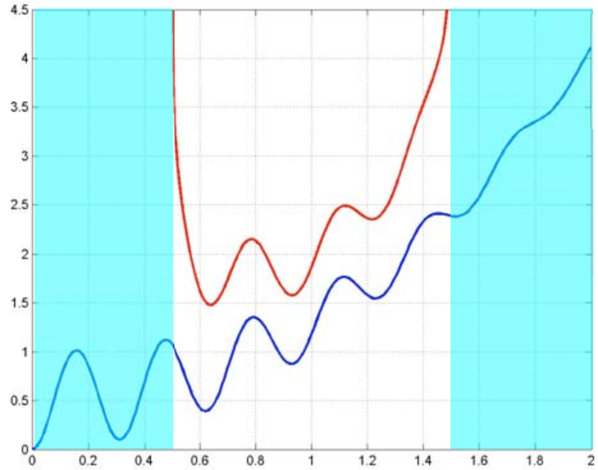


$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \in [a, b] \end{array}$$

$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \le b \\ & x \ge a \end{array}$$

Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/2$



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Logarithmic barrier: $\varepsilon = 1/4$

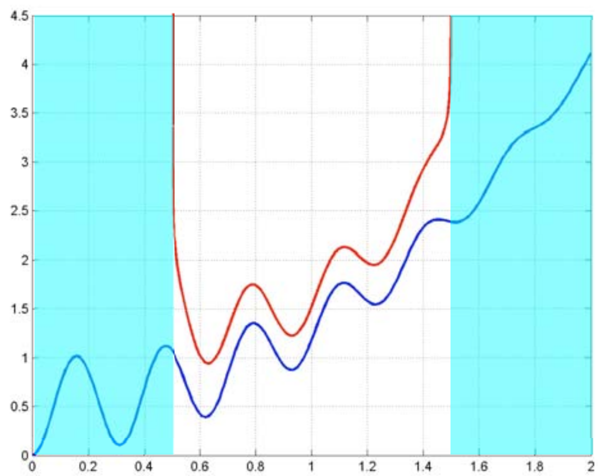

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Illustration of the convergence of the log barrier

Logarithmic barrier: $\varepsilon = 1/8$

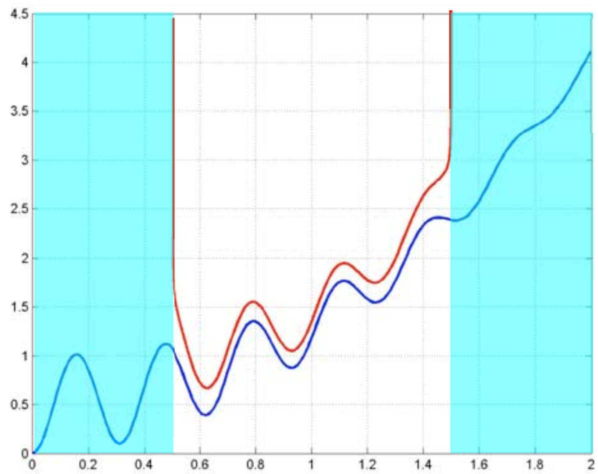

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Illustration of the convergence of the log barrier

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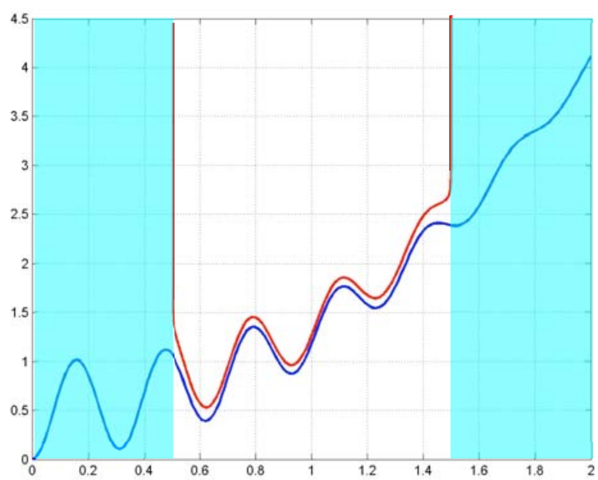

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Illustration of the convergence of the log barrier

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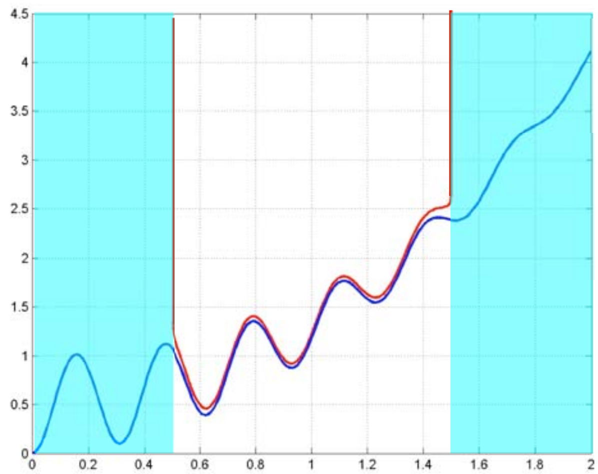

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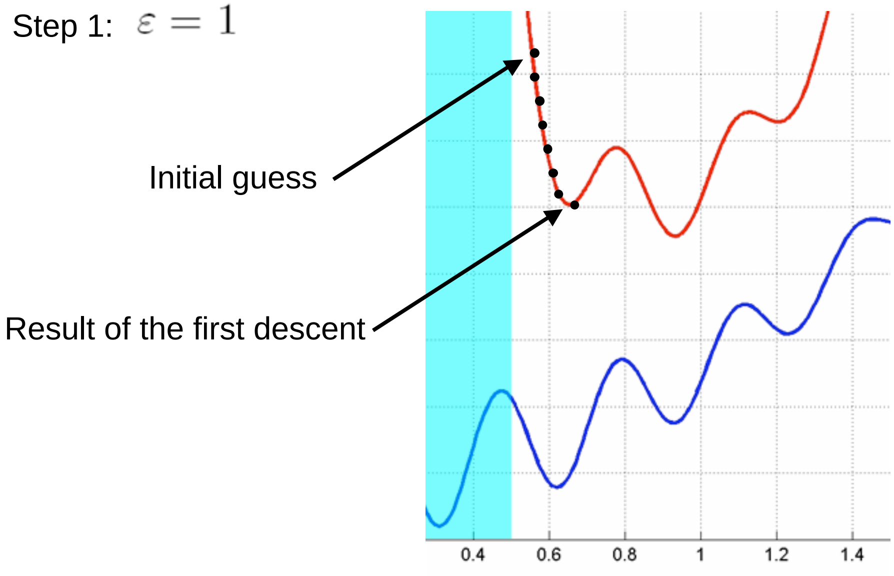
Illustration of the convergence of the log barrier

Make a guess inside the constraint set
Start with epsilon not too small
repeat
    minimize the augmented function (using previous chapter)
    use the result as the guess for the next step
    decrease the log barrier
Until barrier is almost zero inside the constraint set

One can prove that the result of this method converges to a minimum of the original problem

Illsustration of the algorithm

Step 1: $\varepsilon = 1$



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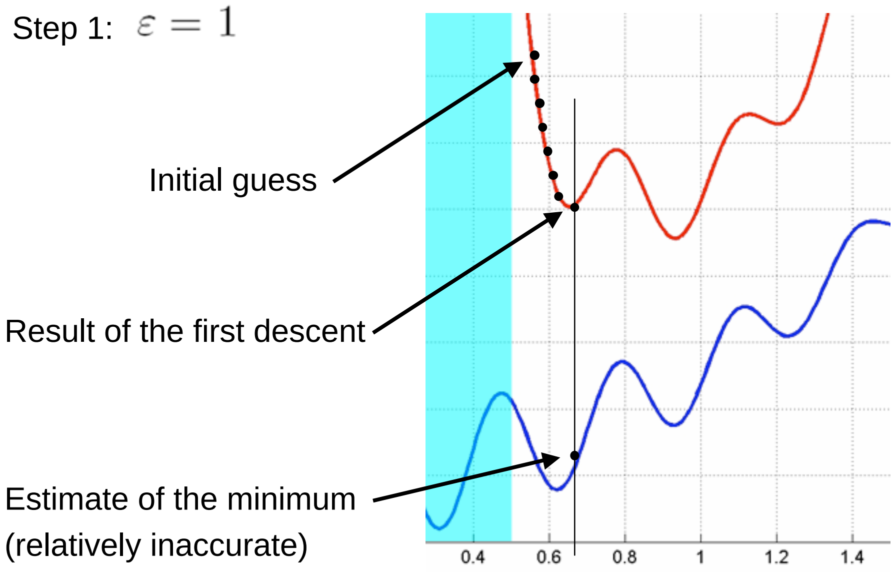


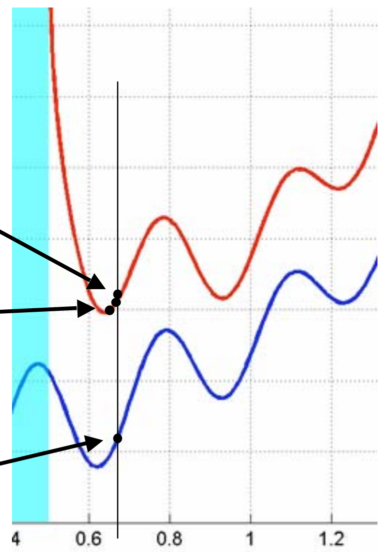
Illsustration of the algorithm

Step 2: $\varepsilon = 1/2$

New guess
(old result)

Result of the
second descent

Estimate of the minimum
(relatively inaccurate)



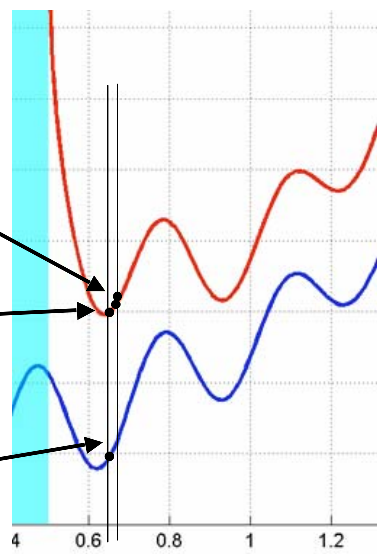
Illsustration of the algorithm

Step 2: $\varepsilon = 1/2$

New guess
(old result)

Result of the
second descent

New estimate of the
minimum (a little better)



Illsustration of the algorithm

Step 2: $\varepsilon = 1/4$

New guess (old result)

Result of the second descent

New estimate of the minimum (a little better)

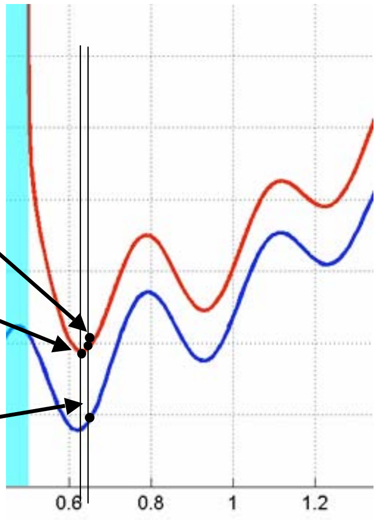


Illustration of the algorithm

Step 2: $\varepsilon = 1/4$

New guess (old result)

Result of the second descent

New estimate of the minimum (a little better)

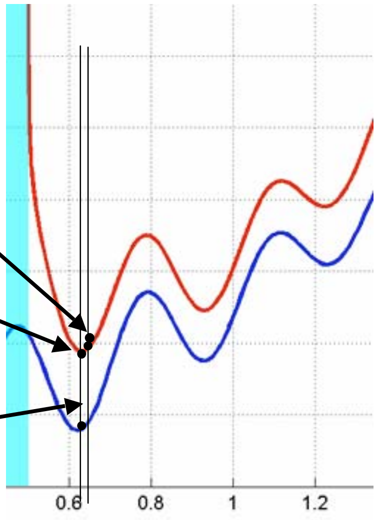


Illustration of the convergence of the algorithm

Logarithmic barrier: $\varepsilon = 1$

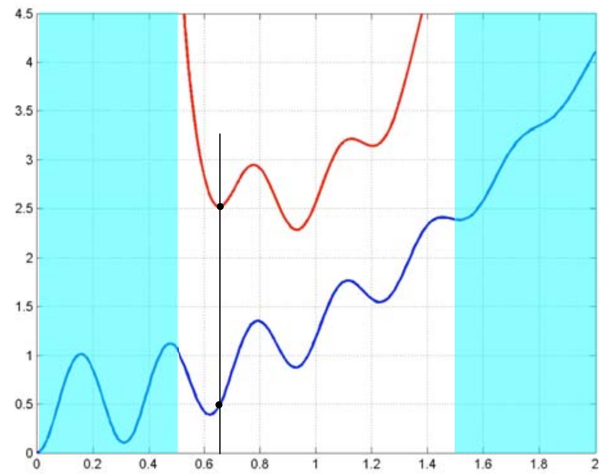


Illustration of the convergence of the algorithm

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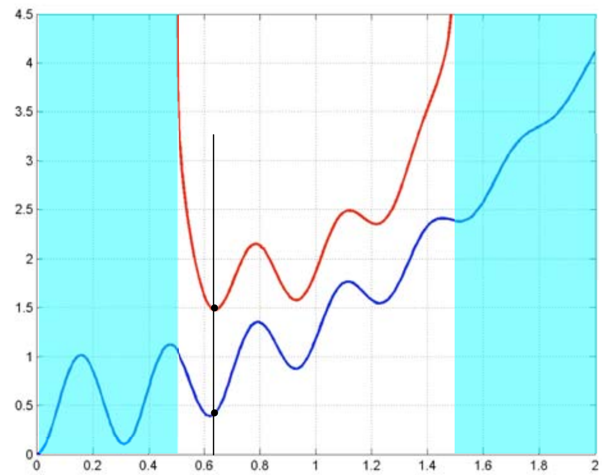

min: $f(x)$
s.t. $x \in [a, b]$

min: $f(x)$
s.t. $x \le b$
$x \ge a$

Illustration of the convergence of the algorithm

Logarithmic barrier: $\varepsilon = 1/4$

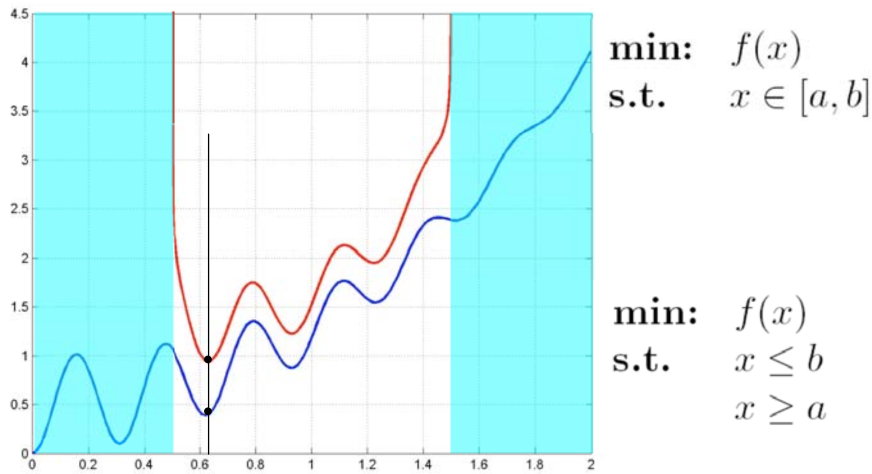

Illustration of the convergence of the algorithm

Logarithmic barrier: $\varepsilon = 1/8$

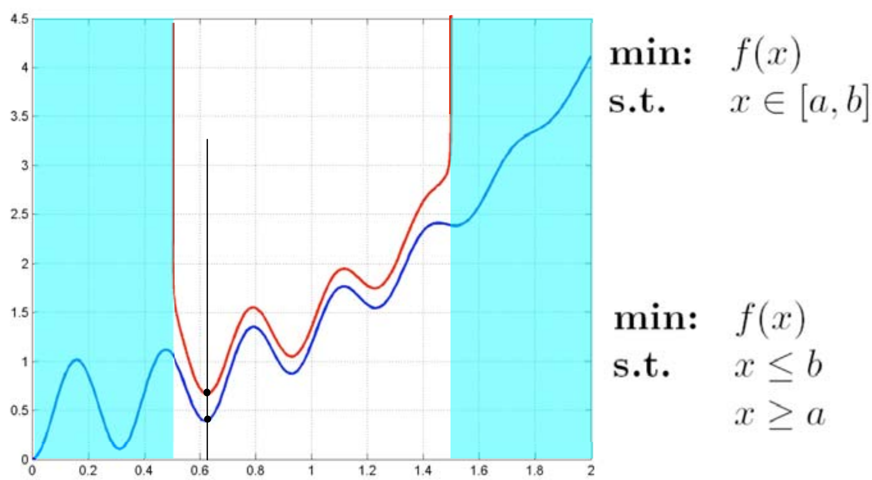

Illustration of the convergence of the algorithm

Logarithmic barrier: $\varepsilon = 1/16$

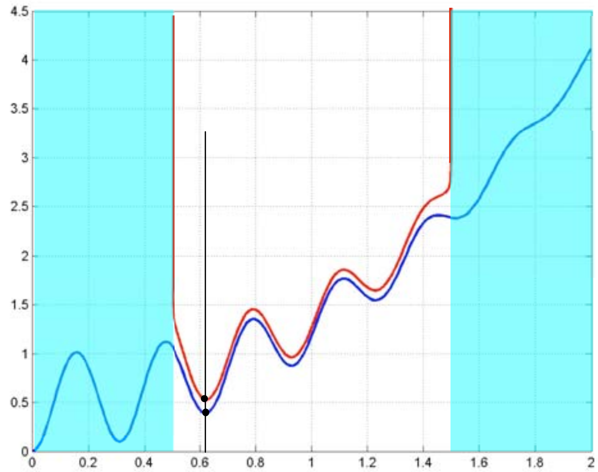

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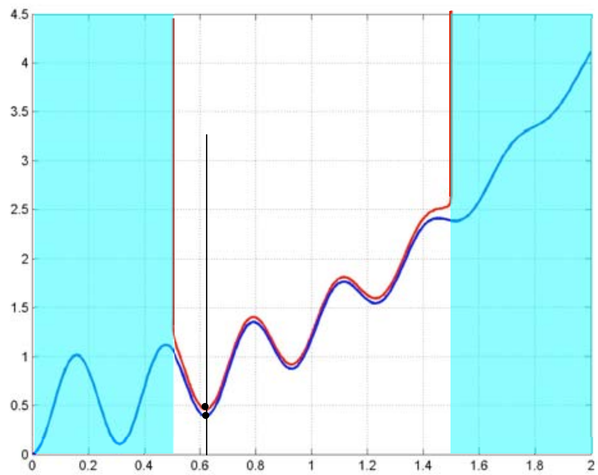
Illustration of the convergence of the algorithm

Logarithmic barrier: $\varepsilon = 1/32$



$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \in [a, b] \end{array}$$

$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & x \leq b \\ & x \geq a \end{array}$$

Formal description of the algorithm

Start with epsilon not too small

repeat

solve $$\begin{array}{ll} \textbf{min:} & f(x) - \varepsilon b(x) \\ \textbf{s.t.} & \text{no constraints} \end{array}$$

use the result as the guess for the next step

decrease the log barrier $\varepsilon := \varepsilon / 2$ or similar

Until barrier is almost zero inside the constraint set

Generalization to multiple dimensions

Transformation of a constrained problem into an unconstrained problem

$$\begin{array}{ll} \textbf{min:} & f(x) \\ \textbf{s.t.} & g(x) \leq 0 \end{array}$$

Introduce logarithmic barrier

$$b(x) = -\log(-g(x))$$

Problem to solve becomes (in the limit ε goes to zero)

$$\begin{array}{ll} \textbf{min:} & f(x) - \varepsilon b(x) \\ \textbf{s.t.} & \text{no constraints} \end{array}$$