Viability-Based Computations of Solutions to the Hamilton-Jacobi-Bellman Equation

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Abstract. This article proposes a new capture basin algorithm for computing the numerical solution of a class of *Hamilton-Jacobi-Bellman* (HJB) *partial differential equations* (PDEs) [3], based on a Lax-Hopf formula [2]. The capture basin algorithm is derived and implemented to perform numerical computations. Its performance is measured with highway data obtained for interstate I80 in California.

Assumptions. We posit the following assumptions

1. A concave function $\psi : X \mapsto \mathbb{R}$ on $[0,\omega]$, which vanishes at 0 and ω, equal to $\psi'(0)v$ for $v \leq 0$ and $\psi'(\omega)(\omega - v)$ for $v \geq \omega$.
2. A bounded continuous function $v : \mathbb{R}_+ \mapsto \mathrm{Dom}(\psi)$,
3. An upper semicontinuous initial datum $\mathbf{N}_0 : X \mapsto \mathbb{R}_+$. We set $\mathbf{N}_0(0,x) := \mathbf{N}_0(x)$ and $\mathbf{N}_0(t,x) := -\infty$ if $t > 0$.
4. A Lipschitz function $\mathbf{b} : \mathbb{R}_+ \times X \mapsto \mathbb{R} \cup \{-\infty\}$ setting the upper constraint.

We set $\forall\, x \in \partial K,\ \ \forall\, t \geq 0,\ \ \gamma(t,\xi) \ := \ 0$ and $\forall\, x > \xi,\ \ \gamma(t,x) \ = \ -\infty$. This is required to satisfy consistency assumptions

$$\begin{cases} (i)\ \ \forall\, t \geq 0,\ \forall\, x \in K,\ \ \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \ \leq \ \mathbf{b}(t,x) \\ (ii)\ \forall x \in K,\ \ \mathbf{N}_0(x) \ \leq \ \inf\limits_{s \geq 0} \left(\dfrac{x-\xi}{s} \displaystyle\int_0^s v(\tau) d\tau \right) \end{cases} \tag{1}$$

When the function $v(\cdot) \equiv v$ is constant, condition (1)(ii) boil down to

$$\forall\, x \geq \xi,\ \ \mathbf{N}_0(x) \ \leq \ v(x-\xi)$$

Problem statement. Under the above mentioned assumptions, that are assumed all along this paper, we shall solve the existence of a solution to the *non-homogenous HJB PDE:*

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$$\forall\, t > 0,\;\; x \in \mathrm{Int}(K),\;\; \frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \;=\; \psi(v(t)) \tag{2}$$

satisfying the *initial and Dirichlet conditions*

$$\begin{cases} (i)\;\; \forall\, x \in K,\;\; \mathbf{N}(0,x) = \mathbf{N}_0(x) \text{ (initial condition)} \\ (ii)\; \forall\, t \geq 0,\; \mathbf{N}(t,\xi) = 0 \text{ (Dirichlet boundary condition)} \end{cases} \tag{3}$$

and the user defined *viability constraints.*

$$\forall\, t \geq 0,\; x \in K,\;\; \mathbf{N}(t,x) \;\leq\; \mathbf{b}(t,x) \text{ (upper inequality constraint)} \tag{4}$$

Flux functions. The assumption that the flux function ψ is concave and upper semicontinuous plays a crucial role for defining the viability hyposolution. Indeed, since ψ is concave, the function $\varphi(p) := -\psi(p)$ is convex and its Fenchel transform is defined by:

$$\varphi^*(u) \;:=\; \sup_{p\in \mathrm{Dom}(\varphi)} [p\cdot u - \varphi(p)] \;=\; \sup_{p\in \mathrm{Dom}(\psi)} [p\cdot u + \psi(p)] \tag{5}$$

Recall that the fundamental theorem of convex analysis states that $\varphi = \varphi^{**}$ if and only if φ is convex, lower semicontinuous, and non trivial (i.e. $\mathrm{Dom}(\varphi) := \{p \mid \varphi(p) < +\infty\} \neq \emptyset$). Therefore we can recover the function ψ from φ^* by

$$\psi(p) \;:=\; \inf_{u\in \mathrm{Dom}(\varphi^*)} [\varphi^*(u) - p\cdot u] \tag{6}$$

Proposition 1. *Let us consider a concave flux function ψ_0 defined on a neighborhood of the interval $[0,\omega]$ and satisfying $\psi_0(0) = \psi_0(\omega) = 0$. We assume for simplicity that ψ is differentiable at 0 and ω, and we set $\nu^\flat = \psi'(0) \geq 0$ and $\nu^\sharp = -\psi'(\omega) \geq 0$ We associate with it the continuous concave function ψ:*

$$\psi(p) \;=\; \begin{cases} \nu^\flat p & \text{if } p \leq 0 \\ \psi_0(p) & \text{if } p \in [0,\omega] \\ \nu^\sharp(\omega - p) & \text{if } p \geq \omega \end{cases}$$

Then the Fenchel transform φ^* is bounded above, and its domain $Dom(\varphi^*) = [-\nu^\flat, +\nu^\sharp]$ is bounded:

$$\varphi^*(u) \;=\; \begin{cases} \varphi_0^*(u) & \text{if } u \in [-\nu^\flat, +\nu^\sharp] \\ +\infty & \text{if } u \notin [-\nu^\flat, +\nu^\sharp] \end{cases}$$

Viability hyposolution of the HJB equation [1]**.** We define a target $\mathcal{C} := \mathcal{H}yp(\mathbf{c})$ as the subset of triples $(T,x,y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{c}(T,x)$ (which is the *hypograph* of the function $\mathbf{c}$), where the function $\mathbf{c}(t,x)$ is defined (here) by:

$$\mathbf{c}(t,x) \;:=\; \begin{cases} -\infty & \text{if } t > 0 \text{ and } x > \xi \\ \mathbf{N}_0(x) & \text{if } t = 0 \text{ and } x \geq \xi \\ 0 & \text{if } t \geq 0 \text{ and } x = \xi \end{cases}$$

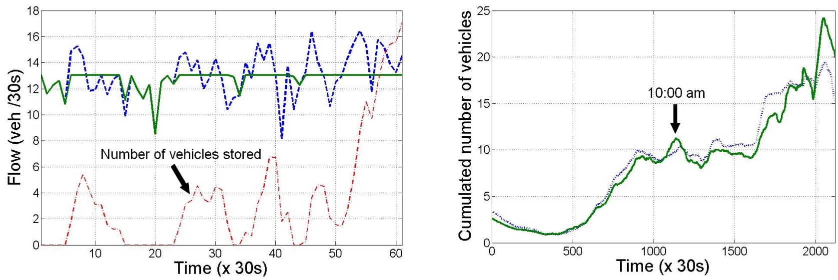

Fig. 1. Left: Flow-time curves. The (actual) measured inflow on the boundary $x = \xi$ is represented by the dashed curve. The continuous curve shows the simulated inflow through boundary $x = \xi$, taking into account the highway capacity. The number of corresponding (remaining) vehicles stored at the $x = \xi$ boundary is shown on the dash dotted curve. **Right:** Comparison between experimental values and simulated values for the cumulated vehicle number $\mathbf{N}(t, \xi + L)$ between ξ and L.

The environment $\mathcal{K} := \mathcal{H}yp(\mathbf{b})$ is the subset of triples $(T, x, y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{b}(T, x)$, which is a user-defined function (this is the *hypograph* of the function $\mathbf{b}$). We define the auxiliary control system :

$$\begin{cases} \tau'(t) &= -1 \\ x'(t) &= u(t) \\ y'(t) &= \varphi^*(u(t)) - \psi(v(\tau(t))) \end{cases} \qquad \text{where } u(t) \in [-\nu^\flat, +\nu^\sharp] \tag{7}$$

where φ^* is the Fenchel conjugate function of ψ, as defined previously. To be rigorous, we have to mention *once and for all* that the controls $u(\cdot)$ are measurable integrable functions with values in $\text{Dom}(\varphi^*)$, and thus, ranging $L^1(0, \infty; \text{Dom}(\varphi^*))$, and that the above system of differential equations is valid for almost all $t \geq 0$.

Definition 1. ***The Viability Hyposolution.*** *The capture basin* $\text{Capt}_{(7)}(\mathcal{K}, \mathcal{C})$ *of a target* $\mathcal{C}$ *viable in the environment* $\mathcal{K}$ *under control system (7) is the subset of initial states* (t, x, y) *such that there exists a measurable control* $u(\cdot)$ *such that the associated solution*

$$s \mapsto \left(t - s, x + \int_0^s u(\tau) d\tau, y + \int_0^s (\varphi^*(u(\tau)) - \psi(v(t - \tau))) d\tau \right)$$

to system (7) is viable in $\mathcal{K} = \mathcal{H}yp(\mathbf{b})$ *until it reaches the target* $\mathcal{C} = \mathcal{H}yp(\mathbf{c})$*. The* viability hyposolution $\mathbf{N}$ *is defined by*

$$\mathbf{N}(t, x) := \sup_{(t,x,y) \in \text{Capt}_{(7)}(\mathcal{K}, \mathcal{C})} y \tag{8}$$

Theorem 1. Non-homogenous Dirichlet/Initialvalue Problem with inequality contraints. *The viability hyposolution* $\mathbf{N}$ *defined by (8) is the* largest *upper semicontinuous solution to Hamilton-Jacobi equation (2) satisfying initial and Dirichlet conditions (3) and inequality constraints (4). If the functions ψ, φ^* and v are furthermore Lipschitz, then the viability hyposolution* $\mathbf{N}$ *is its* unique *upper semicontinuous solution in both the contingent Frankowska sense and in the Barron-Jensen/Frankowska sense.*

Other numerical illustrations of the article [2], such as a sup-morphism property, have been carried out using the viability algorithm. The results can be compared with explicit analytical solutions obtained from Lax-Hopf formulas extended to the case of boundary-value problems. The viability kernel algorithm [4] is adapted to the case in which the target $\mathcal{C}$ and the environment $\mathcal{K}$ are hypographs, which allows us to take some specificities of the problem into account. An example of a solution computed with the algorithm is provided in Figure 2. In this Figure, one can see two computations: one of an unconstrained solution (thick line), and one of a constrained solution (thin lines). A cap of $\mathbf{b}(t,x) = 40$ is imposed on the constrained solution, as can be seen in this Figure. From Figure 2, one can observe that the solution with constraints is not the supremum of the solution without constraints and the function $\mathbf{b}$.

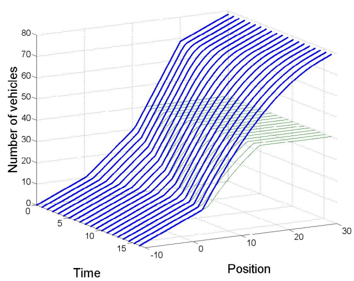

Fig. 2. Plots of $\mathbf{N}(t,x)$ versus t and x. The unconstrained solution is represented by a thick line. The thin line represents the constrained solution of the same system (we set $\mathbf{b}(t,x) = 40$ in this example). Result obtained using a Greenshields [2] flux function.

We use the same experimental set up as in earlier work [5] to assess the performance of the algorithm with highway traffic data: we use three loop detectors in interstate I80 at Emeryville. When the measured inflow (upstream) exceeds the modeled highway capacity (because of noise in the measurements or model inaccuracy), we "store" the corresponding vehicles at $x = \xi$ until they can be released into the highway. The resulting curves are shown in Figure 1 (left): the cutoff happens at $\psi(v(t)) = \delta$ above which the vehicles have to be stored until the highway capacity allows them to enter at $x = \xi$. In this Figure, all numbers of vehicles are per lane. The corresponding number of "stored" vehicles is shown in the same subfigure, and the corresponding $\mathbf{N}(t, \xi + L)$ curves are shown in the right subfigures, where L represents the length of the corresponding highway stretch. The evolution $\mathbf{N}(t, \xi + L)$ thus represents the evolution of the cumulated number of vehicles between ξ and $\xi + L$ as a function of time. Both simulated and measured curves are represented on this plot and show remarkable agreement. The differences between simulation and theory are mainly linked with uncertainties on the numerical values of the parameters of the model.

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