Mixed Initial-Boundary Value Problems for Scalar Conservation Laws: Application to the Modeling of Transportation Networks

Issam S. Strub and Alexandre M. Bayen

Civil Systems, Department of Civil and Environmental Engineering,
University of California, Berkeley, CA, 94720-1710
strub@ce.berkeley.edu
Civil Systems, Department of Civil and Environmental Engineering,
University of California, Berkeley, CA, 94720-1710
bayen@ce.berkeley.edu
Tel.: (510)-642-2468; Fax:(510)-643-5264

Abstract. This article proves the existence and uniqueness of a weak solution to a scalar conservation law on a bounded domain. A weak formulation of hybrid boundary conditions is needed for the problem to be well posed. The boundary conditions are represented by a hybrid automaton with switches between the modes determined by the direction of characteristics of the system at the boundary. The existence of the solution results from the convergence of a Godunov scheme derived in this article. This weak formulation is written explicitly in the context of a strictly concave flux function (relevant for highway traffic). The numerical scheme is then applied to a highway scenario with data from the I210 highway obtained from the California PeMS system. Finally, the existence of a minimizer of travel time is obtained, with the corresponding optimal boundary control.

Keywords: Weak solution of scalar conservation laws, Weak hybrid boundary conditions, LWR PDE, Highway traffic modeling, Boundary control.

1 Introduction

This article is motivated by recent research efforts which investigate the problem of controlling highway networks with metering strategies that can be applied at the on-ramps of the highway (see in particular [46] and references therein). The seminal models of highway traffic go back to the 1950's with the work of Lighthill-Whitham [36] and Richards [43] who tried to use fluid dynamics equations to model traffic flow. The resulting theory, called *Lighthill-Whitham-Richards* (LWR) theory relies on a scalar hyperbolic conservation law, with a concave flux function. Very few approaches have tackled the problem of boundary control of scalar conservation laws in bounded domains in an explicit manner directly applicable for engineering. Unlike the viscous Burgers equation, which has been is focus of numerous ongoing studies, very few results exist for the

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inviscid Burgers equation, which is traditionally used as a model problem for hyperbolic conservation laws. Differential flatness [42] and Lyapunov theory [30] have been explored and appear as promising directions to investigate.

The proper notion of weak solution for the LWR partial differential equation (PDE), called *entropy solution* was first defined by Oleinik [39] in 1957. Even though this work was known to the traffic community, it does not (as far as we know) appear explicitly in the transportation literature before the 1990's with the work of Ansorge [4]. The entropy solution has been since acknowledged as the proper weak solution to the LWR PDE [18] for traffic models. Unfortunately the work of Oleinik in its initial form [39] does not hold for bounded domains, i.e. it would only work for infinitely long highways with no on-ramps or off-ramps. Bounded domains, i.e. highways of finite length (required to model on and off-ramps) imply the use of boundary conditions, for which the existence and uniqueness of a weak solution is not straightforward.

The first result of existence and uniqueness of a weak solution of the LWR PDE in the presence of boundary conditions follows from the work of Bardos, Leroux and Nedelec [8], in the more general context of a first order quasilinear PDE on a bounded open set of $\mathbb{R}^n$. In particular, they introduce a weak formulation of the boundary conditions for which the initial-boundary value problem is well-posed.

We begin this article by explaining that in general, one cannot expect the boundary conditions to be fulfilled pointwise a.e. and we provide several examples to illustrate this fact. We then turn to the specific case of highway traffic flow, for which we are able to state a simplified weak hybrid formulation of the boundary conditions, and prove the existence and uniqueness of a weak solution to the LWR PDE, the former resulting from the convergence of the associated Godunov scheme to the entropy solution of the PDE. This represents a major improvement from the existing traffic engineering literature, where boundary conditions are expected to be fulfilled pointwise and therefore existence of a solution and convergence of the numerical schemes to this solution are not guaranteed. We illustrate the applicability of the method and the numerical scheme developed in this work with a highway scenario, using data for the I210 highway, obtained from the California PeMS system. In particular, we show that the model is able to reproduce flow variations on the highway with a good accuracy over a period of five hours. The last part of the article is devoted to the boundary control of the LWR PDE and its application to a highway optimization problem, in which boundary control is used to minimize travel time on a given stretch of the highway.

2 The Need for a Weak Formulation of Hybrid Boundary Conditions

This section shows three examples of the sort of trouble one runs into when prescribing the boundary conditions in the strong sense. Numerous articles solve a discrete version of this type of problems. Regardless of the numerical schemes used (Godunov [22], Jameson-Schmidt-Turkel [25, 26], Daganzo [17, 18]), these methods suffer from the same difficulties: the authors solve a discrete problem with strong boundary conditions which entails that the corresponding continuous problem is usually ill-posed, i.e. does not have a solution. While the numerical schemes listed above might still yield a numerical output, this numerical data would be meaningless since the initial boundary-value problem does not have a solution in the first place. The object of this work is not to make an endless list of engineering articles which exhibit such shortcomings: we will just mention a previous paper from one of the authors [9] and let the reader discover that this is far from being an exception... To sum up, boundary conditions may only be prescribed on the part of the boundary where the characteristics are incoming, that is entering the domain.

Example 1: Advection equation. We start by considering the simple example where the propagation speed is a constant c,

$$\frac{\partial \rho}{\partial t} + c\frac{\partial \rho}{\partial x} = 0 \text{ for } (x,t) \in (a,b) \times (0,T).$$

In that case, one can clearly see that the boundary condition is either prescribed on the left ($x = a$) if the speed c is positive or the right ($x = b$) if the speed is negative. While finding the sign of the speed is quite simple in the linear case, it becomes more subtle when dealing with a nonlinear conservation law such as the LWR PDE as this sign is no longer constant.

Example 2: LWR PDE, shock wave back-propagation due to a bottleneck. For this example, we consider the LWR PDE with a Greenshields flux function [24]:

$$\frac{\partial \rho}{\partial t} + v\left(1 - \frac{2\rho}{\rho^*}\right)\frac{\partial \rho}{\partial x} = 0 \tag{1}$$

where $\rho = \rho(x,t)$ is the vehicle density on the highway, ρ^* is the *jam density* and v is the *free flow density* (see [17, 18] for more explanations on the interpretation of these parameters). We consider a road of length $L = 30$, $\rho^* = 4$ and $v = 1$ (dummy values), and an initial density profile given by $\rho_0(x) \triangleq \rho(x,0) = 2$ if $x \in [0,10]$, $\rho_0(x) \triangleq \rho(x,0) = 4$ if $x \in (10,20]$, $\rho_0(x) \triangleq \rho(x,0) = 1$ if $x > 20$. The highway might be bounded or unbounded on the right at $x = L = 30$ (it does not matter for our problem). We assume free flow conditions at $x = L$, that we can control the inflow at $x = 0$, and we try to prescribe it pointwise, i.e. $\rho(0,t) = 2$ for all t (this corresponds to sending the maximum flow onto the highway). The solution to this problem can easily be computed by hand (for example by the method of characteristics, see Figure 1, left). The solution to this problem reads

$$\begin{cases} \rho(x,t) = 2 & \text{if } t \leq 2(10-x) & \text{AC: shock} \\ \rho(x,t) = 4 & \text{if } 2(10-x) \leq t \leq 20-x & \text{BC: left edge of exp. wave} \\ \rho(x,t) = 2(1-(x-20)/t) & \text{if } t \geq \max\{20-x, 2(x-20)\} & \text{CBD is an expansion wave} \\ \rho(x,t) = 1 & \text{if } t \leq 2(x-20) & \text{BD: right edge of exp. wave} \end{cases}$$

As can be seen, $\lim_{x\to 0^+} \rho(x,t) = 2$ for $t \leq 20$ and $\lim_{x\to 0^+} \rho(x,t) = 2(1+20/t)$ for $t > 20$. Thus, the boundary condition $\rho(0,t) = 2$ is no longer verified as soon

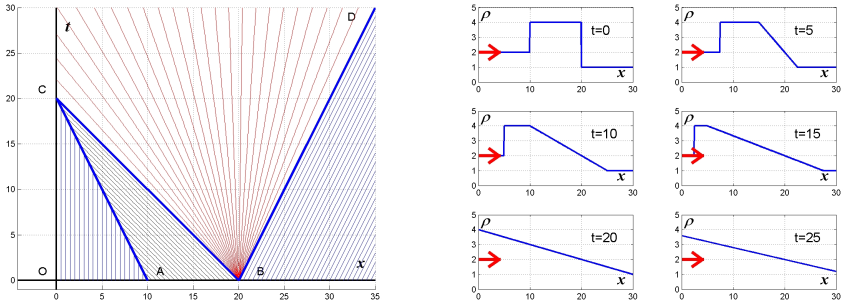

Fig. 1. Left: Characteristics for the solution of the LWR PDE for Example 2. **Right:** corresponding value of the solution at successive times. The arrow represents the value of the input at $x = 0$, which becomes irrelevant for $t \geq 20$.

as $t \geq 20$. This phenomenon is crucial in traffic flow models: it represents the back-propagation of congestion (i.e. upstream). If the location $x = 0$ was the end of a link merging into the highway (that we could potentially control), the case when $\rho(0^+, t) > \rho^*$ is congested would correspond to a situation in which the upstream flow ($x = 0^-$) is imposed by the downstream flow ($x = 0^+$), i.e. the boundary condition on the left becomes irrelevant. When $\rho(0^+, t) < \rho^*$ is not congested, the boundary condition is relevant and can be imposed pointwise.

Example 3: Burgers equation. We now consider the inviscid Burgers equation on $(0, 1) \times (0, T)$. If we try to prescribe strong boundary conditions at both ends, the problem becomes ill-posed. Burgers equation reads:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \tag{2}$$

The initial value is $u(x, 0) = 1$, and the boundary conditions $u(0, t) = u(1, t) = 0$ on $[0, 1]$. The solution of (2) with these boundary conditions is for $t < 1$:

$$\begin{cases} u(x,t) = \frac{x}{t} \text{ if } x < t \text{ self similar expansion wave} \\ u(x,t) = 1 \ \text{ if } x > t \text{ convection to the right with speed } 1 \end{cases}$$

We notice that the boundary condition is not satisfied at $x = 1$. Since the data propagates at speed u, they are leaving $[0, 1]$ at $x = 1$ while they stay in $[0, 1]$ as a rarefaction wave at $x = 0$.

3 Traffic Flow Equation with Hybrid Boundary Conditions

We consider a mixed initial-boundary value problem for a scalar conservation law on $(a, b) \times (0, T)$.

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = 0 \tag{3}$$

with the initial condition

$$\rho(x,0)=\rho_0(x) \text{ on } (a,b)$$

and the boundary conditions

$$\rho(a,t)=\rho_a(t) \text{ and } \rho(b,t)=\rho_b(t) \text{ on } (0,T).$$

As usual with nonlinear conservation laws, in general there are no smooth solutions to this equation and we have to consider weak solutions (see for example [10], [19], [45]). In this article we use the space BV of functions of bounded variation which appears very often when dealing with conservation laws. A function of bounded variation is a function in L^1 such that its weak derivative is uniformly bounded. We refer the intrigued readers to the book from Ambrosio, Fusco and Pallara [1] for many more properties and applications of BV functions. Other valuable references on BV functions include the article by Vol'pert [48] and the book from Evans and Gariepy [20].

In our problem, we make the assumption that the flux q is continuous and that the initial and boundary conditions ρ_0, ρ_a, ρ_b are functions of bounded variation. When the flux q models the flux of cars in terms of the car density ρ we obtain the LWR PDE. As explained earlier on, boundary conditions may not be fulfilled pointwise a.e., thus following [8], we shall require that an entropy solution of (3) satisfy a weak formulation of the boundary conditions:

$$L(\rho(a,t),\rho_a(t))=0 \text{ and } R(\rho(b,t),\rho_b(t))=0$$

where

$$L(x,y)=\sup_{k\in I(x,y)}(sg(x-y)(q(x)-q(k)) \text{ and}$$
$$R(x,y)=\inf_{k\in I(x,y)}(sg(x-y)(q(x)-q(k)) \text{ for } x,y\in\mathbb{R}$$

and $I(x,y)=[\inf(x,y),\sup(x,y)]$ with sg denoting the sign function. In the case of a strictly concave flux (such as the Greenshields [24] and Greenberg [23] models used in traffic flow modeling), the boundary conditions can be written as (Le Floch gives analogous conditions in the case of a strictly convex flux in [33]):

$$\begin{cases}\rho(a,t)=\rho_a(t) \text{ or}\\ q'(\rho(a,t))\leqslant 0 \text{ and } q'(\rho_a(t))\leqslant 0 \text{ or}\\ q'(\rho(a,t))\leqslant 0 \text{ and } q'(\rho_a(t))\geqslant 0 \text{ and } q(\rho(a,t))\leqslant q(\rho_a(t))\end{cases} \tag{4}$$

Similarly, the boundary condition at b is:

$$\begin{cases}\rho(b,t)=\rho_b(t) \text{ or}\\ q'(\rho(b,t))\geqslant 0 \text{ and } q'(\rho_b(t))\geqslant 0 \text{ or}\\ q'(\rho(b,t))\geqslant 0 \text{ and } q'(\rho_b(t))\leqslant 0 \text{ and } q(\rho(b,t))\geqslant q(\rho_b(t))\end{cases} \tag{5}$$

As noticed in [33], we can always assume the boundary data are entering the domain at both ends. Indeed, if for example $q'(\rho_a(t))<0$ on a subset I of $\mathbb{R}_+$ of positive measure, the boundary data:

$$\tilde{\rho}_a(t) = \begin{cases} q'^{-1}(0) \text{ if } t \in I \\ \rho_a(t) \text{ otherwise} \end{cases} \tag{6}$$

will yield the same solution. With this assumption the boundary conditions can be written as:

$$\begin{cases} \rho(a,t) = \rho_a(t) \text{ or} \\ q'(\rho(a,t)) \leqslant 0 \text{ and } q(\rho(a,t)) \leqslant q(\rho_a(t)) \end{cases} \tag{7}$$

and

$$\begin{cases} \rho(b,t) = \rho_b(t) \text{ or} \\ q'(\rho(b,t)) \geqslant 0 \text{ and } q(\rho(b,t)) \geqslant q(\rho_b(t)) \end{cases} \tag{8}$$

We can now define an notion of entropy solution for a scalar conservation law (3) with initial and boundary conditions.

Interpretation of the hybrid automaton for concave flux functions. Figure 2 (left) shows the three-mode automaton corresponding to (4). The first mode, $\rho(a,t) = \rho_a(t)$ corresponds to the situation in which the boundary condition $\rho_a(t)$ is effectively applied (as in the strong sense). The second mode $q'(\rho(a,t)) \leqslant 0$ and $q'(\rho_a(t)) \leqslant 0$ corresponds to a situation in which the characteristics exit the domain at $x = a$ for both the solution $\rho(a,t)$ and the prescribed boundary condition $\rho_a(t)$ (therefore the boundary condition does not 'affect' the solution). The third mode corresponds to a supercritical $\rho(a,t)$, i.e. $\rho(a,t) \geq \rho_c$ (see Figure 3 and [17, 18]), a subcritical $\rho_a(t)$, i.e. i.e. $\rho_a(t) \leq \rho_c$, and a prescribed inflow $q(\rho_a(t))$ greater than the actual flow $q(\rho(a,t))$ at $x = a$. This corresponds to a shock moving to the left (to see this, plug the previous quantities in the Rankine-Hugoniot conditions), which means that the prescribed boundary condition does

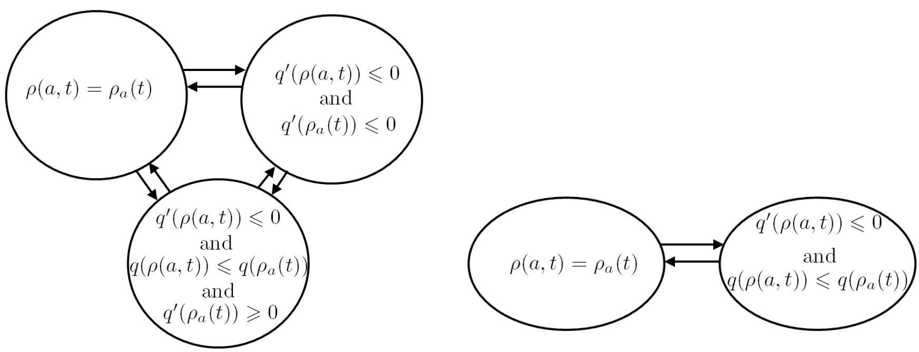

Fig. 2. Left: Hybrid automaton encoding the boundary conditions at $x = a$, corresponding to (4). A similar automaton can be constructed for (5). **Right:** Simplification of the automaton corresponding to the transformation of (4) into (7). A similar automaton can be constructed for (8) from (5).

not ‘affect’ the solution. The guards for this hybrid systems are thus determined by the sign of the flux derivative $q'(\cdot)$ and the values of the flux $q(\cdot)$ at $x = a$.

Definition: A solution of the mixed initial-boundary value problem for the PDE (3) is a function $\rho \in L^\infty((a,b)\times(0,T))$ such that for every $k \in \mathbb{R}$, $\varphi \in C_c^1((0,T))$, the space of C^1 functions with compact support, and $\psi \in C_c^1((a,b)\times(0,T))$ with φ and ψ nonnegative:

$$\int_a^b \int_0^T (|\rho - k|\frac{\partial \psi}{\partial t} + sg(\rho - k)(q(\rho) - q(k))\frac{\partial \psi}{\partial x})dxdt \geqslant 0$$

and there exist E_0, E_L, E_R three sets of measure zero such that :

$$\lim_{t\to 0, t\notin E_0} \int_a^b |\rho(x,t) - \rho_0(x)|dx = 0$$

$$\lim_{x\to a, x\notin E_L} \int_0^T L(\rho(x,t), \rho_a(t))\varphi(t)dt = 0$$

$$\lim_{x\to b, x\notin E_R} \int_0^T R(\rho(x,t), \rho_b(t))\varphi(t)dt = 0$$

With this definition, we now establish the uniqueness by proving an L^1- semigroup property following the method introduced by Kružkov [31] (see also the articles from Keyfitz [28] and Schonbek [44]).

Let ρ, σ be two solutions of (3), φ and ψ two test functions in $C_c^1((0,T))$ and $C_c^1((a,b))$ respectively and nonnegative; the aforementioned definition yields:

$$\int_a^b \int_0^T (|\rho(x,t) - \sigma(x,t)|\psi(x)\varphi'(t) + sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t)))\varphi(t)\psi'(x)))dxdt \geqslant 0$$

For ψ approximating $\chi\mid_{[a,b]}$, the characteristic function of the interval $[a,b]$, we have:

$$\int_a^b \int_0^T |\rho(x,t) - \sigma(x,t)|\varphi'(t)dt \geqslant \liminf_{x\to b} \int_0^T sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t) - q(\sigma(x,t)))\varphi(t)dt$$
$$- \limsup_{x\to a} \int_0^T sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t)))\varphi(t)dt.$$

For a fixed $x \notin E_L$ and $t \in (0,T)$, we can always define $k(x,t) \in I(\sigma(x,t), \rho_a(t)) \cap I(\rho(x,t), \rho_a(t))$ such that:

$$sg(\rho(x,t) - \sigma(x,t))(q(\rho(x,t)) - q(\sigma(x,t))) = sg(\rho(x,t) - \rho_a(t))(q(\rho(x,t)) - q(k(x,t)))$$
$$+ sg(\sigma(x,t) - \rho_a(t))(q(\sigma(x,t)) - q(k(x,t))) \leqslant L(\rho(x,t), \rho_a(x,t)) + L(\sigma(x,t), \rho_a(x,t)).$$

The situation is similar in a neighborhood of b which eventually yields:

$$\int_a^b \int_0^T |\rho(x,t) - \sigma(x,t)|\varphi'(t)dtdx \geqslant 0.$$

Therefore, for $0 < t_0 < t_1 < T$,

$$\int_a^b |\rho(x,t_1) - \sigma(x,t_1)|dx \leqslant \int_a^b |\rho(x,t_0) - \sigma(x,t_0)|dx$$

which proves the L^1-semigroup property from which the uniqueness follows.

4 Numerical Methods for the Initial-Boundary Value Problem

In this section, we prove the existence of a solution to equation (3) through the convergence of the Godunov scheme. Let $h = \frac{b-a}{M}$ and $I_i = [a + h(i - \frac{1}{2}), a + h(i + \frac{1}{2}))$ for $i \in \{0, ..., M\}$. For $r > 0$, let $J_n = [(n - \frac{1}{2})rh, (n + \frac{1}{2})rh)$ with $n \in \{0, 1, ..., N = E(1 + \frac{T}{rh})\}$. We approximate the solution ρ by ρ_i^n on each cell $I_i \times J_n$, with ρ_h the resulting function on $[a, b] \times [0, T]$. The initial and boundary conditions can be written as:

$$\begin{cases} \rho_i^0 = \frac{1}{h} \int_{I_i} \rho_0(x)dx \ , \ 0 \leqslant i \leqslant M \\ \rho_0^n = \frac{1}{rh} \int_{J_n} \rho_a(t)dt \text{ and } \rho_M^n = \frac{1}{rh} \int_{J_n} \rho_b(t)dt \ , \ 0 \leqslant n \leqslant N \end{cases}$$

According to the Godunov scheme [22], ρ_i^{n+1} is computed from ρ_i^n by the following algorithm:

$$\begin{cases} \rho_{i+\frac{1}{2}}^n \text{ is an element } k \text{ of } I(\rho_i^n, \rho_{i+1}^n) \text{ such that } sg(\rho_{i+1}^n - \rho_i^n)q(k) \text{ is minimal} \\ \rho_i^{n+1} = \rho_i^n - r(q(\rho_{i+\frac{1}{2}}^n) - q(\rho_{i-\frac{1}{2}}^n)) \end{cases}$$

Let $M_0 = \max(\|\rho_0\|_\infty, \|\rho_a\|_\infty, \|\rho_b\|_\infty)$; if the CFL (Courant-Friedrichs-Lewy) condition ([35])

$$r \sup_{|k|<M_0} |q'(k)| \leqslant 1$$

is verified, ρ_h converges in $L^1((a,b)\times(0,T))$ to a solution $\rho \in BV((a,b)\times(0,T))$. The CFL condition yields the following estimates:

$$|\rho_i^{n+1}| \leqslant (1 + C_0 h) \sup(|\rho_{i-\frac{1}{2}}^n|, |\rho_i^n|, |\rho_{i+\frac{1}{2}}^n|) + C_1 h \text{ for every } i \in \mathbb{Z}$$

$$\sum_{1\leqslant i\leqslant M} |\rho_{i+1}^{n+1} - \rho_i^{n+1}| \leqslant (1 + C_2 h) \sum_{|i|\leqslant M+1} |\rho_{i+1}^n - \rho_i^n| + C_3 M h^2 \text{ for every } M \in \mathbb{N}$$

$$\sum_{|i|\leqslant M} |\rho_i^{n+1} - \rho_i^n| \leqslant \sum_{|i|\leqslant M+1} |\rho_{i+1}^n - \rho_i^n| + C_4 M h(1 + \sup_{i\in\mathbb{Z}} |\rho_i^n|) \text{ for every } M \in \mathbb{N}$$

from which we can deduce that a subsequence ρ_{h_n} converges strongly to a function $\rho \in L^\infty((a,b)\times(0,T))$ of bounded variation and verifying the initial condition. We also have for k of $I(\rho_i^n, \rho_{i+1}^n)$

$$|\rho_i^{n+1} - k| \leqslant |\rho_i^n - k| - r(sg(\rho_{i+\frac{1}{2}}^n - k)(q(\rho_{i+\frac{1}{2}}^n) - q(k)) - sg(\rho_{i-\frac{1}{2}}^n - k)(q(\rho_{i-\frac{1}{2}}^n) - q(k)))$$

which shows that ρ is a weak solution of (3). If $\varphi^n = \frac{1}{rh}\int_{I_n}\varphi(t)dt$ for $\varphi \in C_c^1((0,T))$, non negative, we have:

$$\sum_{0\leqslant n\leqslant N} sg(\rho^n_{i+\frac{1}{2}} - k)(q(\rho^n_{i+\frac{1}{2}}) - q(k))\varphi^n rh \leqslant \sum_{0\leqslant n\leqslant N} sg(\rho^n_{\frac{1}{2}} - k)(q(\rho^n_{\frac{1}{2}}) - q(k))\varphi^n rh +$$
$$+ih\|\varphi'\|_\infty T(M_0 + |k|).$$

Let $\lambda(t)$ be the weak * limit in $L^\infty((0,T))$ of a subsequence of $q(\rho^i_{\frac{1}{2}})$; the following inequality holds:

$$\int_0^T sg(\rho(x,t) - k)(q(\rho(x,t)) - q(k))\varphi(t)dt \leqslant \int_0^T sg(\rho_a(t) - k)(\lambda(t) - q(k))\varphi(t)dt +$$
$$+|x-a|\|\varphi'\|_\infty T(M_0 + |k|),$$

using that $sg(\rho^n_{\frac{1}{2}} - k)(q(\rho_{\frac{1}{2}}) - q(k)) \leqslant sg(\rho^n_0 - k)(q(\rho_{\frac{1}{2}}) - q(k))$.

$\rho(x,.)$ is of bounded variation, therefore it converges strongly in L^1 sense to a limit $\alpha \in L^\infty((0,T))$ and it verifies:

$$sg(\alpha(t) - k)(q(\alpha(t)) - q(k)) \leqslant sg(\rho_a(t) - k)(\lambda(t) - q(k))$$

for every $k \in \mathbb{R}$ and a.e. $t \in (0,T)$. This inequality shows that $\lambda = q(\alpha)$ a.e. and $L(\alpha(t), \rho_a(t)) \leqslant 0$ and ρ verifies the weak boundary condition at $x = a$. Similarly, ρ verifies the corresponding condition at $x = b$ and the existence is proved.

5 Implementation and Simulations for I201W

We now turn to the practical implementation of the Godunov scheme for the LWR PDE. The scheme is written as follows:

$$\rho_i^{n+1} = \rho_i^n - r(q_G(\rho_i^n, \rho_{i+1}^n) - q_G(\rho_{i-1}^n, \rho_i^n))$$

If the flux q is strictly concave, which is often the case in traffic flow modeling, it reaches its only maximum at a point ρ_c (see Figure 3) and the numerical flux is defined by:

$$q_G(\rho_1,\rho_2) = \begin{cases} \min(\rho_1,\rho_2) & \text{if } \rho_1 \leqslant \rho_2, \\ q(\rho_1) & \text{if } \rho_2 < \rho_1 < \rho_c, \\ q(\rho_c) & \text{if } \rho_2 < \rho_c < \rho_1, \\ q(\rho_2) & \text{if } \rho_c < \rho_2 < \rho_1. \end{cases}$$

The boundary conditions are treated via the insertion of a ghost cell on the left and on the right of the domain, that is:

$$\rho_0^{n+1} = \rho_0^n - r(q_G(\rho_0^n, \rho_1^n) - q_G(\rho_{-1}^n, \rho_0^n))$$

with $\rho_{-1}^n = \frac{1}{rh}\int_{J_n}\rho_a(t)dt$, $0 \leqslant n \leqslant N$ for the left boundary condition and

$$\rho_M^{n+1} = \rho_M^n - r(q_G(\rho_M^n, \rho_{M+1}^n) - q_G(\rho_{M-1}^n, \rho_M^n))$$

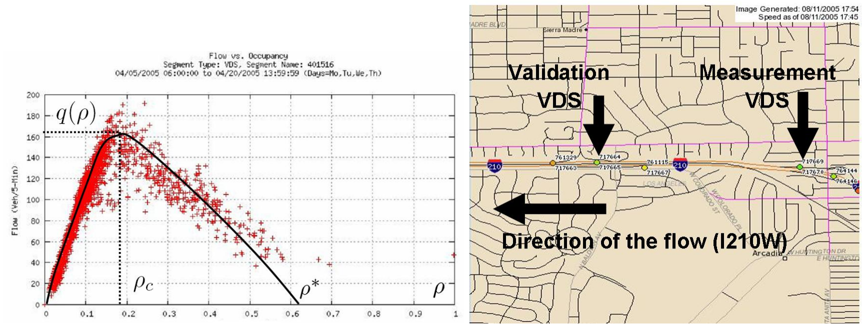

Fig. 3. Left: Illustration of the empirical data obtained from the PeMS system. The horizontal axis represents the normalized density ρ (i.e. occupancy, see [41, 38] for more details). The vertical axis represents the flux $q(\cdot)$. Each track corresponds to a loop detector measurement. This data can easily be modelled with a strictly concave flux function (solid fit), for which we display the critical density ρ_c and the jam density, ρ^*. **Right:** Location of the loop detectors used for measurement and validation purposes.

with $\rho^n_{M+1} = \frac{1}{rh}\int_{J_n} \rho_b(t)dt$, $0 \leqslant n \leqslant N$ on the right of the domain. We illustrate an application of this Godunov scheme to the simulation of highway traffic. A comparison of the density obtained numerically with the corresponding experimental density measured by the loop detectors is performed. We consider I210 West in Los Angeles and focus on a stretch going from the Santa Anita on-ramp 1 to the Baldwin on-ramp 2 in free-flow conditions between midnight and 05:00 a.m. The data measured by the loop detectors is accessible through the PeMS system (Performance Measurement System [41]); in our case the two detectors ID are 764669 and 717664.

We measure the flow at the loop detector 764669 (left subfigure in Figure 4). The need for signal processing is quite visible; for this example, it was done using Fast Fourier Transform methods. Noise levels are a very important issue with PeMS measurements, that has been covered extensively in the literature and is out of the scope of this work. The comparison with the actual measurements is performed at the next downstream loop detector (detector 717664), see right subfigure in Figure 4. The results shown in this figure illustrate the fact that the method is able to reproduce traffic flow patterns over an extended period of time (5 hours in the present case). The numerical simulation was done with FORTRAN codes from the CLAWPACK software developed by LeVeque and available at [12], implemented on a Sun Blade workstation. Further model refinements would be needed to obtain an enhanced matching of the two curves. This is also out of the scope of this article (the reader is referred to [38] for more on this topic).

6 Optimization of Travel Time Via Boundary Control

Our next endeavor is directed towards the minimization of the mean time spent by cars traveling through a stretch of highway between $x = x_0$ and $x = x_1$ via

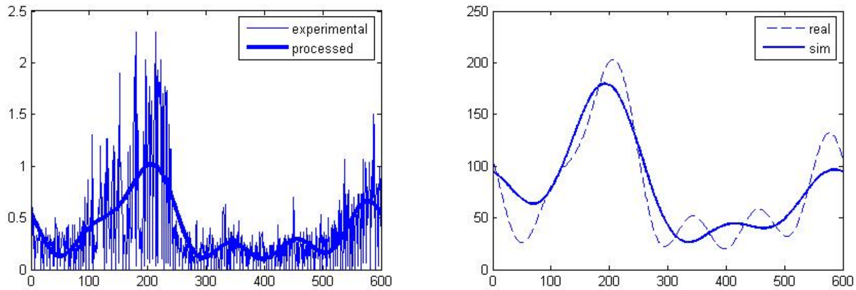

Fig. 4. Left: PeMS data used for the simulation, measured at loop detector 717669. The horizontal axis represents time, the vertical axis represents the inflow at the left boundary. **Right:** Comparison between loop detector measurements 717664 and flux simulations predicted by the model at the same location. The horizontal axis is time; the vertical axis is the vehicle flux. Source [41] .

the adjustment of the density of cars entering the highway. The results from Ancona and Marson ([2], [3]) enable us to solve this problem. The first step consists in studying the attainable set at a fixed point in space x_1:

$A(x_1,\mathcal{C}) = \{\rho(x_1,.)\}$, ρ being a solution of the LWR PDE with $\rho_0 = 0$ and $\rho_a \in \mathcal{C}$ for a given set of admissible controls $\mathcal{C} \subset L^1_{loc}$.

Using the method of generalized characteristics introduced by Dafermos ([15], [16]), the attainable set is shown to be compact, the key argument being that the set of fluxes $\{q(\rho_a), \rho_a \in \mathcal{C}\}$ is weakly compact in L^1 (see [32] for functional analysis in L^p spaces). The compactness of the attainable set in turn yields the existence of a solution to the optimal control problem

$$\min_{\rho_a \in \mathcal{C}} F(S_{(.)}\rho_a(x_1))$$

for $F : L^1([0,T]) \to \mathbb{R}$ a lower semicontinuous functional and $\mathcal{C}$ a set of admissible controls. We use the semigroup notation $S_t\rho_a$ to designate the unique solution of the LWR PDE at time t (we refer to the textbook [19] for more on semigroup theory). In the case of traffic modeling on a highway, we wish to minimize the difference between the average incoming time of cars at $x = x_1$ and at $x = x_0$ which can be written as:

$$\min_{\rho_a \in \mathcal{C}} \left(\int_0^{+\infty} tq(S_t\rho_a(x_1))dt - \int_0^{+\infty} tg(t)dt \right) \left(\int_0^{+\infty} g(t)dt \right)^{-1}$$

where $g(t)$ represents the number of cars entering the stretch of highway per unit of time. This amounts to solving the equivalent problem:

$$\min_{\rho_a \in \mathcal{C}} \int_0^{+\infty} tq(S_t\rho_a(x_1))dt$$

For this particular problem, we make the following additional assumptions:

– the net flux of cars entering the highway is equal to the total number of cars arriving at the entry:

$$\int_0^{+\infty} q(\rho_a(s))ds = \int_0^{+\infty} g(s)ds$$

– for every time $t > 0$ the total number of cars which have entered the highway is smaller than or equal to the total number of cars that have arrived at the entry from time 0 to t:

$$\int_0^t q(\rho_a(s))ds \leqslant \int_0^t g(s)ds$$

– the number of cars entering the highway is at most equal to the maximum density of cars on the highway:

$$\rho_a(t) \in [0, \rho_m]$$

– after a given time T no cars enter the highway:

$$\rho_a(t) = 0 \text{ for } t > T.$$

The map $F : \rho \to \int_0^T q(\rho(t))dt$ is obviously a continuous functional on L^1_{loc} $([0,T])$, hence the existence of a solution of an optimal control ρ_a.

Furthermore, a comparison principle for solutions of scalar nonlinear conservation laws with boundary conditions established by Terracina in [47] will allow us to find an explicit expression of the optimal control. Indeed if $\rho(x,t)$ is a weak solution of the LWR PDE, $u(x,t) = -\int_x^{+\infty} \rho(y,t)dy$ is the viscosity solution ([14]) of the Hamilton-Jacobi equation

$$\frac{\partial u}{\partial t} + q(\frac{\partial u}{\partial x}) = q(0).$$

Since viscosity solutions verify a comparison property [13], so will the solution of the LWR PDE.

Since $\int_0^T tq(S_t\rho_a(x_1))dt = T\int_0^T q(S_t\rho_a(x_1))dt - \int_0^T\int_0^t q(S_s\rho_a(x_1))dsdt$, the boundary control problem can be rewritten as:

$$\max_{\rho_a \in \mathcal{C}} \int_0^T \int_0^t q(S_s\rho_a(x_1))dsdt.$$

As we can assume that the boundary data is always incoming, the comparison principle shows that the optimal control $\tilde{\rho}$ should verify:

$$\int_0^t q(\tilde{\rho}(s))ds \geqslant \int_0^t q(\rho_a(s))ds, \text{ for every } t > 0 \text{ and } \rho_a \in \mathcal{C}.$$

Eventually we obtain the following expression of the optimal control $\tilde{\rho}$:

$$\tilde{\rho}(t) = \begin{cases} q^{-1}(\rho_m) & \text{if } g(t) \leqslant q(\rho_m) \text{ and } \int_0^t q(\tilde{\rho}(s))ds < \int_0^t g(s)ds \text{ or } g(t) > q(\rho_m) \\ q^{-1}(g(t)) & \text{if } g(t) \leqslant q(\rho_m) \text{ and } \int_0^t q(\tilde{\rho}(s))ds = \int_0^t g(s)ds \end{cases}$$

7 Conclusion

We have proved the existence and uniqueness of a weak solution to a scalar conservation law on a bounded domain. The proof relies on the weak formulation of the hybrid boundary conditions which is necessary for the problem to be well posed. For strictly concave flux functions, the simplified expression of the weak formulation of the hybrid boundary conditions was written explicitly. The corresponding Godunov scheme was developed and applied on a highway traffic flow application, using PeMS data for the I210W highway in Pasadena. The numerical scheme and the parameters identified for this highway were validated experimentally against measured data. Finally, the existence of a minimizer of travel time was obtained, with corresponding optimal boundary control.

The hybridness of the boundary conditions is closely linked to the one-dimensional nature of the problem (i.e. to the direction of the characteristics and the corresponding values of the fluxes). The switches between the modes occur based on the value of the solution, which itself acts as a guard. The boundary conditions derived in this article should thus be viewed as an instantiation of the more general weak boundary conditions given in [8], for which a clear hybrid structure appears in the one dimensional case, through a modal behavior.

This article should be viewed as a first step towards building sound metering control strategies for highway networks: it defines the mathematical solution, and appropriate hybrid boundary conditions to apply in order to pose and solve the optimal control problem properly. Not using the framework developed here while computing numerical solutions of the LWR PDE would lead to ill-posed problems and therefore the data obtained through a numerical scheme would be meaningless.

Our result is crucial for highway performance optimization, since by nature, in most highways, traffic flow control is achieved by on-ramp metering, i.e. boundary control. However, results are still lacking in order to generalize our approach to a real highway network. For such a network, PDEs are coupled through boundary conditions, which makes the problem harder to pose. Furthermore, optimization problems arising in transportation networks often cannot be solved as the problem derived in the last section of this article. In fact, several approaches have to rely on the computation of the gradient of the optimization functional, which for example could be achieved using adjoint-based techniques. Obtaining the proper formulation of the adjoint problem, and the corresponding proofs of existence and uniqueness of the resulting solutions represents a challenge for which the present result is a building block.

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