# Network Congestion Alleviation Using Adjoint Hybrid Control: Application to Highways*

Alexandre M. Bayen**, Robin L. Raffard, and Claire J. Tomlin

Hybrid Systems Laboratory, Department of Aeronautics and Astronautics
Stanford University, Stanford, CA 94305-4035
{bayen,rraffard,tomlin}@stanford.edu

**Abstract.** This paper derives an optimization-based control methodology for networks of switched and hybrid systems in which each mode is governed by a *partial differential equation* (PDE). We pose the continuous controller synthesis problem as an optimization program with PDEs in the constraints. The proposed algorithm relies on an explicit formulation of the gradient of the cost function, obtained via the adjoint of the PDE operator. First, we show how to use the result of the optimization to synthesize on/off control strategies. Then, we generalize the method to optimal switching control of hybrid systems over PDEs: the system is allowed to switch from one mode (or PDE) to another at times which we synthesize to minimize a given cost. We derive an explicit expression of the gradient of the cost with respect to the switching times. We implement our techniques on a highway congestion control problem using *Performance Measurement System* (PeMS) data for the California I210 for a 9 mile long strip with 26 on-ramps (controllable with red/green metering lights) and off-ramps (uncontrollable).

## 1 Introduction

Physical systems governed by *partial differential equations* (PDEs) abound in science and engineering: fluid mechanics, biology, control of processes, and integrated circuits are four examples. Within the realm of PDE driven systems, we are interested in the class of systems governed by one dimensional networked PDEs; this class includes highway networks [6], the air traffic control system [15], and irrigation networks [14]. One common feature of these networked PDE systems is that the governing PDE for each portion of the network is linked to neighboring PDEs through boundary conditions. However, the actuation available to control these systems depends on the problem: for irrigation channels, one controls the boundary conditions (water inflow), using dams; in air traffic control, the control is the velocity field prescribed by air traffic controllers. For the

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** Corresponding author

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present case, in which we are interested in controlling congestion on the highway, a standard actuation scheme consists of controlling boundary conditions with *metering lights* which delay the entrance of cars onto the highway [16]. An alternate control scheme [9] uses time varying speed limits which prevent the creation of traffic jams in congested areas.

Numerous approaches which attempt to control highway systems rely on the well-known *Lighthill-Whitham-Richards* (LWR) model [13,17], which describes the evolution of the car density on the highway using a PDE. This PDE relates the time derivative of the car density to the space derivative of the *flux function*, where the flux function is an empirically determined function which relates the number of cars traveling through a given section of the highway per unit of time to the local car density. To our best knowledge, no approach has ever tackled the problem of controlling the LWR PDE directly. Rather, most of the research focuses on controlling the discretized LWR PDE [6,16] using classical optimal control techniques for discrete time dynamical systems; the technique is easier, but the underlying discrete time dynamical system sometimes exhibits discrepancies from the original continuous model [13,17].

Recent mathematical results have enabled the characterization of the entropy solution [8] as the correct weak solution of the LWR PDE for non-convex flux functions [1]. Modern numerical analysis techniques have enabled accurate computations of this solution [11]. Finally, recent development of adjoint-based techniques have enabled the control of nonlinear first order PDEs [10,3].

A mathematical difficulty appears naturally in the treatment of the highway control mentioned above: it is by its nature a hybrid problem. The metering lights are a set of on/off systems, which one tries to regulate. Determining the on/off sequences of actuators for distributed systems is a difficult task in general. For the alternate control scheme, realistic time dependent speed limits require the system to switch between modes in which the maximum allowable speed is one of three possible values (typically 45mph, 55mph and 65mph). As will be seen, in each of the modes (45, 55 or 65), the governing equation of the system is a different PDE. Other approaches [16] have also characterized the highway system as hybrid by nature and modeled different modes of the highway (congested mode, free flow mode, etc.), each of them governed by a discrete time dynamical system. These approaches make the problem easier to control. However, we are interested in deriving control based on the continuous PDE directly, before performing any discretization. This paper contains contributions pertinent to several different aspects of the problem.

1. Concerning the model and the numerical simulations, this work is to the best of our knowledge the first to attack the problem of accurate simulations of the continuous LWR PDE. We demonstrate the efficiency of numerical schemes against analytically constructed entropy solutions [1,8]. In particular, we show excellent performance of the *Jameson-Schmidt-Turkel* (JST) [11] and the *Daganzo* [6] schemes, which are both nonlinear. An important advantage of the JST scheme is that it works for any flux function as well as for the adjoint problem, which we will demonstrate here. We also show that linear

numerical schemes such as Lax-Friedrichs, by contrast, exhibit extremely poor performance, which raises questions about the use of these schemes.
2. We construct an adjoint based method to solve an optimization program formulation of the control problem, which is applicable to highway networks. We have already applied this approach successfully to air traffic control [2], here we show how to apply this method to on/off systems.
3. We generalize the notion of optimal control to hybrid systems for which the modes are governed by PDEs. We show how to compute gradients with respect to the switching times between the different PDEs.
4. We apply our results to actual highway data obtained from the *Performance Measurement System* (PeMS) [12] database to I210 in Los Angeles. We successfully control a highway portion containing 26 on- and off-ramps.

This paper is organized as follows. In Section 2, we derive the network model and validate the computational tools we will use against an analytically derived entropy solution of the LWR PDE. In Section 3, we set up the optimal control problem as an optimization program with PDE constraints, and derive an explicit expression of the gradient through the adjoint problem (Formula 1). We embed this result into a gradient descent algorithm to solve the optimization problem. We explain how to use the result to synthesize on/off switching sequences and apply it to the I210 example. In Section 4, we generalize these results and compute a gradient with respect to the switching times (Formula 2).

## 2 Eulerian Highway Network Model

### 2.1 PDE Model

We consider a network of $N$ connected highway segments, indexed by $i$. The density of cars on link $i$ is denoted $\rho_i$. We call $L_i$ the length of link $i$, and $x_i \in [0, L_i]$ the coordinate on this link. Several models exist for describing the evolution of car density on the highway. We use the *Lighthill-Whitham-Richards* (LWR) model in the present study. In this model, the density obeys the LWR *partial differential equation* (PDE):

$$\mathcal{N}_i(\rho_i) \triangleq \frac{\partial \rho_i}{\partial t} + \frac{\partial q_i(\rho_i)}{\partial x_i} = 0 \tag{1}$$

in which $q_i(\cdot)$ represents a flux function relating the flux of cars (number of cars through a given section of the highway during a time unit) to the car density at that location. This equation expresses that the local rate of change of car density is equal to the space derivative of the flux of cars, i.e. conservation of mass. $q_i(\cdot)$ is identified empirically from highway data. Several models have been proposed for $q_i(\cdot)$, such as the Greenshield model [1], trapezoidal or triangular models [6, 16]. As will be seen in the next section, the computational method that we use can handle any $q_i(\cdot)$.

The links can merge and can diverge, have on- and off-ramps. In the context of the present work, we will be interested in a highway portion connecting two

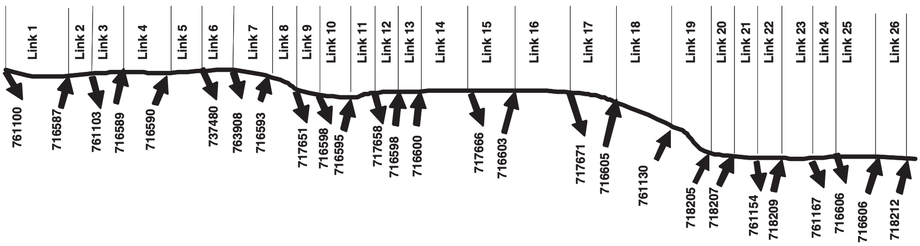


**Fig. 1.** Example of interest for this study: portion of highway I210 East between I5 and I605 in Pasadena, decomposed into 26 links. Each of the arrows denotes an on- or off-ramp. The numbers refer to the link detectors [12], which measure the flow through the on- or off-ramps. The total length of this strip is 9 miles. The loop detector labeled 761100 is at mile 26 (from a given reference point upstream); the loop detector labeled 718212 is at mile 35. The flow is going East (in increasing order of the links).

highways, with a total of $N$ on- or off-ramps (see Figure 1). The governing equations for this system are given by:

$$\begin{cases} \mathcal{N}_i(\rho_i) = \frac{\partial \rho_i}{\partial t} + \frac{\partial q_i(\rho_i)}{\partial x_i} = 0 & 1 \leq i \leq N \\ \rho_i(x_i, 0) = \rho_i^\circ(x_i) & 1 \leq i \leq N \\ q_i(\rho_i(0,t)) = q_{i-1}(\rho_{i-1}(L_{i-1},t)) + q_i^{\text{on}}(t) & \forall i \in \mathcal{ON} \\ q_i(\rho_i(0,t)) = (1 - \beta_{i-1}(t)) q_{i-1}(\rho_{i-1}(L_{i-1},t)) & \forall i \in \mathcal{OFF} \end{cases} \tag{2}$$

In the previous equation, $\rho_i^\circ(x_i)$ denotes the density of cars at time 0. $\mathcal{ON}$ denotes the set of links with merging on-ramps (in Figure 1, $\mathcal{ON} = \{2, 4, 5, \cdots\}$); $\mathcal{OFF}$ denotes the set of links with diverging off-ramps ($\mathcal{OFF} = \{1, 3, 6, \cdots\}$). $q_i^{\text{on}}(t)$ denotes the inflow of cars into link $i$, $\beta_i(t) \in [0,1]$ denotes the proportion of cars leaving link $i$ through an off-ramp. Every $x_i$ ranges in $[0, L_i]$, and $t \in [0, T]$. The interpretation of the two last equations in (2) is as follows: the third equation in (2) expresses the conservation of flow at an on-ramp location (the flow into link $i$ is the flow from link $i-1$ plus the additional flow from the ramp); and the last equation (2) expresses the same with off-ramps. In this last equation, $\beta_i(t)$ represents the proportion of flow leaving link $i$. For the rest of this article, we will use the first order approximation that $\beta_i(t) = \beta_i$ does not depend on time. In order for (2) to be consistent with Figure 1, we need to set $q_0(\rho_0(L_0, t)$ equal to the inflow into the highway, and $\beta_0$ to the portion of this flow leaving the highway through link 761100. Note that this framework encapsulates more general network topologies, as was successfully done in the context of Air Traffic Control [2] (time dependent $\beta_i$ and a more general network).

## 2.2 Numerical Schemes

In this section, we briefly demonstrate the performance of the numerical scheme which will be used for the rest of the article to solve various forms of the PDE model (2). The LWR PDE is a first order hyperbolic PDE, which admits several

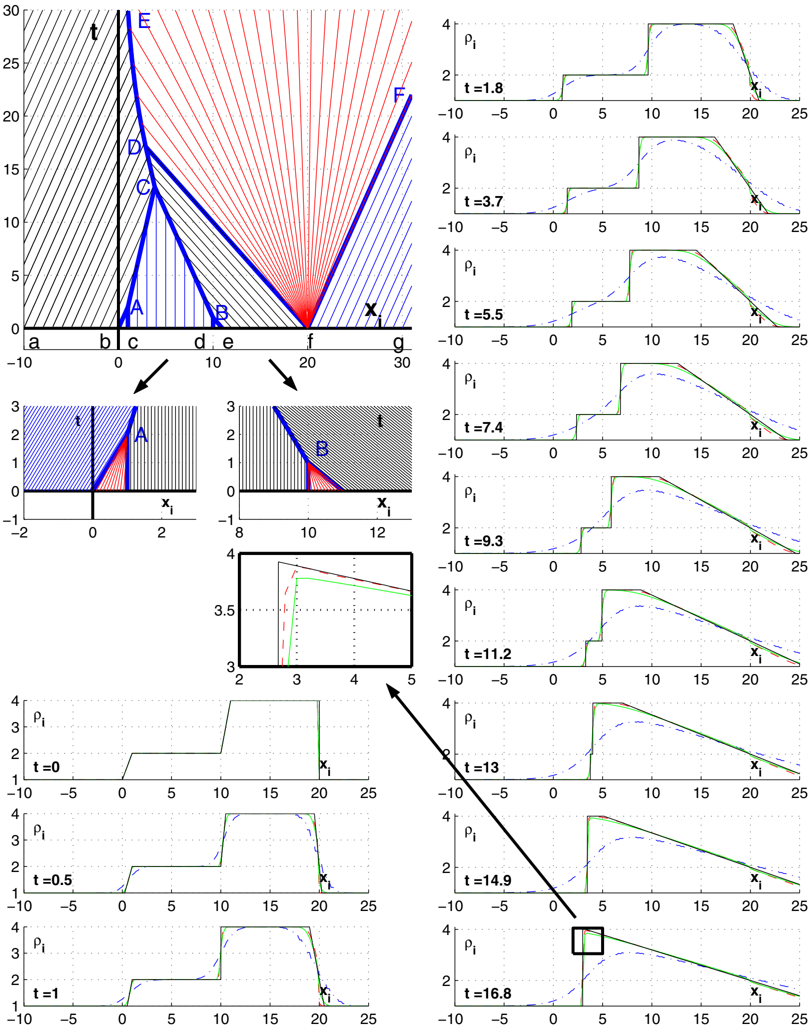


**Fig. 2.** Subplots indexed by time: entropy solution of the LWR PDE (1) with Greenshield flux function [1] and initial condition shown for $t = 0$. Comparisons of the results provided by the different numerical schemes: analytical solution constructed with the method of characteristics (solid, sharp), Daganzo (solid, lightly shaded), JST (−−), visually almost not differentiable from the analytical solution, Lax Friedrichs scheme (−·). As can be seen, there is a traffic jam initially between 11 and 20 (flow is going to the right). It dissolves on the right, because of low density downstream. The low density upstream piles up into the region of medium density, generating a compression wave, which becomes a shock wave. The same happens with the medium density into the region of high density. The two shock waves collapse into a single shock wave after $t = 13$, as the traffic jam further dissolves (which is known as the "N-wave" because of its shape). The upper left plot represents the characteristics [8] in the $(x_i, t)$ plane used to obtain the analytical solution. The two little subplots below are magnified plots of the characteristics around A and B.

weak solutions [8]. These solutions can exhibit features which hinder numerical computation, such as shocks. Among these solutions, [8] shows the existence and uniqueness of a particular solution called the *entropy solution*, which was identified in [1] as the correct solution to the highway problem. We use the *Jameson-Schmidt-Turkel* (JST) scheme [11] to compute the entropy solution numerically. We show convergence of this scheme for a benchmark problem by comparing the numerical results to an analytically computed entropy solution obtained with the method of characteristics [8], and we compute the error as a function of the number of gridpoints.

*Numerical validation example:* We consider one link ($i = 1$). We use the Greenshield model: $q_i(\rho_i) = v_i\rho_i\,(1 - \rho_i/\rho_i^{\max})$, where the speed $v_i$ is a constant and $\rho_i^{\max}$ is called *jam density* (above which cars stop on the highway). In the experiment, $v_i = 1$ [unit of length / unit of time] and $\rho_i^{\max} = 4$ [cars / unit length]. These numbers are nondimensional, but for the applications presented later, we will use highway data. The initial conditions $\rho_i(x_i, 0)$, which are prescribed (infinite domain), and solution $\rho_i(x_i, t)$, which we computed, are outlined below:

$$\begin{cases} \rho_i(x_i,0) = 1 & \text{if } x_i \leq 0 \\ \rho_i(x_i,0) = 1 + x_i & \text{if } x_i \in [0,1] \\ \rho_i(x_i,0) = 2 & \text{if } x_i \in [1,10] \\ \rho_i(x_i,0) = 2 + 2(x_i - 10) & \text{if } x_i \in [10,11] \\ \rho_i(x_i,0) = 4 & \text{if } x_i \in [11,20] \\ \rho_i(x_i,0) = 1 & x_i \geq 20 \end{cases} \qquad \begin{cases} abACDE & \rho_i(x_i,t) = 1 \\ bAc & \rho_i(x_i,t) = \frac{1+x_i-v_it}{1-\frac{v_it}{2}} \\ cACBd & \rho_i(x_i,t) = 2 \\ dBe & \rho_i(x_i,t) = 2(1 + \frac{x_i-10}{1-v_it}) \\ eBCDf & \rho_i(x_i,t) = 4 \\ EDfF & \rho(x_i,t) = 2(1 - \frac{x_i-20}{v_it}) \\ Ffg & \rho_i(x_i,t) = 1 \end{cases}$$

where the polygons of the previous formula are shown in Figure 2; we computed this solution analytically using Rankine-Hugoniot jump conditions and self similar expansion waves [8]. This scenario represents the backwards propagation of a traffic jam into a medium density portion of the highway and its dissolution. Figure 2 shows three different numerical schemes: Lax-Friedrichs (LxF), Daganzo (Dag) and Jameson-Schmidt-Turkel (JST). The numerical validation is summarized in the following array:

| $N_{\text{gridpoints}}$ | $L_2$ rel. error (LxF) | $L_2$ rel. error (JST) | $L_2$ rel. error (Dag) |
|---|---|---|---|
| 40 | 0.2404 | 0.1088 | 0.0981 |
| 80 | 0.1804 | 0.0729 | 0.0641 |
| 160 | 0.1376 | 0.0474 | 0.0430 |
| 320 | 0.1049 | 0.0299 | 0.0291 |
| 640 | 0.0787 | 0.0187 | 0.0198 |

As can be seen, linear numerical schemes such as LxF perform relatively poorly at low resolutions. The excellent performance of JST and Dag (less than 2% relative error) enables us to use them for the rest of this article. We will use the JST, because it not only applies to the problem of interest, but also to the adjoint, which we construct later.

### 2.3 Validation of the Model against PeMS Data

Even though the LWR PDE model is not new and is acknowledged as a valid model of highway traffic [13,17,6,16], we want to demonstrate its accuracy and thus the accuracy of our direct numerical solutions by simulating highway traffic using real data from the PeMS website [12]. From the PeMS database, we can get data sampled at a 5 minute rate for the I210 East in Los Angeles. We simulate a 9 mile long section between I5 and I605, with 26 on- and off-ramps (see Figure 1). The highway has 5 lanes and a carpool lane, which become 4 lanes and a carpool lane at Link 14. We can compute the flux functions $q_i(\cdot)$ empirically from the PeMS data. We have access to both flux and density at the on- and off-ramps as well as on the highway, using algorithms available in [12]. At the on-ramps, we can use this data to compute the $q_i^{\text{on}}(t)$ directly. At the off-ramps, we can evaluate the $\beta_i$ as the ratio of the exiting flow over the flow on the highway. We perform a validation over four hours of traffic. The results are shown in Figure 3 for 35 minutes, and are available in movie format at [19]. The simulation starts at 14:00, captures the increase in density until its peak around 17:00. After 17:00, it stays at the peak and captures an almost steady state car density, jammed upstream. The discrepancies between measurements and simulations observed in Figure 3 have several causes we list: exact location of the sensors on the highway, noise in the data, lack of data, sensor malfunctions, and highway configuration. Despite these discrepancies, we are able to capture the propagation of congestion on the highway, which is our goal; this comparison thus demonstrates the validity of the model for our purposes.

## 3 Gradient Computation via the Adjoint Problem

We can now use the framework presented in the previous section to design a control methodology for the network problem. For this section, the control variables are the $q_i^{\text{on}}(t)$, which we can adjust in order to prevent the density on the highway from becoming too high. We first explain how to synthesize a set of continuous $q_i^{\text{on}}(t)$, and then how to use it to regulate the metering lights on the highway with an on/off control strategy.

### 3.1 PDE Constrained Optimization Formulation of Control Problems

We want to optimize a cost function subject to the constraints inherent to the problem and imposed by the available control. The framework presented below is very general and allows any cost function. We will illustrate this by maximizing the *Vehicle Miles Traveled* (VMT), as defined in [5], which represents the sum of distances driven by cars on a highway section over a time interval. The VMT is very easily expressed as a function of $q_i(\rho_i(\cdot))$, as can be seen in the cost function of the following optimization program:

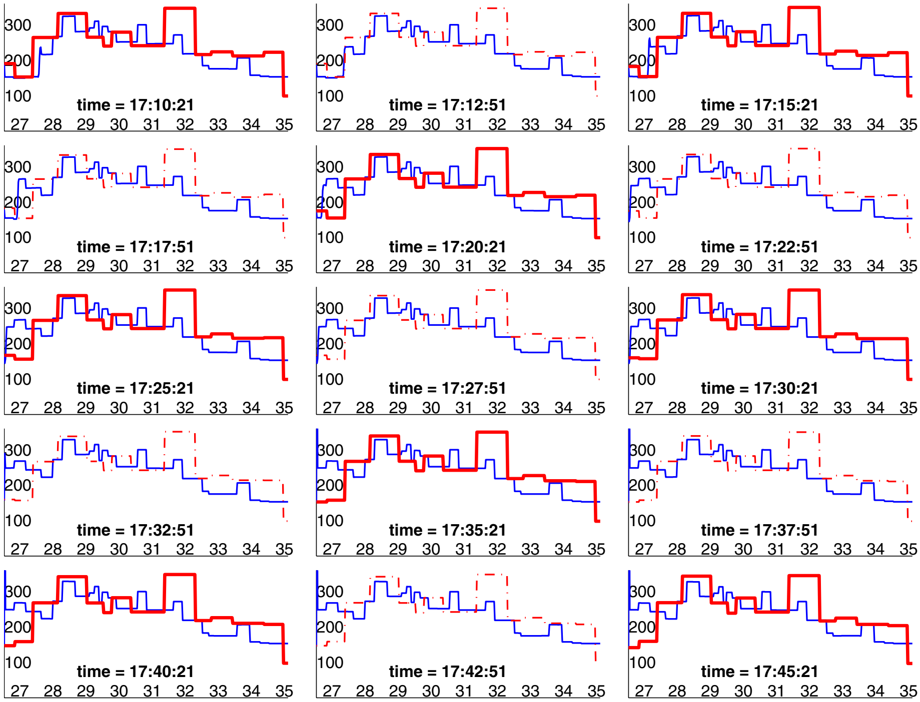


**Fig. 3.** Simulation for 35 minutes of traffic for I210 East. The horizontal axis is the distance (in miles, see Figure 1); the vertical axis is the density (in vehicles per mile). The solid thin curve is the result of the simulation. It is obtained with the JST scheme. The time step is on the order of 0.25 sec., i.e. we compute the (simulated) data every 0.25 sec.; it is displayed every 150 sec., and compared to the measured data. The measured data is available every 5 minutes. It is displayed when measured (thick line), and does not change until the next measurement (therefore the dash-dotted curves are just copies of the previous thick curve). This data is extracted from a movie obtained for four hours of traffic, available at [19].

$$\begin{array}{ll}\textbf{max:} & \sum_{i=1}^{N} \int_0^{L_i} \int_0^T q_i(\rho_i(x_i,t))dx_i dt \\ \textbf{s.t.:} & (2) \\ & 0 \leq q_i^{\text{on}}(t) \leq q_{\max,i}(t)\end{array} \tag{3}$$

The constraints in the previous optimization program are (2), which are the governing equations of the system, and $0 \leq q_i^{\text{on}}(t) \leq q_{\max,i}(t)$, which sets a bound on the number of cars that can be let into the highway ($q_i^{\text{on}}(t)$ is the control variable which we want to compute for this study, and from which we want to extract the on/off sequence for the switching metering light at on-ramp $i$). Using a standard log barrier technique [4] to avoid constraints in the control, we can transform the problem into:

$$\begin{aligned}\textbf{min: } & \mathcal{J} \stackrel{\triangle}{=} -\textstyle\sum_{i=1}^{N}\int_0^{L_i}\int_0^T q_i(\rho_i(x_i,t))dx_i dt \\ & -\frac{1}{M}\sum_{i\in\mathcal{ON}}\int_0^T \log(q_i^{\text{on}}(t)(q_{\max,i}(t)-q_i^{\text{on}}(t))) \\ \textbf{s.t.: } & (2)\end{aligned} \tag{4}$$

From [4], we know that if $M$ is a positive real number, then problems (3) and (4) will be equivalent in the limit $M \to +\infty$. This optimization program is non-convex, nonlinear and has constraints in the form of PDEs, which makes it impossible to use standard optimization tools [4]. We therefore will derive a methodology to compute the gradient of the cost function, in order to perform this optimization.

### 3.2 Derivation of the Adjoint of the LWR Problem

The computation of the gradient is done via the adjoint problem, which we compute as follows. The method is adapted from [10,3] and extended to a network problem involving multiple PDEs coupled by boundary conditions. In order to compute the gradient, we perturb the variables (control $q_i^{\text{on}}$ and state $\rho_i$), and compute the corresponding expression of the perturbed cost function. Each quantity is written as $\rho_i = \overline{\rho}_i + \rho_i'$, $q_i^{\text{on}} = \overline{q}_i^{\text{on}} + {q_i^{\text{on}}}'$, $\mathcal{J} = \bar{\mathcal{J}} + \mathcal{J}'$ etc, where the overline denotes nominal and the prime perturbation. The result reads:

$$\mathcal{J}' = -\sum_{i=1}^{N}\int_0^{L_i}\int_0^T c_i(\overline{\rho}_i)\rho_i' dx_i dt - \frac{1}{M}\sum_{i\in\mathcal{ON}}\int_0^T\left(\frac{{q_i^{\text{on}}}'(t)}{\overline{q}_i^{\text{on}}(t)} - \frac{{q_i^{\text{on}}}'(t)}{q_{\max,i}(t)-\overline{q}_i^{\text{on}}(t)}\right)dt$$

In the previous formula, $c_i$ denotes the first derivative of the flux function $dq_i(\rho_i)/d\rho_i$, called the *celerity*. Written as such, this expression cannot be used practically, since it depends on the state $\rho_i'$ which cannot be controlled directly. We compute the linearized differential operator $\mathcal{N}_i'(\cdot)$ associated with the LWR operator $\mathcal{N}_i(\cdot)$ from (2), and appropriate perturbed boundary and initial conditions:

$$\begin{cases}\mathcal{N}_i'(\overline{\rho}_i)\rho_i' := \frac{\partial \rho_i'}{\partial t} + \frac{\partial}{\partial x_i}\left(c_i(\overline{\rho}_i)\rho_i'\right) = 0 & 1 \le i \le N \\ \rho_i'(0,x_i) = 0 & 1 \le i \le N \\ c_i(\overline{\rho}_i(0,t))\rho_i'(0,t) = c_{i-1}(\overline{\rho}_{i-1}(L_{i-1},t))\rho_{i-1}'(L_{i-1},t) + {q_i^{\text{on}}}'(t) & \forall i \in \mathcal{ON} \\ c_i(\overline{\rho}_i(0,t))\rho_i'(0,t) = (1-\beta_{i-1}(t))c_{i-1}(\overline{\rho}_{i-1}(L_{i-1},t))\rho_{i-1}'(L_{i-1},t) & \forall i \in \mathcal{OFF}\end{cases}$$

where dependencies in $x_i$ and $t$ are omitted when trivial. As usual [10,3], the linearized operator depends on the nominal flow $\overline{\rho}_i$. We denote by $\langle\cdot|\cdot\rangle_i$ the inner product, defined for any two functions $\rho_i^*$ and $\rho_i'$ by:

$$\langle \rho_i^* | \rho_i \rangle_i := \int_0^T\int_0^{L_i} \rho_i^*(x_i,t)\rho_i(x_i,t)dx_i dt \tag{5}$$

We denote by $\rho_i^*$ the adjoint variable of $\rho_i'$, and by $\mathcal{N}_i^*(\overline{\rho}_i)$ the adjoint operator of $\mathcal{N}_i'(\overline{\rho}_i)$, defined algebraically by the adjoint identity:

$$\langle \rho_i^* | \mathcal{N}_i'\rho_i' \rangle_i \stackrel{\triangle}{=} \langle \mathcal{N}_i^*\rho_i^* | \rho_i' \rangle_i + b_i \tag{6}$$

where $b_i$ denotes the boundary conditions. The adjoint operator and the boundary conditions can be computed explicitly:

$$\begin{cases} \mathcal{N}_i{}^* = -\frac{\partial(.)}{\partial t} - c_i(\bar{\rho}_i)\frac{\partial(.)}{\partial x_i} \\ b_i = \int_0^{L_i} \rho_i^* \rho_i'|_0^T \, dx_i + \int_0^T \rho_i^* c_i(\overline{\rho}_i)\rho_i'|_0^{L_i} \, dt \end{cases} \tag{7}$$

Note that the adjoint depends on $\overline{\rho}_i$. Let $\rho_i^*$ be the solution of the following PDE:

$$\mathcal{N}_i{}^*(\rho_i^*) = c_i(\bar{\rho}_i(x_i, t)) \tag{8}$$

Then, using the adjoint identity, we can plug $\mathcal{N}_i{}^*(\rho_i^*)$ into $\mathcal{J}'$; using the boundary conditions of (7), we can eliminate all perturbed states from $\mathcal{J}'$ provided:

$$\begin{cases} \rho_i^*(x_i, T) = 0 & 1 \leq i \leq N \\ \rho_i^*(L_i, t) = 0 & i = N \\ \rho_{i-1}^*(L_{i-1}, t) = \rho_i^*(0, t) & i \in \mathcal{ON} \\ \rho_{i-1}^*(L_{i-1}, t) = (1 - \beta_{i-1})\rho_i^*(0, t) & i \in \mathcal{OFF} \end{cases} \tag{9}$$

which provides the following formula:

**Formula 1 (Reduced gradient formulation):** The perturbation of the cost function can be expressed as the inner product of the gradient and the control variable as follows

$$\mathcal{J}' = \sum_{i \in \mathcal{ON}} \left\langle -\rho_i^*(0, \cdot) - \frac{1}{M}\left(\frac{1}{\overline{q}_i^{\text{on}}(\cdot)} - \frac{1}{q_{\max,i}(\cdot) - \overline{q}_i^{\text{on}}(\cdot)}\right) \middle| \, q_i^{\text{on}\prime}(\cdot) \right\rangle_{[0,T]} \tag{10}$$

where $\langle \cdot | \cdot \rangle_{[0,T]}$ denotes the inner product w.r.t. $t$ only.

We can now use Formula 1 to embed the gradient computation into an algorithm to solve the control problem through the optimization program (4):

**Algorithm 1 (adjoint based gradient optimization):** The following algorithm converges to a minimum of the optimization program (4):

0 Start with a guess $q_i^{\text{on}} \leq q_{\max,i}$;
1 Compute the $\bar{\rho}_i$ with JST from the known $\rho_i^\circ$ for all $i$, for a small $M$;
2 Compute the solution $\bar{\rho}_i^*$ of (8)-(9) using $\bar{\rho}_i$ computed in 1, for all $i$;
3 Compute the optimal update $\{q_i^{\text{on}'}\}_{i \in \{1, \cdots, N\}}$ using (10) from Formula 1;
4 Use steepest descent with backtracking to update $q_i^{\text{on}}$ for all $i$;
5 Compute $\bar{\rho}_i$ with updated $q_i^{\text{on}}$. Unless converged, go back to 2;
6 Unless log term is negligible, increase $M$ and go back to 2.

The solution of this algorithm converges [4] to a solution of the initial optimization problem (3) in the limit $M \to \infty$. The backtracking procedure is sometimes referred to by the name of the more general *Armijo condition* [7,4].

The entropy solution of the LWR equation is known to be discontinuous (as was shown in the validation example). The gradient derivation however is only

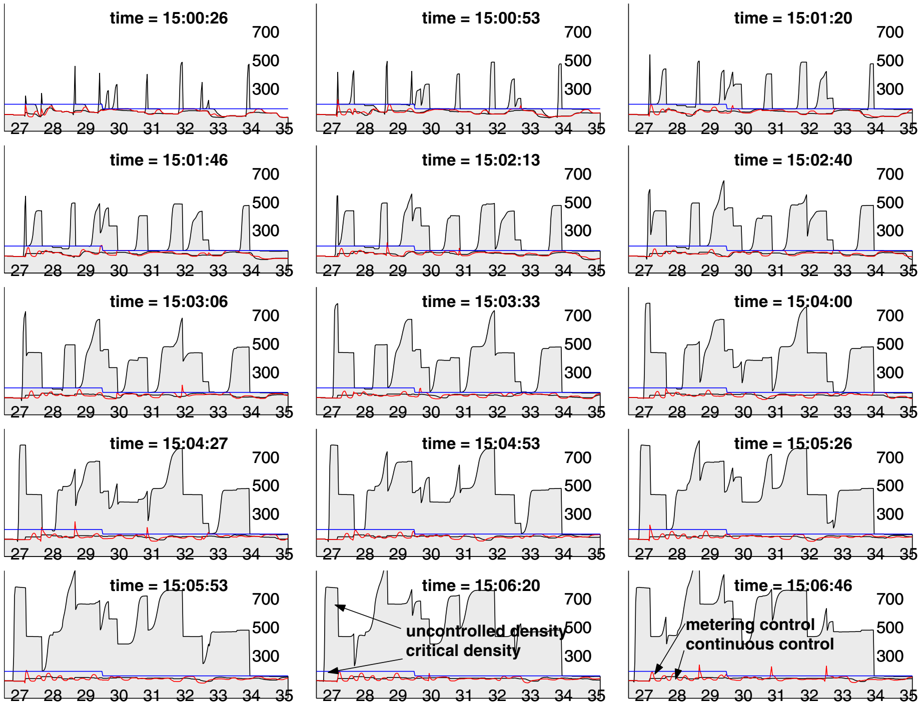


**Fig. 4.** Congestion control results for the highway portion shown in Figure 1. The horizontal axis is the distance (in miles, see Figure 1); the vertical axis is the density (in vehicles per mile). The critical density, above which $q_i$ is the horizontal line with one step (its value decreases when the highway becomes 4 lanes between mile 29 and 30). The uncontrolled density is the solid line enclosing the shaded area. The density controlled with continuous control is the dark solid line below the critical density; the density controlled with metering control is the bright solid below the critical density, with small wiggles. The full simulation is available as a movie at [19].

valid for infinitesimal $\rho'$, which we know is not true in presence of shocks. In our simulations, we add a logarithmic barrier to the density, in order to avoid exceeding the critical density. Given the triangular expression of our fit of $q_i$, we can prove very easily that this will prevent the appearance of shocks. We will also show that this method works in practice in presence of shocks, even though a more careful analysis of the first order perturbation is necessary to mathematically validate this approach.

### 3.3 Application to On/Off Switching for Metering Traffic

From the previous section, we now can compute the continuous $q_i^{\text{on}}$ in order to prevent congestion of the network. We call the corresponding control input *continuous control*. Because these continuous functions must be implemented as on/off metering lights, we can construct a set of $t_n$ from these continuous functions defined by $\int_{t_n}^{t_{n+1}} q_i^{\text{on}}(t)dt = q_{\max,i}T_{\text{green}}$. Here, $T_{\text{green}}$ is the duration of the green light metering, which, after satisfying a minimum practical requirement, we choose arbitrarily (in practice five seconds); $t_0$ is the time at which we want to start the metering, $t_{n+1}$ can be solved recursively from $t_n$, knowing $q_i^{\text{on}}$. We set the light to red in $[t_n, t_{n+1} - T_{\text{green}}]$ and to green in $[t_{n+1} - T_{\text{green}}, t_{n+1}]$. By construction, $t_n \leq t_{n+1} - T_{\text{green}}$. We call *metering control* the resulting control. It is clear that the same amount of flow is released into the network over a time interval of $[t_n, t_{n+1}]$ as with the continuous control, but the corresponding solution is suboptimal, since it does not release all the flow according to $q_i^{\text{on}}$. Figure 4 shows the results of the simulations obtained for the highway system shown in Figure 1, for 10 minutes of traffic. The goal of this simulation is to demonstrate the capability of maintaining the density below the critical density. For this, we take real on-ramp flows (from PeMS data), and multiply them by 10. As can be seen, in absence of control, the density (solid lines above the critical density) almost immediately exceeds the critical density (thick horizontal line) at the on-ramps. Within less than one minute, this density spreads out and progressively jams the rest of the highway (the jam density for this portion of the highway is on the order of 1000 cars / mile). The result of the continuous control can be seen to always remain below the critical density. The metering control only exceeds the critical density for very small amounts of time at locations dispersed over the highway strip. The effect of the continuous control is to increase the VMT by 42% with respect to traffic without metering; in presence of sampling (metering control), the VMT is increased by 41%. If the inflows were only multiplied by 2 or 3, we might be able to get similar results by running longer experiments (it takes a significant amount of time for the highway to become saturated), and the benefit of the metering control only appears once the highway is jammed.

## 4 Hybrid PDE Switching

In the previous section, we derived adjoint-based control for a continuous problem, and implemented it as a discrete approximation in order to use metering lights; in this section, we consider a true hybrid problem, which is based on regulation of car speed limits [9]. In this context, each link $i$ can be in one of finitely many modes, indexed by $j$, and can switch between these modes. The goal of this section is to derive a counterpart to Formula 1 for the case in which the control variables are switching times instead of boundary conditions. Mode $j$ is represented by an LWR PDE which incorporates the speed limit in that mode for link $i$. Consider an infinitely long road, with density $\rho$. Let $T > 0$ be given, and consider $0 = \tau_0 < \tau_1 < \cdots < \tau_M < \tau_{M+1} = T$, where the $\tau_j$, $j \in \{1, \cdots, M\}$ are not known a priori. Assume that the car density on the highway is governed by the following PDE:

$$\mathcal{N}_\tau(\rho) \triangleq \frac{\partial \rho}{\partial t} + \sum_{j=0}^{M} \chi_{[\tau_j, \tau_{j+1}]}(t) \frac{\partial q_j(\rho)}{\partial x} - \epsilon \frac{\partial^2 \rho}{\partial x^2} = 0 \tag{11}$$

In the previous formula, $\chi_{[\tau_j,\tau_{j+1}]}(t) = 1$ for $t \in [\tau_j, \tau_{j+1}]$, and 0 otherwise; the $q_j(\cdot)$ are functions indexed by $j$, representing the flux function of the road in different modes (i.e. with different speed limits). Finally, $\epsilon$ is a small diffusion coefficient, which enables us to eliminate shocks, while preserving the large scale shape of the solution. As a result, we expect the solution of (11) to be smooth.[1] The interpretation of (11) is as follows: for $t \in [\tau_j, \tau_{j+1}]$, $\rho$ is governed by the LWR PDE corresponding to $q_j(\cdot)$ (flux function incorporating maximum allowable speed for mode $j$) augmented by a small diffusion operator to smooth the solution. Note that the order of the switching problem is known a priori.

This problem falls into the class of hybrid systems, where each mode is modeled by a different PDE governing the state. The application of adjoint-based optimization to this problem is, to our best knowledge, new. A counterpart for the conventional hybrid system case in which each mode is governed by an ODE is the problem of optimal control of switching times, for which recent results are presented in [18,7]. In the context of discrete linear hybrid dynamical systems for the same highway problem, observability and controllability results are available in [16].

The control problem consists in maximizing the following cost function: $\sum_{j=0}^{M} \int_{\mathbb{R}} \int_{\tau_j}^{\tau_{j+1}} p(x)q(\rho(x,t))dxdt$, where the control variables are the $\tau_j$, $j \in \{1, \cdots, M\}$, and $p(x)$ is a penalty function, which is arbitrary (for example we might penalize more heavily locations in which we would like fewer cars). In mathematical terms,

$$\begin{array}{ll} \textbf{min:} & \mathcal{J} \triangleq -\sum_{j=1}^{M} \int_{\mathbb{R}} \int_{\tau_j}^{\tau_{j+1}} p(x)q_j(\rho(x,t))dxdt \\ \textbf{s.t.:} & (11), \text{ with } \rho(x,0) = \rho^\circ(x) \text{ given} \\ & 0 = \tau_0 < \tau_1 < \cdots < \tau_M < \tau_{M+1} = T \end{array} \tag{12}$$

We apply the same technique as before: we compute the first variation of $\mathcal{J}$, obtained by perturbation of the $\tau_j$: call $\rho$ the solution of $\mathcal{N}_\tau(\rho) = 0$, and $\overline{\rho}$ the solution of $\mathcal{N}_{\overline{\tau}}(\overline{\rho}) = 0$ (nominal flow). Let $\rho' \triangleq \rho - \overline{\rho}$. Defining

$$\mathcal{N}'_{\overline{\tau}}(\overline{\rho})\rho' \triangleq \frac{\partial \rho'}{\partial t} + \sum_{j=0}^{M} \chi_{[\overline{\tau}_j,\overline{\tau}_{j+1}]}(t) \frac{\partial c_j(\overline{\rho})\rho'}{\partial x} - \epsilon \frac{\partial^2 \rho'}{\partial x^2}$$

it can be shown that $\rho'$ satisfies the following relation:

$$\mathcal{N}'_{\overline{\tau}}(\overline{\rho})\rho' = -\sum_{j=0}^{M} \left(\sigma_{[\tau_j,\overline{\tau}_j]}(t) + \sigma_{[\overline{\tau}_{j+1},\tau_{j+1}]}(t)\right) \frac{\partial q_j(\rho)}{\partial x} \tag{13}$$

[1] Analytical solutions computed for benchmark examples with three links and one switch, available at [19] have shown that numerous undesirable phenomena occur at boundaries of the links and switching surfaces (shocks and expansion waves are generated). In order to cast this problem in a mathematically sound framework, it is necessary to add this diffusion operator to the PDE, which "smooths" the solution and avoids problems of differentiability of the solution.

where $\sigma_{[\tau_j,\overline{\tau}_j]} = \chi_{[\tau_j,\overline{\tau}_j]}$ if $\tau_j < \overline{\tau}_j$ and $\sigma_{[\tau_j,\overline{\tau}_j]} = -\chi_{[\tau_j,\overline{\tau}_j]}$ otherwise. The first variation of $\mathcal{J}$ can be computed in terms of the nominal, perturbed, and perturbation variables: calling $\tau'_j \stackrel{\triangle}{=} \tau_j - \overline{\tau}_j$, we have

$$\mathcal{J}' = \sum_{j=1}^{M} \tau'_j \int_{\mathbb{R}} p(x)[q_j(\overline{\rho}) - q_{j-1}(\overline{\rho})]dx - \sum_{j=1}^{M} \int_{\mathbb{R}} \int_{\overline{\tau}_j}^{\overline{\tau}_{j+1}} p(x)c_j(\overline{\rho})\rho' dxdt$$

Following the steps of the previous section, we can define the adjoint of $\mathcal{N}'_{\overline{\tau}}(\overline{\rho})$ by the identity

$$\langle \rho^* | \mathcal{N}'_{\overline{\tau}}(\overline{\rho})\rho' \rangle \stackrel{\triangle}{=} \langle \mathcal{N}^*_{\overline{\tau}}(\overline{\rho})\rho^* | \rho' \rangle + b$$

A double integration by parts provides the following explicit form of the adjoint:

$$\mathcal{N}^*_{\overline{\tau}}(\overline{\rho})\rho^* = -\frac{\partial \rho^*}{\partial t} - \sum_{j=1}^{M} \chi_{[\overline{\tau}_j,\overline{\tau}_{j+1}]} q_j(\overline{\rho}) \frac{\partial \rho^*}{\partial x} - \epsilon \frac{\partial^2 \rho^*}{\partial x^2}$$

Using the continuity of $\rho'$ and $\rho^*$ at $t = \overline{\tau}_j$, the fact that $\rho'(x, 0) = 0$, the "good choice" $\rho^*(x, T) = 0$, and the assumption that $\lim_{x\to\pm\infty} \rho(x, t) = 0$, as well as its derivatives, we get $b = 0$. Now making the "good choice" $\mathcal{N}^*_{\overline{\tau}}(\overline{\rho})\rho^* = -p(x)c_j(\overline{\rho})$, we can substitute $\langle \mathcal{N}^*_{\overline{\tau}}(\overline{\rho})\rho^* | \rho' \rangle$ into $\mathcal{J}'$. After using the adjoint identity, we obtain:

$$\mathcal{J}' = \sum_{j=1}^{M} \tau'_j \int_{\mathbb{R}} p(x)[q_j(\overline{\rho}) - q_{j-1}(\overline{\rho})]dx + \langle \rho^* | \mathcal{N}'_{\overline{\tau}}(\overline{\rho})\rho' \rangle$$

which we can evaluate using (13). The result provides us with the following expression of the first variation of $\mathcal{J}$.

**Formula 2 (Gradient with respect to switching):** The perturbation of the cost function can be expressed as the inner product of the gradient and the control variables as follows

$$\mathcal{J}' = \sum_{j=1}^{M} \tau'_j \int_{\mathbb{R}} \left\{ p(x)[q_j(\overline{\rho}) - q_{j-1}(\overline{\rho})] + \rho^*(x, \overline{\tau}_j) \left\{ \frac{\partial q_j(\overline{\rho})}{\partial x} - \frac{\partial q_{j-1}(\overline{\rho})}{\partial x} \right\} \right\}_{t=\overline{\tau}_j} dx$$

Formula 2 is a first step towards computing the gradient with respect to switching for the network problem of Section 2. For this, one major difficulty inherent to the network needs to be overcome: the discontinuities of the solutions generated by boundary conditions and perturbation of the inflows (which do not appear in the infinite road problem). This is to our best knowledge a known open problem in fluid mechanics, whose solution will enable the computation of the first variation of the cost function.

## 5 Conclusion

We have shown how to synthesize hybrid controllers for systems governed by PDEs, via an adjoint-based computation of the gradient, which we have embedded in an optimization algorithm. We have shown how to make use of very efficient numerical schemes to compute these gradients accurately, and have performed simulations on a highway congestion problem with PeMS data. We also are interested in computing the Hessian of the optimization problem via the adjoint, in order to avoid the use of steepest descent, for obvious efficiency reasons. We are still in the process of determining when this is possible. We are also interested in higher order models of highway traffic (using second order PDEs). Finally, we will apply these techniques to other networks of PDEs, such as irrigation channels [14], in order to demonstrate the generality of the method.

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