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Guaranteed bounds for traffic flow parameters estimation using mixed Lagrangian-Eulerian sensing.

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Abstract— **This article proposes a new method combining convex optimization and viability theory for estimating traffic flow conditions on highway segments. Traffic flow is modeled by a *Hamilton-Jacobi* equation. Using a Lax-Hopf formula, we formulate the necessary and sufficient conditions for a mixed boundary and internal conditions problem to be well posed. The well-posedness conditions result in a system of linear inequalities, which enables us to compute upper and lower bounds on traffic flow parameters as the solution to a linear program. We illustrate the capabilities of the method with a data assimilation problem for the estimation of the travel time function using Eulerian and Lagrangian measurements generated from *Next Generation Simulation* (NGSIM) traffic data.**

I. INTRODUCTION

A. Motivation

The emergence of smartphones as sensors for the environment has opened the door to several applications in the field of large scale infrastructure monitoring, in particular for transportation systems [23], [38]. *Global Positioning System* (GPS) technology is progressively penetrating the smartphone fleet in use, enabling the ubiquitous mobile monitoring of transportation systems [23], [17].

Transportation networks can be monitored using a combination of *Eulerian* (fixed) or *Lagrangian* (mobile) sensors. Lagrangian sensors are attractive for the transportation infrastructure because their deployment does not imply the usual costs of a public monitoring infrastructure such as loop detectors embedded in the pavement, which comes with maintenance costs. It relies on the communication infrastructure which is market driven and has penetrated the United States at a much more rapid pace than the traffic monitoring infrastructure. Large scale automotive systems such as highways are traditionally monitored using *Eulerian* (or fixed) sensors such as loop detectors [26], cameras, or speed radars. In contrast, *Lagrangian* (or mobile) sensing can be performed using GPS onboard vehicles interfaced with any communication network (the cell phone infrastructure in particular) or transponders [22], [39].

The fundamental challenge of integrating these different types of sensing data is the proper use of a constitutive model of the system, which in the case of the highway is a traffic flow model. The process of integrating sensing data (Eulerian or Lagrangian) into a flow model is called *data assimilation* or *inverse modeling* [25], [33]. Models commonly used in traffic engineering include first-order macroscopic traffic flow models [28], [35]. First-order traffic flow models are of great interest for the traffic flow assimilation problem since they only depend on a few parameters easily identifiable, inherent to the features of the physical system. Second-order models are potentially more accurate than first order models when reproducing a given situation [6], [19], [40], but are requiring the knowledge of more parameters, and are thus less robust. First-order flow models describe the evolution of a vehicle density function, which is related to the degree of congestion of a highway. An alternate first order formulation of traffic flow, known as the *cumulative number of vehicles*, or Moskowitz function [31], [29] appears in this context as a very appropriate framework to use in order to incorporate Lagrangian data into a flow model. The Moskowitz function has been vastly adopted in the transportation engineering community as a good model for following "tagged" vehicles, *i.e.* vehicles to which labels have been added to follow them during their trips. This function has been shown by Newell to satisfy a *Hamilton-Jacobi* (HJ) *partial differential equation* (PDE) [32]. The proper mathematical solution to this equation was defined in [3], using viability [1], [2], and concepts key to control theory such as capturability (capture basins). The concept of *component of the solution* was introduced later in [10], to be able to decompose the solutions into different pieces which enable one to include Lagrangian sensing data, *i.e.* data obtained inside the physical domain (not at the boundaries as commonly done when dealing with PDEs). In the same article, necessary and sufficient conditions for a *mixed initial-boundary-internal condition problem* to be well posed were also derived, leading to a mathematical framework capable of integrating Lagrangian sensing information.

Contributions of the article. The first main contribution of this article is the derivation of the necessary and sufficient conditions on the boundary and internal conditions for a given *mixed boundary and internal conditions problem* to be properly formulated in the Barron-Jensen/Frankowska sense. These conditions result in a set of linear inequalities on some traffic parameters. These linear inequalities encode the constraints on the degrees of liberty of the reconstruction problem. The second contribution is the construction of a *Linear Program* (LP) which enables us to compute bounds on the accumulation of vehicles on the highway at the initial time. These bounds can be used to compute the smallest and largest travel time functions compatible with the boundary and internal conditions. This procedure is common in data

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assimilation problems in which the initial state is unknown in general. The third contribution is the implementation of the linear programs using real traffic flow data obtained in the framework of the NGSIM experiment. We also use the results to evaluate the trade-off between accuracy and privacy inherent to Lagrangian sensing.

II. MATHEMATICAL FRAMEWORK USED IN THIS ARTICLE

We define the set X as $X := [\xi, \chi] \subset \mathbb{R}$ where ξ represents the upstream boundary and χ represents the downstream boundary of the physical (and computational) domain. The state of a first order hyperbolic conservation law is described by a density function [27], [19], which for the application of interest represents an aggregated number of vehicles per unit length. The density function is traditionally denoted $\rho(t,x)$ at time t and location x, and is the solution of the following first order hyperbolic conservation law:

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} = 0 \tag{1}$$

In the above equation, $\psi(\cdot)$ is a concave function defined on $[0,\omega]$ known as the *flux function* or *fundamental diagram* [36], [3], [10] where ω is called *jam density*. In the context of traffic flow, the flux function satisfies $\psi'(0) = \nu^\flat$ and $\psi'(\omega) = -\nu^\sharp$, where $\nu^\flat > 0$ and $\nu^\sharp > 0$.

Instead of describing traffic flow in terms of a density function [27], [36], a possible alternate formulation known as the *Moskowitz function* uses a Hamilton-Jacobi equation for an integral of the function ρ [12], [3], [10], [11].

Definition 2.1: [Moskowitz function] Let consecutive integer labels be assigned to vehicles entering the highway at location $x = \xi$. The Moskowitz function $\mathbf{M}(\cdot,\cdot)$ is a continuous function satisfying $\lfloor \mathbf{M}(t,x) \rfloor = n$ where n is the label of the vehicle located in x at time t [15], [16]. Hence, $\mathbf{M}(t,x)$ represents the label of the vehicle located at x at time t, counted from the reference point $(0,\xi)$ corresponding to the vehicle numbered 0. The Moskowitz function is traditionally denoted $\mathbf{M}(t,x)$ at time t and location x, and solves the following equation:

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) = 0 \tag{2}$$

The proper notion of weak solution used in the present article is the Barron-Jensen/Frankowska solution [7], [18]. The key feature of this weak solution (also used in [10], based on [3]) is the lower semicontinuity of the solution. The formal link between this class of weak solutions and the viscosity solution [14], [13] has been formally established by Frankowska [18].

The formal link between the density function $\rho(\cdot,\cdot)$ and the Moskowitz function $\mathbf{M}(\cdot,\cdot)$ is given by:

$$\mathbf{M}(t_2,x_2) - \mathbf{M}(t_1,x_1) = \int_{x_1}^{x_2} -\rho(t_1,x)dx + \int_{t_1}^{t_2} \psi(\rho(t,x_2))dt \tag{3}$$

One of the fundamental contributions of the present article as well as [10], [11] is the use of control theoretic methods (in the present case viability theory [1] and set-valued analysis [4]) to construct the proper solutions to the problem (26) of Definition 4.2. This solution is called viability episolution. To construct it, we need to define a convex transform of the flux function $\psi(\cdot)$ as follows:

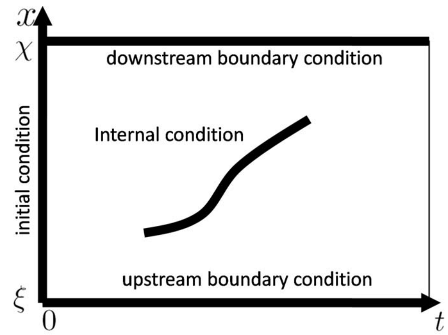

Fig. 1. **Illustration of the domains of the possible value conditions used to construct the solution of the Moskowitz HJ PDE**.

Definition 2.2: [Convex transform] For a concave function $\psi(\cdot)$ defined as previously, the convex transform φ^* is given by:

$$\varphi^*(u) := \begin{cases} \sup_{p \in \mathrm{Dom}(\psi)} [p \cdot u + \psi(p)] & \text{if } u \in [-\nu^\flat, \nu^\sharp] \\ +\infty & \text{otherwise} \end{cases} \tag{4}$$

The function $\varphi^*(\cdot)$ is convex as the pointwise supremum of affine functions [9], and is defined on the interval $\mathrm{Dom}(\varphi^*) := [-\nu^\flat, \nu^\sharp]$. Note that since $\psi(\cdot)$ is concave and satisfies $\psi'(0) = \nu^\flat$, the function $\varphi^*(\cdot)$ satisfies $\varphi^*(-\nu^\flat) := \sup_{p \in \mathrm{Dom}(\psi)}[-p\nu^\flat + \psi(p)] = 0$. Since $\psi(0) = 0$ and $0 \in [0,\omega]$, we have by definition (4) that $\varphi^*(\cdot) \geq 0$. Since $\varphi^*(\cdot)$ is convex, it is subdifferentiable [9] on $[-\nu^\flat, \nu^\sharp]$, and its subderivative satisfies the Legendre inversion formula [3]:

$$u \in -\partial_+ \psi(\rho) \quad \text{if and only if} \quad \rho \in \partial_- \varphi^*(u)$$

One contribution of the articles [3], [10], [11] was to propose a solution of equation (2) (*i.e.* problem (26)) using a new mathematical framework for this problem based on viability theory. For this, we define an auxiliary dynamical system F associated to the HJ PDE (2) as follows, referred to as characteristic system [3], [10]:

Definition 2.3: [Auxiliary dynamical system] We define an auxiliary dynamical system F associated to the HJ PDE (2):

$$F := \begin{cases} \tau'(t) = -1 \\ x'(t) = u(t) \\ y'(t) = -\varphi^*(u(t)) \end{cases} \quad \text{where } u(t) \in \mathrm{Dom}(\varphi^*) \tag{5}$$

This dynamical system is both Marchaud and Lipschitz [3]. The function $u(\cdot)$ is called auxiliary control of the dynamical system F.

Definition 2.4: [Environment,Target] We define an environment set $\mathcal{K}$ as $\mathcal{K} := \mathbb{R}_+ \times X \times \mathbb{R}$, which will describe constraints of the problem. We also define a target $\mathcal{C}$ in epigraphical form by $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$, where $\mathbf{c}(\cdot,\cdot)$ is a given lower continuous function defined on $X \times \mathbb{R}_+$. The set $\mathcal{E}pi(\mathbf{c})$ is the *epigraph* of the function $\mathbf{c}$, and corresponds to the subset of triples $(t, x, y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \geq \mathbf{c}(t, x)$.

Definition 2.5: [Viability episolution] Following [3], [10], we define the viability episolution $\mathbf{M_c}(\cdot,\cdot)$ associated to the target $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$ by:

$$\mathbf{M_c}(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{C})\} \qquad (6)$$

where $\text{Capt}_F(\mathcal{K}, \mathcal{C})$ represents the capture basin of $\mathcal{C}$ in $\mathcal{K}$ by F defined in [1], [2], [3]: denoting $\mathcal{S}_F(X_0)$ the set of solutions $X(\cdot)$ to the differential inclusion $X'(\cdot) \in F(X(\cdot))$ satisfying $X(0) = X_0$, the capture basin of $\mathcal{C}$ in $\mathcal{K}$ by F is given by:

$$\begin{array}{l} \text{Capt}_F(\mathcal{K}, \mathcal{C}) := \{X_0 \in \mathcal{K} \mid \exists X(\cdot) \in \mathcal{S}_F(X_0) \text{ and } \exists T \geq 0 \\ \text{such that } X(T) \in \mathcal{C} \text{ and } \forall t \in [0, T], \;\; X(t) \in \mathcal{K}\} \end{array} \qquad (7)$$

Capture basins are a common tool in control theory which appear in different forms in articles related to capture problems, see for example [30], [5], [21], [34], [24].

The episolution $\mathbf{M_c}$ associated to any given lower semicontinuous target function $\mathbf{c}$ is a solution to the Moskowitz HJ PDE (2) in the Barron-Jensen/Frankowska sense [3], [10].

III. Componentwise construction of the viability episolution

A. Inf-morphism property

It is well known [1], [2], [3] that for a given environment $\mathcal{K}$, the capture basin of a finite union of targets is the union of the capture basins of these targets:

$$\text{Capt}_F\left(\mathcal{K}, \bigcup_{i \in I} \mathcal{C}_i\right) = \bigcup_{i \in I} \text{Capt}_F(\mathcal{K}, \mathcal{C}_i) \qquad (8)$$

where I is a finite set. This property can be translated in epigraphical form:

Proposition 3.1: [Inf-morphism property] [3] Let $\mathbf{c}_i$ (i belongs to a finite set I) be a family of functions whose epigraphs are the targets $\mathcal{C}_i$. Since the epigraph of the minimum of the functions $\mathbf{c}_i$ is the union of the epigraphs of the functions $\mathbf{c}_i$, the target $\mathcal{C} := \bigcup_{i \in I} \mathcal{C}_i$ is the epigraph of the function $\mathbf{c} := \inf_{i \in I} \mathbf{c}_i$. Hence, equation (6) implies the following property:

$$\forall\, t \geq 0, \; x \in X, \;\; \mathbf{M_c}(t, x) \; = \; \inf_{i \in I} \mathbf{M}_{\mathbf{c}_i}(t, x) \qquad (9)$$

Using the inf-morphism property, we can construct the episolution associated to the mixed initial, boundary and internal conditions problem using the concept of *components*. The components are encoding the influence of each of the value conditions (initial, boundary and internal conditions) on the solution.

B. Components of the Moskowitz function

Definition 3.2: [Components of the Moskowitz function] The component $\mathbf{M}_{\mathbf{c}_i}$ associated to the target function $\mathbf{c}_i$ is defined by:

$$\mathbf{M}_{\mathbf{c}_i}(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{E}pi(\mathbf{c}_i))\} \qquad (10)$$

Example 3.3: [Initial, boundary and internal condition components] We consider four functions[1] $\mathbf{M}_0(\cdot,\cdot)$, $\gamma(\cdot,\cdot)$, $\beta(\cdot,\cdot)$ and $\mu(\cdot,\cdot)$, satisfying the following properties:

$$\mathbf{M}_0(t, x) := \begin{cases} \mathbf{M}_0(0, x) \text{ (given)} & \text{for } t = 0 \text{ and } x \in X \\ +\infty & \forall t \neq 0 \text{ or } \forall x \notin X \end{cases} \qquad (11)$$

$$\gamma(t, x) := \begin{cases} \gamma(t, \xi) \text{ (given)} & \text{for } x = \xi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \xi \text{ or } \forall t < 0 \end{cases} \qquad (12)$$

$$\beta(t, x) := \begin{cases} \beta(t, \chi) \text{ (given)} & \text{for } x = \chi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \xi \text{ or } \forall t < 0 \end{cases} \qquad (13)$$

$$\mu(t, x) := \begin{cases} \overline{\mathbf{M}} \text{ (given)} & \text{if } (t, x) \in \text{Dom}(\mu) \text{ (given)} \\ +\infty & \text{otherwise} \end{cases} \qquad (14)$$

In the remainder of this article, we assume that the domain of the trajectory label function $\mu(\cdot,\cdot)$ is of the form $\text{Dom}(\mu) := \text{Graph}(\overline{x}(\cdot))$. The function $\overline{x}(\cdot)$ is the trajectory function associated to $\mu(\cdot,\cdot)$. The initial condition component $\mathbf{M}_{\mathbf{M}_0}$, left boundary condition component $\mathbf{M}_\gamma$, right boundary condition component $\mathbf{M}_\beta$, and internal condition component $\mathbf{M}_\mu$ associated to the target functions $\mathbf{M}_0$, γ. β and μ respectively, are defined by the following formulas:

$$\begin{cases} \mathbf{M}_{\mathbf{M}_0}(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{E}pi(\mathbf{M}_0))\} \\ \mathbf{M}_\gamma(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{E}pi(\gamma))\} \\ \mathbf{M}_\beta(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{E}pi(\beta))\} \\ \mathbf{M}_\mu(t, x) \; := \; \inf \{y \mid (t, x, y) \in \text{Capt}_F(\mathcal{K}, \mathcal{E}pi(\mu))\} \end{cases} \qquad (15)$$

The domains of the initial, boundary and internal conditions are illustrated in Figure 1. The component $\mathbf{M}_c$ associated to a given target function $\mathbf{c}$ can be computed using the following *generalized Lax-Hopf formula*:

Theorem 3.4: [Generalized Lax Hopf formula] The component $\mathbf{M_c}$ associated to a target $\mathcal{C} := \mathcal{E}pi(\mathbf{c})$, for a given lower semicontinuous function $\mathbf{c}$, and defined by equation (10) can be expressed as:

[1]The dependency of $\mathbf{M}_0$, γ and β on two arguments has been added for notational consistency. Note that $\mathbf{M}_0(t, x)$ is only defined when $t = 0$ and $x \in X$, that $\gamma(t, x)$ is only defined when $t \geq 0$ and $x = \xi$, and that$\beta(t, x)$ is only defined when $t \geq 0$ and $x = \chi$.

$$\mathbf{M_c}(t,x) = \inf_{(u,T)\in \mathrm{Dom}(\varphi^*)\times\mathbb{R}_+} \left(\mathbf{c}(t-T,x+Tu)+T\varphi^*(u)\right) \quad (16)$$

Proof — The proof is available in [10]. ■

Proposition 3.5: [Barron-Jensen/Frankowska property] [3], [10] The viability episolution $\mathbf{M_c}(t,x)$ associated to a given lower semicontinuous function $\mathbf{c}$ is a Barron-Jensen/Frankowska (BJ/F) solution to the Hamilton-Jacobi PDE (2).

Proof — See [10] for a proof of this property. ■

C. Proper formulation conditions

Definition 3.6: [Proper formulation of a component] The component $\mathbf{M}_{\mathbf{c}_i}$ associated to a target function $\mathbf{c}_i$ is said to be *properly formulated* if the following condition is satisfied:

$$\forall (t,x)\in \mathrm{Dom}(\mathbf{c}_i),\ \ \mathbf{M}_{\mathbf{c}_i}(t,x) = \mathbf{c}_i(t,x) \qquad (17)$$

Remark — It is well known [1] that for any environment $\mathcal{K}$ and target $\mathcal{C}$, we have $\mathcal{C}\subset \mathrm{Capt}_F(\mathcal{K},\mathcal{C})$, which implies the following inequality:

$$\forall (t,x)\in \mathrm{Dom}(\mathbf{c}_i),\ \ \mathbf{M}_{\mathbf{c}_i}(t,x) \leq \mathbf{c}_i(t,x) \qquad (18)$$

A component is thus properly formulated if and only if the converse inequality is true. □

Example 3.7: The initial condition and internal condition components defined by equation (15) are always properly formulated. The left and right boundary condition components are properly formulated if and only if the functions γ and β satisfy:

$$\begin{array}{l} \forall t\in\mathbb{R}_+,\ \forall T\in[0,T], \gamma\left(t-T,\xi\right)+T\varphi^*(0)\geq \gamma\left(t,\xi\right) \\ \forall t\in\mathbb{R}_+,\ \forall T\in[0,T], \beta\left(t-T,\chi\right)+T\varphi^*(0)\geq \beta\left(t,\xi\right) \end{array} \qquad (19)$$

The proof of these properties is available in [10]. In the remainder of this article, we assume that all the components are properly formulated.

D. Mixed initial, boundary and internal conditions problem

Definition 3.8: [Mixed initial, boundary and internal conditions problem] We consider a initial condition function $\mathbf{M}_0$ as defined in equation (11), a left boundary condition function γ as defined in equation (12), a right boundary condition function β as defined in equation (13), and multiple trajectory functions $\overline{x}_i(\cdot),\ \ i\in I$ defined in the time intervals $[\overline{t}_{\min_i},\overline{t}_{\max_i}]$ and associated to the vehicles labeled $\overline{\mathbf{M}}_i$. The trajectory label functions μ_i are defined by:

$$\mu_i(t,x) := \begin{cases} \overline{\mathbf{M}}_i & \text{if } \ (t,x)\in \mathrm{Graph}(\overline{x}_i) \\ +\infty & \text{otherwise} \end{cases} \qquad (20)$$

The solution $\mathbf{M}$ to the associated *mixed initial, boundary and internal conditions problem* is defined as:

$$\begin{cases} \mathbf{M} \text{ is a BJ/F solution to equation } (2) \\ \mathbf{M}(0,x) = \mathbf{M}_0(x) \quad \forall x\in X \\ \mathbf{M}(t,\xi) = \gamma(t,\xi) \quad \forall t\in\mathbb{R}_+ \\ \mathbf{M}(t,\chi) = \beta(t,\chi) \quad \forall t\in\mathbb{R}_+ \\ \mathbf{M}(t,\overline{x}_i(t)) = \overline{\mathbf{M}}_i \quad \forall i\in I,\ \ \forall t\in[\overline{t}_{\min_i},\overline{t}_{\max_i}] \end{cases} \qquad (21)$$

Definition 3.9: [Target function] For the mixed initial, boundary and internal conditions problem, we define the target function $\mathbf{c}$ as:

$$\mathbf{c} = \min\left(\mathbf{M}_0,\gamma,\beta,\min_{i\in I}\mu_i\right) \qquad (22)$$

Proposition 3.5 states that the episolution $\mathbf{M_c}$ associated to target $\mathbf{c}$ defined by equation (22) is a Barron-Jensen/Frankowska solution to the Moskowitz HJ PDE.

Proposition 3.10: [Properties of the episolution associated to $\mathbf{c}$*]* The episolution $\mathbf{M_c}$ associated to the target $\mathbf{c}$ defined by equation (22) satisfies the following equalities:

$$\begin{cases} \mathbf{M_c}(0,x) = & \min(\mathbf{M}_0(0,x),\mathbf{M}_\gamma(0,x),\mathbf{M}_\beta(0,x), \\ & \min_{i\in I}\mathbf{M}_{\mu_i}(0,x)) \\ & \forall x\in X \\ \mathbf{M_c}(t,\xi) = & \min(\mathbf{M}_{\mathbf{M}_0}(t,\xi),\gamma(t,\xi),\mathbf{M}_\beta(t,\xi), \\ & \min_{i\in I}\mathbf{M}_{\mu_i}(t,\xi)) \\ & \forall t\in\mathbb{R}_+ \\ \mathbf{M_c}(t,\chi) = & \min(\mathbf{M}_{\mathbf{M}_0}(t,\chi),\mathbf{M}_\gamma(t,\xi),\beta(t,\chi), \\ & \min_{i\in I}\mathbf{M}_{\mu_i}(t,\chi)) \\ & \forall t\in\mathbb{R}_+ \\ \mathbf{M_c}(t,\overline{x}_i(t)) = & \min(\mathbf{M}_{\mathbf{M}_0}(t,\overline{x}_i(t)),\mathbf{M}_\gamma(t,\overline{x}_i(t)) \\ & \mathbf{M}_\beta(t,\overline{x}_i(t)),\overline{\mathbf{M}}_i,\ \min_{j\in I\setminus\{i\}}\mathbf{M}_{\mu_j}(t,\overline{x}_i(t))) \\ & \forall i,\ \ \forall t\in[\overline{t}_{\min_i},\overline{t}_{\max_i}] \end{cases} \qquad (23)$$

Proof — The inf-morphism property implies that $\mathbf{M_c} = \min\left(\mathbf{M}_{\mathbf{M}_0},\mathbf{M}_\gamma,\mathbf{M}_\beta,\min_{i\in I}\mathbf{M}_{\mu_i}\right)$. Since the components $\mathbf{M}_{\mathbf{M}_0}$, $\mathbf{M}_\gamma$, $\mathbf{M}_\beta$ and $\mathbf{M}_{\mu_i}$ are properly formulated, we have the following equalities:

$$\begin{cases} \mathbf{M}_{\mathbf{M}_0}(0,x) = \mathbf{M}_0(0,x) & \forall x\in X \\ \mathbf{M}_\gamma(t,\xi) = \gamma(t,\xi) & \forall t\in\mathbb{R}_+ \\ \mathbf{M}_\beta(t,\chi) = \beta(t,\chi) & \forall t\in\mathbb{R}_+ \\ \mathbf{M}_{\mu_i}(t,\overline{x}_i(t)) = \overline{\mathbf{M}}_i & \forall t\in[\overline{t}_{\min_i},\overline{t}_{\max_i}] \end{cases} \qquad (24)$$

This last property implies equation (23). ■

Proposition 3.11: [Solution to the mixed initial, boundary and internal conditions problem] The function $\mathbf{M}$ satisfying equation (26) is the episolution associated to the target $\mathbf{c} := \min\left(\mathbf{M}_0,\gamma,\beta,\mu_i\right)$ if and only if the following conditions are satisfied:

$$
\begin{array}{lll}
(i) & \mathbf{M}_\gamma(0,x) \geq \mathbf{M}_0(0,x) & \forall x \in X \\
(ii) & \mathbf{M}_{\mathbf{M}_0}(t,\xi) \geq \gamma(t,\xi) & \forall t \in \mathbb{R}_+ \\
(iii) & \mathbf{M}_\beta(0,x) \geq \mathbf{M}_0(0,x) & \forall x \in X \\
(iv) & \mathbf{M}_{\mathbf{M}_0}(t,\chi) \geq \beta(t,\chi) & \forall t \in \mathbb{R}_+ \\
(v) & \mathbf{M}_\gamma(t,\chi) \geq \beta(t,\chi) & \forall t \in \mathbb{R}_+ \\
(vi) & \mathbf{M}_\beta(t,\xi) \geq \gamma(t,\xi) & \forall t \in \mathbb{R}_+ \\
(vii) & \mathbf{M}_{\mathbf{M}_0}(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\
(viii) & \mathbf{M}_{\mu_i}(0,x) \geq \mathbf{M}_0(0,x) & \forall i \in I, \forall x \in X \\
(ix) & \mathbf{M}_\gamma(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\
(x) & \mathbf{M}_{\mu_i}(t,\xi) \geq \gamma(t,\xi) & \forall i \in I, \forall t \in \mathbb{R}_+ \\
(xi) & \mathbf{M}_\beta(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\
(xii) & \mathbf{M}_{\mu_i}(t,\chi) \geq \beta(t,\chi) & \forall i \in I, \forall t \in \mathbb{R}_+ \\
(xiii) & \mathbf{M}_{\mu_j}(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall j \in I \backslash \{i\} \\
 & & \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]
\end{array} \tag{25}
$$

Proof — Equation (25) is a direct consequence of equations (23) and (21). ■

IV. TRAVEL TIME ESTIMATION USING MIXED BOUNDARY AND INTERNAL CONDITIONS

A. Computation of the travel time function

In the remainder of the article, we assume that vehicles are not overtaking each other. Since no overtaking occurs, the vehicles keep their label with time, the trajectory of a given vehicle $\overline{\mathbf{M}}$ is included in the set $\{(t,x) \text{ s.t. } \mathbf{M}(t,x) = \overline{\mathbf{M}}\}$ (this is a level-set [10] of the Moskowitz function[2] $\mathbf{M}$. Figure 2 illustrates this property.

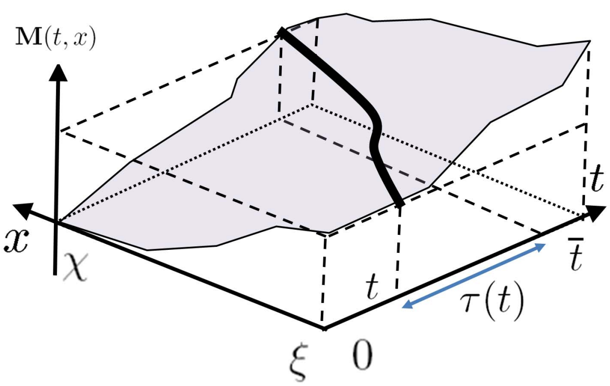

Fig. 2. **Isolevel of the Moskowitz function**. The projection of this level set in the (t,x) plane is the trajectory of vehicle $\overline{\mathbf{M}}$.

We assume for simplicity in the remainder of the article that $\beta(\cdot,\cdot)$ is injective. The travel time between the upstream and downstream boundary (*i.e.* the travel time at time t is defined as follows.

Definition 4.1: [Travel time] Let us set $\overline{\mathbf{M}}$ as $\overline{\mathbf{M}} := \mathbf{M}(t,\xi) = \gamma(t,\xi)$ ($\overline{\mathbf{M}}$ corresponds to the label of the entering vehicle at t). We assume that there exists $\overline{t}$ such that $\overline{\mathbf{M}} = \beta(\overline{t},\chi)$ (this amounts to saying that the vehicle $\overline{\mathbf{M}}$ exits the highway at a finite time $\overline{t}$). Since $\beta(\cdot,\cdot)$ is injective, $\overline{t}$ is uniquely defined (when it exists). The travel time $\tau(t)$ required to cross the highway section at time t is defined by $\tau(t) = \overline{t} - t$ (this corresponds to the duration of the trip of the vehicle $\overline{\mathbf{M}}$ between the boundaries ξ and χ). This property is illustrated in Figure 2.

[2] The case in which the interior of this set is not empty corresponds to a locally empty highway, in which by default the speed is the free flow speed.

B. Mixed boundary and internal conditions problem

Because in highway state estimation problems, initial condition are rarely known, we cannot prescribe $\mathbf{M}_0(\cdot,\cdot)$ in practice. Using the inf-morphism property, we can circumvent this difficulty by omitting $\mathbf{M}_0(\cdot,\cdot)$ and computing the BJ/F solution to the resulting *mixed boundary and internal conditions problem* only.

Definition 4.2: [Mixed boundary and internal conditions problem] We consider a left boundary condition function γ as defined in equation (12), a right boundary condition function β as defined in equation (13), and multiple functions $\mu_i,\ i \in I$ (where I is a finite set) as defined in equation (20).

The solution $\mathbf{M}$ to the associated mixed boundary and internal conditions problem is defined as:

$$
\left\{
\begin{array}{lll}
\multicolumn{3}{l}{\mathbf{M} \text{ is a BJ/F solution to the HJ PDE (2)}} \\
\mathbf{M}(t,\xi) = \gamma(t,\xi) & \forall t \in \mathbb{R}_+ & \text{LBC} \\
\mathbf{M}(t,\chi) = \beta(t,\chi) & \forall t \in \mathbb{R}_+ & \text{RBC} \\
\mathbf{M}(t,x) = \mu_i(t,x) & \forall (t,x) \in \text{Dom}(\mu_i) & \text{IBCs}
\end{array}
\right. \tag{26}
$$

Definition 4.3: [Target function] For the mixed boundary and internal conditions problem, we define the target function $\mathbf{c}$ as:

$$
\mathbf{c} = \min\left(\gamma, \beta, \min_{i \in I} \mu_i\right) \tag{27}
$$

Proposition 3.5 states that the episolution $\mathbf{M}_\mathbf{c}$ associated to target $\mathbf{c}$ defined by equation (27) is a Barron-Jensen/Frankowska solution to the Moskowitz HJ PDE.

Proposition 4.4: [Properties of the episolution associated to $\mathbf{c}$*]* The episolution $\mathbf{M}_\mathbf{c}$ associated to the target $\mathbf{c}$ defined by equation (22) satisfies the following equalities:

$$
\left\{
\begin{array}{ll}
\mathbf{M}_\mathbf{c}(t,\xi) = & \min(\gamma(t,\xi), \mathbf{M}_\beta(t,\xi), \min_{i \in I} \mathbf{M}_{\mu_i}(t,\xi)) \\
 & \forall t \in \mathbb{R}_+ \\
\mathbf{M}_\mathbf{c}(t,\chi) = & \min(\mathbf{M}_\gamma(t,\xi), \beta(t,\chi), \min_{i \in I} \mathbf{M}_{\mu_i}(t,\chi)) \\
 & \forall t \in \mathbb{R}_+ \\
\mathbf{M}_\mathbf{c}(t,\overline{x}_i(t)) = & \min(\mathbf{M}_\gamma(t,\overline{x}_i(t)), \mathbf{M}_\beta(t,\overline{x}_i(t)), \overline{\mathbf{M}}_i, \\
 & \min_{j \in I \backslash \{i\}} \mathbf{M}_{\mu_j}(t,\overline{x}_i(t))) \\
 & \forall i,\ \ \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]
\end{array}
\right. \tag{28}
$$

Proof — The inf-morphism property implies that $\mathbf{M}_\mathbf{c} = \min\left(\mathbf{M}_\gamma, \mathbf{M}_\beta, \min_{i \in I} \mathbf{M}_{\mu_i}\right)$. Since the components $\mathbf{M}_\gamma$, $\mathbf{M}_\beta$ and $\mathbf{M}_{\mu_i}$ are properly formulated, we have the following equalities:

$$
\left\{
\begin{array}{ll}
\mathbf{M}_\gamma(t,\xi) = \gamma(t,\xi) & \forall t \in \mathbb{R}_+ \\
\mathbf{M}_\beta(t,\chi) = \beta(t,\chi) & \forall t \in \mathbb{R}_+ \\
\mathbf{M}_{\mu_i}(t,\overline{x}_i(t)) = \overline{\mathbf{M}}_i & \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]
\end{array}
\right. \tag{29}
$$

This last property implies equation (28). ∎

Proposition 4.5: [Solution to the mixed boundary and internal conditions problem] The episolution $\mathbf{M_c}$ associated to the target $\mathbf{c}$ defined by equation (22) satisfies equation (26) if and only if the following conditions are satisfied:

$$\begin{array}{lll}
(i) & \mathbf{M}_\gamma(t,\chi) \geq \beta(t,\chi) & \forall t \in \mathbb{R}_+ \\
(ii) & \mathbf{M}_\beta(t,\xi) \geq \gamma(t,\xi) & \forall t \in \mathbb{R}_+ \\
(iii) & \mathbf{M}_\gamma(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\
(iv) & \mathbf{M}_{\mu_i}(t,\xi) \geq \gamma(t,\xi) & \forall i \in I, \forall t \in \mathbb{R}_+ \\
(v) & \mathbf{M}_\beta(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\
(vi) & \mathbf{M}_{\mu_i}(t,\chi) \geq \beta(t,\chi) & \forall i \in I, \forall t \in \mathbb{R}_+ \\
(vii) & \mathbf{M}_{\mu_j}(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall j \in I\backslash\{i\} \\
 & & \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]
\end{array} \tag{30}$$

Proof — Equation (30) is a direct consequence of equations (12), (13) and (26). ∎

V. Linear programming methods for travel time estimation using flow and trajectory data

A. Framework used for the reconstruction

In this section, we assume that the inflow and outflow functions are obtained via Eulerian sensing (for instance using loop detectors at ξ and χ). The inflow and outflow functions q_{inflow} and q_{outflow} represent the number of vehicles respectively entering and exiting the highway per unit time, and satisfy the following properties:

$$q_{\text{inflow}}(t) := \frac{\partial \gamma(t,\xi)}{\partial t} \quad \text{and} \quad q_{\text{outflow}}(t) := \frac{\partial \beta(t,\chi)}{\partial t} \tag{31}$$

Proposition 5.1: The functions $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are given by the following formulas:

$$\begin{array}{l}
\gamma(t,x) := \left\{ \begin{array}{ll} \int_0^t q_{\text{inflow}}(\tau)d\tau & \text{for } x=\xi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \xi \text{ or } \forall t<0 \end{array} \right. \\
\beta(t,x) := \left\{ \begin{array}{ll} \int_0^t q_{\text{outflow}}(\tau)d\tau + \Delta & \text{for } x=\chi \text{ and } t \geq 0 \\ +\infty & \forall x \neq \chi \text{ or } \forall t<0 \end{array} \right.
\end{array} \tag{32}$$

where $-\Delta$ represents the total number of vehicles located between ξ and χ at the initial time, which is unknown a priori.

Proof — Equation (32) is a direct consequence of equations (28) and (31). Note that the Moskowitz function satisfies $\mathbf{M}(0,\xi) = 0$. Note also that the parameter $-\Delta$ corresponds to $\mathbf{M}(0,\xi) - \mathbf{M}(0,\chi)$, which represents the total number of vehicles located between ξ and χ at time $t=0$. The parameter $-\Delta$ is the unknown of the problem, over which the optimization is run, to perform the optimal travel time estimation. ∎

Remark — The functions $q_{\text{inflow}}(\cdot)$ and $q_{\text{outflow}}(\cdot)$ are positive when all vehicles flow towards the χ direction. We will assume that this is the case for the remainder of the article. Equation (32) thus implies that $\gamma(\xi,\cdot)$ and $\beta(\chi,\cdot)$ are increasing. □

Remark — Equation (32) implicitly assumes that no integration errors occur. In practice, real sensors readings $q_{\text{inflow}}(\cdot)$ and $q_{\text{outflow}}(\cdot)$ are affected by both systematic and random errors. Equation (32) is thus accurate on short time horizons (typically a few minutes) only. □

We also assume that we can use n pieces of trajectory functions $\overline{x}_i(\cdot), \; i \in I$ for the estimation, where I is a finite set. These trajectories represent vehicles which could be locally (for short periods of time) tracked. Note that the labels $\overline{\mathbf{M}}_i$ of the vehicles are unknown, and are therefore also treated as decision variables.

Proposition 5.2: [Lax-Hopf formulas associated to the flow and trajectory constraints] The components $\mathbf{M}_\gamma$, $\mathbf{M}_\beta$ and $\mathbf{M}_{\mu_i}$ can be computed for all $(t,x) \in \mathbb{R}_+ \times X$ using the following Lax-Hopf formulas:

$$\left\{ \begin{array}{l}
\mathbf{M}_\gamma(t,x) = \\
\quad \inf_{u \in \text{Dom}(\varphi^*) \text{ s. t. } \frac{\xi-x}{u} \geq 0} \left(\gamma\left(t - \frac{\xi-x}{u}, \xi\right) + \frac{\xi-x}{u}\varphi^*(u) \right) \\
\mathbf{M}_\beta(t,x) = \\
\quad \inf_{u \in \text{Dom}(\varphi^*) \text{ s. t. } \frac{\chi-x}{u} \geq 0} \left(\beta\left(t - \frac{\chi-x}{u}, \chi\right) + \frac{\chi-x}{u}\varphi^*(u) \right) \\
\mathbf{M}_{\mu_i}(t,x) = \\
\quad \inf_{T \in \mathbb{R}_+ \cap [t-\overline{t}_{\max_i}, t-\overline{t}_{\min_i}]} \left(\overline{\mathbf{M}}_i + T\varphi^*\left(\frac{\overline{x}_i(t-T)-x}{T} \right) \right)
\end{array} \right. \tag{33}$$

Remark — As can be seen from equations (32) and (33), the components $\mathbf{M}_\gamma$, $\mathbf{M}_\beta$ and $\mathbf{M}_{\mu_i}$, as well as the functions γ, β and μ_i are affine functions of the parameters Δ and $\overline{\mathbf{M}}_1, \ldots, \overline{\mathbf{M}}_{i_{\max}}$. We explicit this fact in the following definition. □

Definition 5.3: [Functions associated to γ, β, μ_i, $\mathbf{M}_\gamma$, $\mathbf{M}_\beta$ and $\mathbf{M}_{\mu_i}$] We define the following functions:

$$\left\{ \begin{array}{l}
f_\gamma(t,x) = \gamma(t,x) \\
f_\beta(t,x) = \beta(t,x) - \Delta \\
f_{\mu_i}(t,x) = \mu_i(t,x) - \overline{\mathbf{M}}_i \\
g_\gamma(t,x) = \mathbf{M}_\gamma(t,x) \\
g_\beta(t,x) = \mathbf{M}_\beta(t,x) - \Delta \\
g_{\mu_i}(t,x) = \mathbf{M}_{\mu_i}(t,x) - \overline{\mathbf{M}}_i
\end{array} \right. \tag{34}$$

By construction, the functions f_γ, f_β, f_{μ_i}, g_γ, g_β and g_{μ_i} are independent of the parameters Δ and $\overline{\mathbf{M}}_1, \ldots, \overline{\mathbf{M}}_{i_{\max}}$.

B. Linear programming formulation of the travel time estimation problem

Definition 5.4: [Decision variables] We define the following vector of variables X:

$$X := \left(\Delta, \overline{\mathbf{M}}_1, \ldots, \overline{\mathbf{M}}_i, \ldots, \overline{\mathbf{M}}_{i_{\max}}\right)^T \tag{35}$$

The vector X contains $i_{\max}+1$ variables.

Proposition 5.5: [Constraints] Using equation (30), we can write the following set of constraints for X:

$$\begin{array}{lll} (i) & g_\gamma(t,\chi) \geq f_\beta(t,\chi) + \Delta & \forall t \in \mathbb{R}_+ \\ (ii) & g_\beta(t,\xi) + \Delta \geq f_\gamma(t,\xi) & \forall t \in \mathbb{R}_+ \\ (iii) & g_\gamma(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\ (iv) & \overline{\mathbf{M}}_i + g_{\mu_i}(t,\xi) \geq f_\gamma(t,\xi) & \forall i \in I, \forall t \in \mathbb{R}_+ \\ (v) & g_\beta(t,\overline{x}_i(t)) + \Delta \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \\ (vi) & \overline{\mathbf{M}}_i + g_{\mu_i}(t,\chi) \geq f_\beta(t,\chi) + \Delta & \forall i \in I, \forall t \in \mathbb{R}_+ \\ (vii) & \overline{\mathbf{M}}_j + g_{\mu_j}(t,\overline{x}_i(t)) \geq \overline{\mathbf{M}}_i & \forall i \in I, \forall j \in I \backslash \{i\} \\ & & \forall t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}] \end{array} \quad (36)$$

As will be seen later, the optimization will be run using g_γ, g_β, etc. directly. Equation (36) can be regarded as a set of linear inequalities in the decision variables Δ and $\overline{\mathbf{M}}_1, \ldots, \overline{\mathbf{M}}_{i_{\max}}$. Indeed, we can write the above constraints as:

$$\left\{ \begin{array}{lll} (i) & \inf_{t \in \mathbb{R}_+} \left(g_\gamma(t,\chi) - f_\beta(t,\chi)\right) \geq \Delta & \\ (ii) & \Delta \geq \sup_{t \in \mathbb{R}_+} \left(-g_\beta(t,\xi) + f_\gamma(t,\xi)\right) & \\ (iii) & \inf_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]} \left(g_\gamma(t,\overline{x}_i(t))\right) \geq \overline{\mathbf{M}}_i & \forall i \in I \\ (iv) & \overline{\mathbf{M}}_i \geq \sup_{t \in \mathbb{R}_+} \left(f_\gamma(t,\xi) - g_{\mu_i}(t,\xi)\right) & \forall i \in I \\ (v) & \inf_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]} \left(g_\beta(t,\overline{x}_i(t))\right) \geq -\Delta + \overline{\mathbf{M}}_i & \forall i \in I \\ (vi) & \overline{\mathbf{M}}_i - \Delta \geq \sup_{t \in \mathbb{R}_+} \left(f_\beta(t,\chi) - g_{\mu_i}(t,\chi)\right) & \forall i \in I \\ (vii) & \overline{\mathbf{M}}_j - \overline{\mathbf{M}}_i \geq \sup_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]} \left(-g_{\mu_j}(t,\overline{x}_i(t))\right) & \\ & & \forall i \in I, \ \forall j \in I \backslash \{i\} \end{array} \right. \quad (37)$$

Proposition 5.6: [Properties of the travel time function] The travel time function $\tau(\cdot)$ is an decreasing function of the parameter Δ.

Proof — Let us set $\overline{\mathbf{M}}$ as $\overline{\mathbf{M}} := \gamma(t,\xi)$, and chose $\overline{t}$ solution to $\overline{\mathbf{M}} = \int_0^{\overline{t}} q_{\text{outflow}}(\tau) d\tau + \Delta$. Since $q_{\text{outflow}}(\cdot)$ is positive, the solution $\overline{t}$ to the previous equation is decreasing when Δ increases. Hence, the travel time function $\tau(\cdot)$ is a decreasing function of Δ. ■

Proposition 5.7: [Lower and upper bounds on the travel time function] The upper (respectively lower) bound on the travel time function can be found by solving the following linear program:

Minimize (respectively Maximize): Δ
Subject to:

$$\left\{ \begin{array}{ll} \inf_{t \in \mathbb{R}_+} \left(g_\gamma(t,\chi) - f_\beta(t,\chi)\right) \geq \Delta & \\ \Delta \geq \sup_{t \in \mathbb{R}_+} \left(-g_\beta(t,\xi) + f_\gamma(t,\xi)\right) & \\ \inf_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_1}]} \left(g_\gamma(t,\overline{x}_i(t))\right) \geq \overline{\mathbf{M}}_i & \forall i \in I \\ \overline{\mathbf{M}}_i \geq \sup_{t \in \mathbb{R}_+} \left(f_\gamma(t,\xi) - g_{\mu_i}(t,\xi)\right) & \forall i \in I \\ \inf_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_1}]} \left(g_\beta(t,\overline{x}_i(t))\right) \geq -\Delta + \overline{\mathbf{M}}_i & \forall i \in I \\ \overline{\mathbf{M}}_i - \Delta \geq \sup_{t \in \mathbb{R}_+} \left(f_\beta(t,\chi) - g_{\mu_i}(t,\chi)\right) & \forall i \in I \\ \overline{\mathbf{M}}_j - \overline{\mathbf{M}}_i \geq \sup_{t \in [\overline{t}_{\min_i}, \overline{t}_{\max_i}]} \left(-g_{\mu_j}(t,\overline{x}_i(t))\right) & \forall i \in I, \ \forall j \in I \backslash \{i\} \end{array} \right. \quad (38)$$

The solutions to problem (38) yield the minimal and maximal possible values $\Delta_{\min}$ and $\Delta_{\max}$ of Δ (assuming that the feasible set is not empty) compatible with the flow and trajectory data. We then use definition 4.1 to compute the worst-case and best-case travel time functions $tt_{\max}(\cdot)$ and $tt_{\min}(\cdot)$ compatible with the flow and trajectory data. $tt_{\max}(\cdot)$ and $tt_{\min}(\cdot)$ are functions of f_γ, f_β, $\Delta_{\min}$ and f_γ, f_β, $\Delta_{\max}$ respectively. Note that in (38), the functions g_γ, g_β, g_{μ_i} are computed using the Lax-Hopf formula (16).

VI. Applications to travel time estimation

To solve problem (38), we used `CVX`, a `Matlab` package for specifying and solving convex programs [20]. While the previous section can handle infinite horizon problems, only finite horizon problems can be implemented numerically. We thus solve problem (38) on a finite time horizon, which can be done by replacing $\mathbb{R}_+$ by $[0, T_{\max}]$ ($T_{\max}$ represents the time horizon) in equation (38).

The example below uses *Next Generation Simulation* (NGSIM) [22] data from a stretch of Interstate I80 in Emeryville, CA as our main benchmark scenario for this study. This data set contains video extracted trajectories of all vehicles traversing a 0.4 mile long highway section during a period of 45 minutes. Given the accuracy of the video, this set of data can be considered as *ground truth*, i.e. it provides the exact location of vehicles to an accuracy of a few centimeters at a 10Hz rate. The corresponding data is represented in Figure 3.

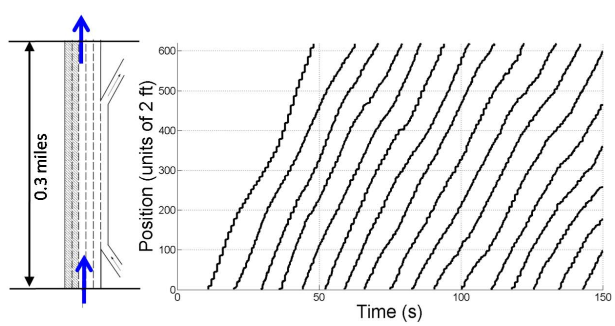

Fig. 3. **NGSIM experimental data**. Decimated representation of NGSIM trajectories. We represent only 5% of the trajectories for the sake of clarity of the figure.

We intentionally degrade the quality of the data to account for the uncertainty linked with the actual Lagrangian sensors (GPS), and use this data as follows:

- We create loop detector data from this NGSIM data, following a standard procedure used in traffic engineering (see [22] for details). This provides us with traffic data similar to what the PeMS loop detector system would record in real life. This yields the boundary condition functions $f_\gamma(\cdot,\cdot)$ and $f_\beta(\cdot,\cdot)$. The associated functions $g_\gamma(\cdot,\cdot)$ and $g_\beta(\cdot,\cdot)$ are computed using the Lax-Hopf formulas (33).
- We extract some trajectories representative of what a GPS tracking device would produce, if traveling onboard of the selected vehicles. This provides us with the trajectories

$\overline{x}_i(\cdot)$ for all $i \in I$. The associated functions $f_{\mu_i}(\cdot,\cdot)$ and $g_{\mu_i}(\cdot,\cdot)$ are computed using equations (20) and (33) respectively.

- We extract the density and flow values using the derivatives of the Moskowitz function. These values are used to compute the parameters of the flux function ψ (we model the flux function using a triangular function [12]).

Note that the first set of data is a typical measurement obtainable from Eulerian sensing (see [22] for a full description of the procedure), while the second data set (the trajectories) is a typical Lagrangian data set, obtainable from multiple probe vehicles.

A. Estimation of the total number of vehicles $-\Delta$ at initial time

We first consider the estimation of the parameter Δ using the time horizon $T_{\max} = 160s$, using the boundary condition functions f_γ and f_β, as well as a single trajectory f_μ of a given vehicle (we use the vehicle labeled 140 for this computation). We display the evolution of the highest and lowest possible travel time functions as a function of the duration of the trajectory used for the estimation in Figure 4. The travel time functions are computed using definition 4.1. As one can see in Figure 4, using a long trajectory improve the quality of the estimation (but also increase the risk of privacy intrusions). The same figure also shows that the estimation is not improved when the duration of the trajectory is too small.

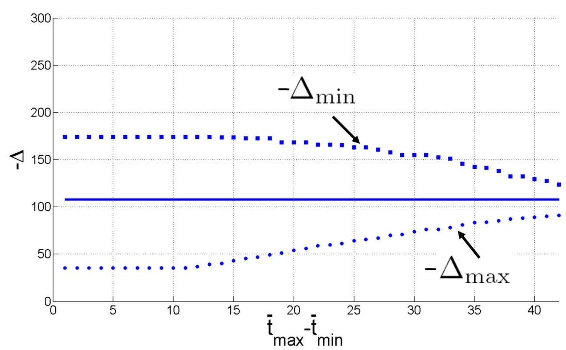

Fig. 4. **Estimation of the total number of vehicles at initial time using boundary condition functions and a single trajectory**. The exact value of the total number of vehicles $-\Delta$ at the initial time is 108 (solid line). The upper and lower bounds on Δ obtained by solving problem (38) are represented by squares and circles respectively. The horizontal axis represents the duration $\overline{t}_{\max} - \overline{t}_{\min}$ of the trajectory function used for the reconstruction (the parameter $\overline{t}_{\min}$ is set to $100s$).

B. Estimation of the travel time function using a single trajectory

Travel time through the section of the highway of interest is a function of time. In this section, we compute guaranteed bounds on the travel time (as a function of time). The way to interpret them is as follows: given Eulerian measurements from loop detectors, and given the model of traffic, the realized travel time through the section of highway is guaranteed to be between $tt_{\min}$ and $tt_{\max}$ throughout the time period considered. We now compute the guaranteed upper and lower possible values of the travel time for the time horizon $T_{\max} = 160s$ using the values obtained previously. Figure 5 displays the realized travel time function as well as the upper and lower possible envelopes if we consider only the boundary condition functions f_γ and f_β.

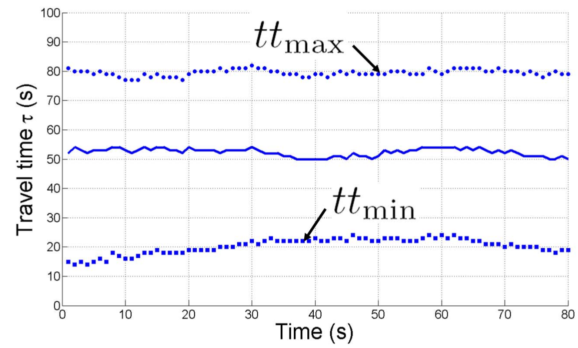

Fig. 5. **Estimation of the travel time function using boundary conditions functions only**. The horizontal axis represents the time t. The exact value of the travel time function $\tau(\cdot)$ is represented using a solid line. The upper and lower bounds on $\tau(t)$ obtained by solving problem (38) and using definition 4.1 are represented by circles and squares respectively.

The bounds provided by Eulerian sensors only are extremely wide. As can be seen in comparison with Figure 6, adding Lagrangian information provides tighter bounds and a better estimate of the travel time function.

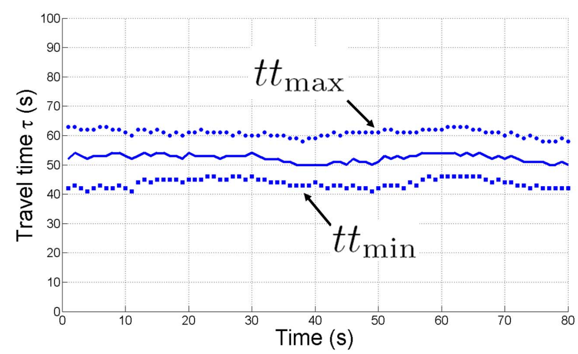

Fig. 6. **Estimation of the travel time function using boundary conditions functions and a single trajectory**. The horizontal axis represents the time t. The exact value of the travel time function $\tau(\cdot)$ is represented using a solid line. The upper and lower bounds on $\tau(t)$ obtained by solving problem (38) and using definition 4.1 are represented by circles and squares respectively. We use the trajectory of vehicle 140 for the reconstruction, and use $\overline{t}_{\max} = 140s$ and $\overline{t}_{\min} = 100s$.

However, sensing vehicles during long intervals of time may increase the risk of privacy intrusion: it enables the tracking of a given vehicle during a large amount of time. In order to prevent the tracking of mobile phone users, the duration $t_{\max} - t_{\min}$ must be as low as possible [37].

C. Influence of the penetration rate

In order to use segments of trajectory as short as possible and yet obtain a good approximation of the travel time function, we now investigate the estimation of the travel time function using multiple trajectory functions.

Definition 6.1: [Penetration rate] The penetration rate R is defined as the ratio of the total number of trajectories

$I_{\max}$ and the total number of vehicles entering the highway between times 0 and $T_{\max}$:

$$R := \frac{I_{\max}}{\mathbf{M}(T_{\max},\xi) - \mathbf{M}(0,\xi)} \tag{39}$$

We consider the same data set as previously, and set the time horizon to $T_{\max} = 160$. We select $I_{\max}$ trajectory functions of fixed duration $\bar{t}_{\max} - \bar{t}_{\min}$ corresponding to random vehicles present on the highway between times 0 and $T_{\max}$, and check the influence of the *penetration rate* for different trajectory durations. We display in Figure 7 the evolution of $\Delta_{\max} - \Delta_{\min}$ as a function of $I_{\max}$ and the duration $\bar{t}_{\max} - \bar{t}_{\min}$. For this particular situation, we have $\mathbf{M}(T_{\max},\xi) - \mathbf{M}(0,\xi) = 363$, and the penetration rate, which can be computed using equation (39) varies between 0.3% and 1.4%.

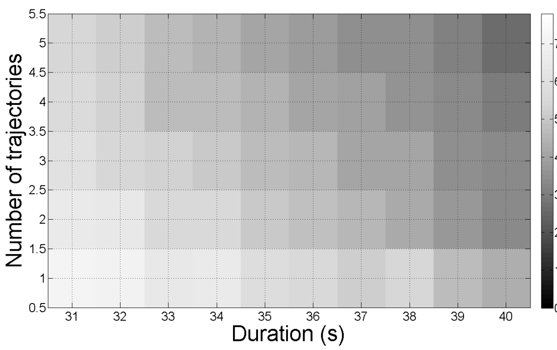

Fig. 7. **Guaranteed range (unit: vehicles) for the estimation of the parameter Δ using randomly chosen trajectories of given duration**. The horizontal axis represents the total duration $\bar{t}_{\max} - \bar{t}_{\min}$ of each trajectory. The vertical axis represents the number of trajectory functions used for the reconstruction (from 1 to 5). The total error $\Delta_{\max} - \Delta_{\min}$ on the parameter Δ is computed by solving problem (38), and is indicated using a gray scale (right). In order for the results to be statistically significant, we averaged the total error on 30 different choices of trajectory functions for each value of the duration and number of trajectory functions. The true value of Δ is $-\Delta = 108$ (vehicles).

As illustrated by Figure 7, the error on the reconstruction depends on the number of trajectory functions used for the reconstruction, but also on the duration of the trajectory function used for the reconstruction. As one can see in this figure, there is a clear trade-off between privacy and accuracy. For instance, using 5 trajectory functions of duration $34s$ yields a greater error on Δ in average than using a single trajectory of duration $40s$! Tracking vehicles during a greater amount of time dramatically reduces the error on the estimation, at the expense of the user's privacy. This trade-off can also be seen in Figure 8, which displays the L_2 error on the travel time function $\tau(\cdot)$ as a function of the duration and the number of trajectories. This figure also shows that we have in average a much lower error on the travel time function using a single trajectory of duration $40s$ than using 5 trajectories of duration $30s$.

The NGSIM data set is to our best knowledge among the most accurate highway traffic data set publicly available in the transportation engineering community. In future work, computations will be realized using the same method for data gathered from the *Mobile Century* experiment. This

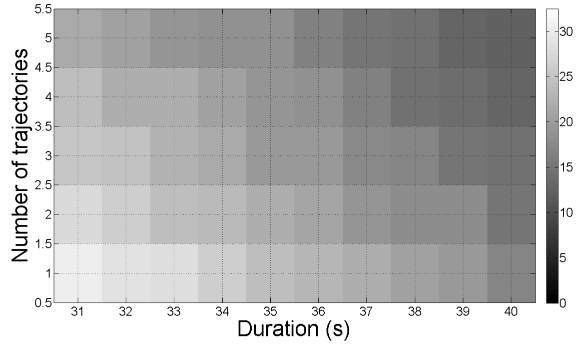

Fig. 8. **Root mean square error (unit: s) on the travel time function using randomly chosen trajectories of given duration**. The horizontal axis represents the total duration $\bar{t}_{\max} - \bar{t}_{\min}$ of each trajectory. The vertical axis represents the number of trajectory functions used for the reconstruction (from 1 to 5). The root mean square error (in s) on the travel time $\tau(\cdot)$ is computed by solving problem 38 and using definition 4.1, and is indicated using a gray scale (right). In order for the results to be statistically significant, we averaged the total error on 30 different choices of trajectory functions for each value of the duration and number of trajectory functions. The true travel time varies around $50s$ for the period of interest.

experiment is part of an ongoing project with Nokia and the California Department of Transportation (Caltrans), to evaluate the use of GPS equipped cellular phones (Nokia N95) onboard 100 vehicles used as probe sensors to monitor the state of traffic in real time. The upcoming *Mobile Millennium* experiment will be a larger deployment of 10000 vehicles starting in November 2008. This *added value* potentially provides state and federal agencies with a new source of information which we have already shown in experiments [22] to work at extremely low penetration rates (between 1% and 5% of equipped vehicles). Given the rate of adoption of GPS-enabled cell phones, we expect this technology and this set of algorithms to have a great impact on highway traffic monitoring within three years, also leading to new products available to travelers, such as travel time info on cell phones.

VII. Conclusion

This article presented a new type of boundary and internal conditions problem in which the initial condition is unknown. This problem is characterized by compatibility conditions, which encode the fact that the solution should satisfy at the same time the model, as well as all the trajectory and flow data. Using a set of variables describing the state of traffic, we derived a linear program yielding guaranteed bounds on the total accumulation of vehicles at initial time compatible with the collected data. Using these bounds, we computed guaranteed bounds on the travel time function using flow data. This work was successfully implemented on NGSIM data, and is in the process of being implemented on Mobile Century data, as well as in the Mobile Millennium system [8].

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conversation. The algorithms developed in this article are using technology produced by the company VIMADES [8].

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