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A Cooperative Distributed Approach to Multi-Agent Eulerian Network Control: Application to Air Traffic Management*

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Based on an Eulerian network model, air traffic flow in the National Airspace System is optimized using second order adjoint methods for hyperbolic PDEs. Multiple airlines with independent cost structures are added to previous work on air traffic flow modeling and optimization. A cooperative, distributed solution methodology is proposed that allows airlines to keep cost information private by distributing the computational workload between the airlines while ensuring efficient outcomes. The Nash Bargaining Solution is presented as a possible solution concept, and a distributed method for its calculation is presented.

I. Introduction

Management of today's National Airspace System (NAS) has become an extremely complicated task. Pilots and controllers must continuously manage numerous complex subsystems and adapt to frequent changes in traffic flow, spacing and routing caused by variable weather conditions and other network delays. Specifically, the national Air Traffic Control System Command Center (ATCSCC) is responsible for continuously updating decisions concerning traffic flow to maximize network throughput. The current methodology is based on a sophisticated set of rules described in *playbooks* that have been developed through years of controller experience. This procedure is neither automated nor optimal, and requires constant human intervention to allow the system to operate.

Airlines are an integral element of the NAS and are developing more and more advanced methods of aircraft management, as supervised by Airline Operations Centers (AOCs). Their stake in aircraft flow decisions is quite high, as their profitability is directly tied to on-time performance, and significant costs are incurred by flight delays, cancellations and rerouting. Furthermore, their preferences change dynamically, based on highly variable parameters such as aircraft load factor and percentage of connecting passengers per flight. Unfortunately, airlines are currently only able to affect decisions over a small part of the network flow problem, by swapping slots with their own aircraft and deciding which flights to delay when required. For smooth flow of their aircraft over the NAS network, the airlines must rely on the savvy and equity of the ATCSCC controllers who attempt to minimize schedule disruption while not necessarily minimizing incurred costs.

This need not be the case. Consider a futuristic scenario for the NAS, where airlines cooperate with each other in a distributed manner, supervised by the FAA, to dynamically reroute and reschedule flights based on restrictions imposed by weather conditions and network delays in such a manner that their individual profits are maximized, while all safety requirements are met and the airspace is used efficiently. In such a scenario, real-time collaboration amongst the airlines will drive the efficient use of resources by incorporating airline specific preferences between routes into the flow optimization and ultimately, the passenger will be better served.

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For this scenario to come to fruition, three key elements are needed. Firstly, a real-time optimization technique is needed for air traffic network flow based directly on the airlines' internal cost structure. Secondly, the optimization process must achieve an equitable allocation for all airlines involved. Finally, the optimization technique must be distributable, so that the airlines can modify their preferences in real time and do not have to reveal sensitive information to their competitors.

In two previous papers[1],[2] we have shown that an Eulerian approach to modeling the airspace, as first proposed by Menon et al.,[3] could be applied to the air traffic flow control problem. This technique divides the airspace into line elements, or links, and associates with each link an aircraft density and velocity profile. The evolution of aircraft density can be modeled as a set of Partial Differential Equations (PDE) that closely resemble the Lighthill-Whitman-Richards(LWR) PDE from highway traffic modeling[4].[5] In order to maximize air traffic flow over the network, an adjoint-based gradient descent optimization of control strategies was performed. The method resulted in an automated, optimal and centralized approach to air traffic flow management, distinguished from similar approaches[3],[6] by its continuous domain formulation and the use of fast numerical techniques specific to first-order hyperbolic PDEs.

This paper seeks to extend the approach in the three distinct directions mentioned above. First, for a novel multi-airline traffic model, a second order optimization technique is proposed that drastically improves the performance of the centralized algorithm and allows for real-time optimization of network flow under varying flow restrictions imposed on each network link. Second, both Aggregate Flux Maximizing (AFM) and Nash Bargaining Solution (NBS) cost structures are proposed as means to achieve fair and efficient solutions, and finally, a cooperative, distributed optimization algorithm is proposed that enables airlines to participate in the process of air traffic network flow regulation and help ensure efficient and equitable use of the NAS.

This paper is organized as follows. The Eulerian formulation of the air traffic flow problem is extended to incorporate multiple airlines in Section II. A development and discussion of possible global cost functions is presented, specifically the relative merits of the AFM and of the NBS are developed. In Section III, three techniques for optimization of hyperbolic PDEs are compared. A review of the adjoint-based first order technique[2] is presented, followed by development of a second order Newton method as well as an approximate second order method. Section IV proposes an algorithm for distributed computation of both AFM and NBS formulations that relies on dual decomposition techniques and an iterative bargaining process to distribute computation of optimal flow control strategies amongst the airline AOCs. Finally, simulation results with both cost functions are presented for a multiple airline scenario in Section V, as well as a comparison of the distributed and centralized solution methods.

II. Problem Formulation

A. Network Description

Consider an air traffic network defined by a fixed set of links $i \in \{1, \ldots, N_{\mathcal{I}}\} = \mathcal{I}$ and a fixed number of airlines $j \in \{1, \ldots, N_{\mathcal{J}}\} = \mathcal{J}$. With each link $i \in \mathcal{I}$ are associated the set of links $\mathcal{M}_i$, which are the links merging into link i, the set of links $\mathcal{D}_{li}$, which are the links diverging to the left into link i, and the set of links $\mathcal{D}_{ri}$, which are the links diverging to the right into link i. $\mathcal{M} = \cup_{i\in\mathcal{I}}\mathcal{M}_i$ represents the set of all merging links and $\mathcal{D} = \cup_{i\in\mathcal{I}}(\mathcal{D}_{li} \cup \mathcal{D}_{ri})$ represents the set of all diverging links. Finally, one subset of links, $\mathcal{F} \subseteq \mathcal{I}$, are defined as sinks.

To make the above definition more concrete, consider a small portion of U.S. airspace as depicted in Figure 1. Inflow into the network originates from Boston and New York, and heads towards Chicago, passing over Detroit on the way. Flights from New York and Detroit can take either the northern or southern routes beyond Detroit, while flights originating in Boston only take the northern route. For this example, the set $\mathcal{I}$ of all links includes 4 elements, labeled Links 1 through 4 in Figure 2. The sink link set, $\mathcal{F}$, includes both links 3 and 4. Aircraft traveling in link 3 come either from converging link 2 or from diverging link 1, therefore $\mathcal{M}_3 = \{2\}$, $\mathcal{D}_{l3} = \emptyset$, $\mathcal{D}_{r3} = \{1\}$. Aircraft traveling along link 4 come from diverging link 1 only, therefore $\mathcal{M}_4 = \emptyset$, $\mathcal{D}_{l4} = \{1\}$, $\mathcal{D}_{r4} = \emptyset$.

Each link is of length L_i, with the coordinate $x_i \in [0, L_i]$ used to refer to any point on link i. Each link has an associated airline specific aircraft density, $\rho_{i,j}(x_i, t) : [0, L_i] \times \mathbb{R}_+ \to \mathbb{R}_+$, and airline specific aircraft velocity profile over each link, $v_{i,j}(x_i, t) : [0, L_i] \times \mathbb{R}_+ \to \mathbb{R}_+$. To facilitate notation, $\rho_j = (\rho_{1,j}, \ldots, \rho_{N_{\mathcal{I}},j})$, $j \in \mathcal{J}$, refers to the vector of link densities for each airline and $\rho = \left(\rho_1^T, \ldots, \rho_{N_{\mathcal{J}}}^T\right)$ is the matrix of all densities, with similar notation for velocity. For any dividing link $k \in \mathcal{D}_{l\,i}$ in which $i \in \mathcal{I}$, $\beta_{k,j}(t) : \mathbb{R}_+ \to [0, 1]$ describes the airline specific portion of $\rho_{k,j}$ which flows from link k into link i, and for any dividing link $k \in \mathcal{D}_{r\,i}$ in which $i \in \mathcal{I}$, $(1 - \beta_{k,j}(t))$ is the portion

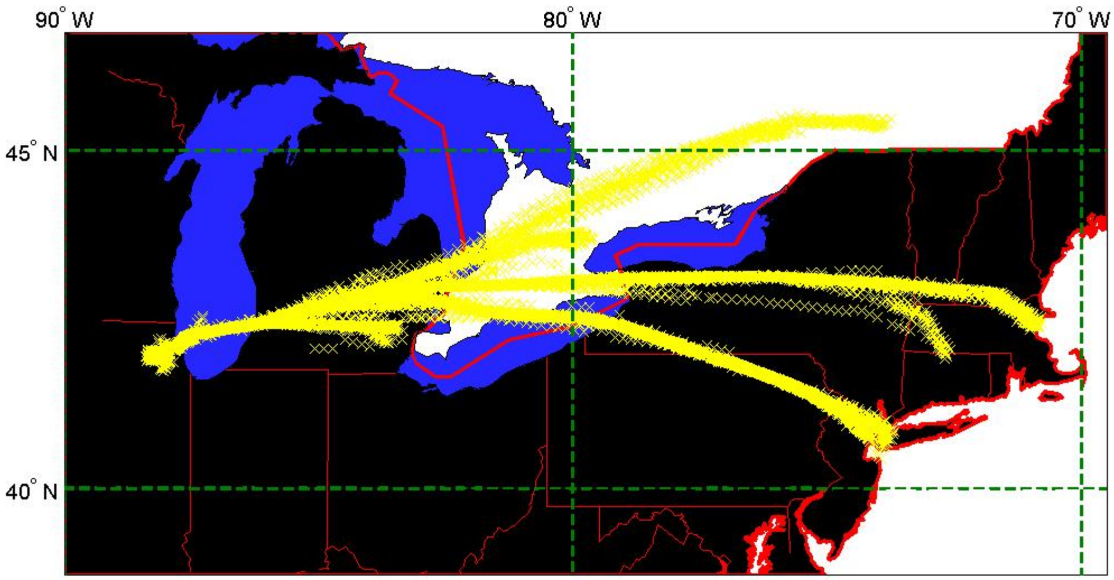

Figure 1. 24 hrs of incoming flight tracks for Chicago (ORD). The upper stream comes from Canada, the middle from Boston (BOS) and the lower from New York (EWR, JFK, LGA). Additional streams merge into the network at Detroit and Hartford Bradley.

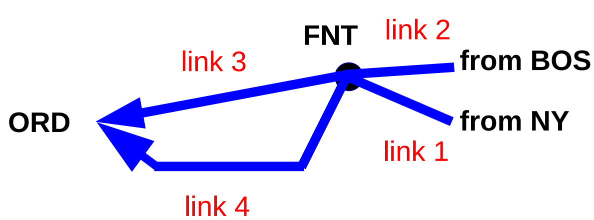

Figure 2. A simplified network model for the tracks shown above, with waypoints labeled. The model includes four links, merging into Chicago (ORD). The inflow terms, q_i^{in}, are defined for Boston (BOS) and New York (EWR, JFK, LGA).

which flows from link k into link i.

Constraints are added to the network flow problem as follows. Each source has associated with it fixed aircraft inflow rates per airline of $q^{in}_{i,j}(t) : \mathbb{R}_+ \to \mathbb{R}_+$. Each link in the network has defined for it an aggregate maximum density, $\bar{\rho}_i(x_i, t) : [0, L_i] \times \mathbb{R}_+ \to \mathbb{R}_+$. Finally, all aircraft must obey the same minimum and maximum velocity limits, $v_{\min,i}(x_i)$ and $v_{\max,i}(x_i)$, defined at each point in the network.

The airline optimization variables are taken to be the velocity profiles $v_{i,j}$ and link splitting parameters, $\beta_{i,j}$. Conservation of mass results in the following Eulerian PDE model for each airline, $j \in \mathcal{J}$, along all links in the network,

$$\mathcal{N}_{i,j}(\rho_{i,j}, v_{i,j}) := \frac{\partial \rho_{i,j}}{\partial t} + \frac{\partial(\rho_{i,j} v_{i,j})}{\partial x_i} = 0\,, \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J} \tag{1}$$

where $\mathcal{N}$ is referred to as the LWR operator and Equation (1) encodes a first order hyperbolic PDE. A complete set of initial and boundary conditions are required to complete the network PDE model. This set includes initial densities for all airlines along all links in the network, link inflow rates for all time and for all links, and continuity conditions for all time between merging and dividing links.

$$\left.\begin{array}{lcl} \rho_{i,j}(x_i, 0) & = & \rho^0_{i,j}(x_i) \\ \rho_{i,j}(0,t) v_{i,j}(0,t) & = & q^{in}_{i,j}(t) \\ & & + \sum_{k \in \mathcal{M}_i} \rho_{k,j}(L_k, t) v_{k,j}(L_k, t) \\ & & + \sum_{k \in \mathcal{D}_{li}} \beta_{k,j}(t) \rho_{k,j}(L_k, t) v_{k,j}(L_k, t) \\ & & + \sum_{k \in \mathcal{D}_{ri}} (1 - \beta_{k,j}(t)) \rho_{k,j}(L_k, t) v_{k,j}(L_k, t) \end{array}\right\} \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J} \tag{2}$$

B. Cost Functions

Airline operating costs are directly tied to on-time performance as many types of expenses are incurred every time a flight is delayed. Fixed costs such as fuel consumption and aircraft usage are increased in a manner relatively consistent across the airlines. Additionally, there are also significant costs that may be incurred whose expense depends significantly on airline policy, such as crew overtime charges, compensation for customer accommodations or rebooking customers on alternate flights if connections are missed. These costs depend explicitly on aircraft load factor and number of connecting passengers per flight, information which is currently not being used in the assessment of air traffic flow.

Based on this airline-centric point of view of NAS traffic flow, formulation of the flow optimization problem proceeds with the definition of a cost function specific to each airline that can embody much of the information not currently taken into consideration. Let

$$J_j : (\rho_j, v_j) \to J_j(\rho_j, v_j) \in \mathbb{R}_+\,, \quad \forall j \in \mathcal{J} \tag{3}$$

define a preference relation over the density and velocity profiles of each airlines vehicles on all links. This cost function can take many forms, and indeed may be different for each airline. A general form for the airline cost functions is adopted for this paper,

$$J_j(\rho_j, v_j) = -\sum_{i \in \mathcal{F}} \omega_i \int_0^T \rho_{i,j}(L_i, t) v_{i,j}(L_i, t) dt \tag{4}$$

where each airline seeks to maximize the flux of aircraft arriving at the airport (end of the sink links) before a fixed point in time, T, and where $\omega_i \in \mathbb{R}_+$, $\sum_{i \in \mathcal{F}} \omega_i = 1$, are individual weights to be applied to each link in the network. A scenario can be envisaged in which the FAA must close a portion of the network due to weather conditions, with an hour's notice. The airlines, therefore, wish to get as many aircraft as possible through prior to the closure, and hence would aim to maximize the throughput over the links that are about to close. In addition, each airline may prefer to delay flights from one direction over another, and as such would apply varied weights to indicate such a preference.

It is certainly true that many other scenarios can be developed, such as deviation from a fixed schedule, minimum flight time, minimum fuel consumption, etc. Each of these scenarios would require a separate cost function, and it is even possible that a combination of these goals is needed by each airline. Fortunately, although the remaining discussion adopts the cost structure defined above, it is by no means limited to this form and can be easily adapted to incorporate any of the alternate aims mentioned.

In this multiple objective situation, one question arises naturally, namely: how does one pose a network flow optimization problem that incorporates the individual costs of each airline that ensures maximal use of the airspace while dividing the available resources fairly amongst the airlines? The simplest approach is to define a constrained optimization problem that takes as cost function the sum of costs of all airlines, referred to as the Aggregate Flux Maximizing (AFM) cost function.

$$J_{AFM}(\rho, v) = \sum_{j \in \mathcal{J}} J_j(\rho_j, v_j) \tag{5}$$

Unfortunately, this cost formulation does not necessarily distinguish between solutions where flux is distributed evenly between airlines and solutions where one airline consumes all flux and the others none. Assuming both such extreme solutions are feasible, the global minimal value returned will be identical, but the nature of the solution, as well as the airlines' satisfaction with the outcome, may vary drastically.

A more elegant approach to the multi-agent cooperative optimization problem was first proposed by John Nash Jr. in 1950 and its result is referred to as the Nash Bargaining Solution[7] (NBS). A refinement of Pareto optimality, the NBS is inherently based upon predefined local cost functions for each airline. For the NBS to exist, a disagreement point, $d = (d_1, \ldots, d_{N_\mathcal{J}}) \in \mathbb{R}_+^{N_\mathcal{J}}$ must be defined, which represents the resulting costs to each airline if an agreement is not reached. Intuitively, the disagreement point defines the worst case solution an airline is willing to accept, and can be considered equal to 0, as the FAA can authorize the grounding of any airline should they be responsible for a lack of agreement.

Based on the following four axioms and the definition of a disagreement point, Nash constructed an elegant proof of existence of a unique, Pareto optimal bargaining solution for convex problems. For the remainder of the treatment of the NBS concept, the term agent shall refer to an independent decision maker, which for the motivating example of this paper can be interpreted as an individual airline.

Assumption 1. *Local cost functions are convex, and local optimization variable spaces are convex and compact.*

Axiom 1. ***Axiom of Rationality****: Each agent prefers the locally optimal solution.*

Axiom 2. ***Axiom of Linear Invariance****: Neither scaling nor offset of any cost function affects the resulting bargaining solution.*

Axiom 3. ***Axiom of Independence of Irrelevant Alternatives****: If the set of feasible solutions is restricted and yet contains the optimal bargaining solution of the original problem, the original optimal bargaining solution remains optimal for the restricted problem.*

Axiom 4. ***Axiom of Symmetry****: If the set of feasible solutions is symmetric with respect to the agents, the bargaining solution must assign equal costs to all.*

The axioms follow from a logical assessment of the bargaining problem. Axiom 1 ensures suboptimal solutions are not selected locally by any of the agents and results in a Pareto optimal bargaining solution for convex cost functions. By the assumption of continuity and compactness of the optimization variables, each agent's cost function is affine invariant; scaling and offset have no effect on the preference ordering of solutions. Axiom 2 requires this property of the bargaining solution as well. As a consequence, the disagreement point is needed to define a common base value for the bargaining process (analogous to an origin), and Axiom 3 is required to fix the relative scaling of utility functions between agents. Finally, the concept of fairness amongst bargainers is embodied in Axiom 4.

Strikingly, the NBS results in a centralized optimization of the product of the difference between the agents' utility and its respective disagreement value. The centralized cost function, J_{NBS}, whose minimum uniquely defines the NBS for convex problems, can be formulated as,

$$J_{NBS} = -\prod_{j \in \mathcal{J}} (d_j - J_j(\rho_j, v_j)) \tag{6}$$

which binds the agents costs, and hence optimization variables, together as a product.

It must be noted that the NBS was developed based on convexity of the agents' cost functions, and as such does not apply directly to the airline flow problem as formulated above. Recent work, by Conley and Wilkie[8] as well as Hougaard and Tvede,[9] has looked at extending bargaining solutions to non-convex problems while maintaining uniqueness of the solution, but the works rely on geometric constructs that are difficult to implement in practice. The convexity requirement of the original formulation, however, merely ensures the bargaining solution is unique and so, in using the NBS cost function in a non-convex environment, this method will find solutions that may not be unique bargaining solutions, but that not only satisfy necessary conditions for Pareto optimality but also the additional necessary conditions imposed by the axioms above. In other words, like all non-convex optimization problems, non-convex NBS may also be trapped in local minima. The advantage to using the NBS cost function over the AFM, however, is that it is more selective. Given the set of all solutions that satisfy necessary conditions for Pareto optimality, NBS solutions are restricted to those that also satisfy Nash's axiomatic notions of fairness and scalability.

C. Centralized Formulation

The multi-airline optimization problem can now be formulated in a centralized manner,

$$\begin{array}{ll} \text{minimize} & J(\rho, v) \\ \text{subject to} & (1) \text{ and } (2) \\ & v_{\min,i} \leq v_{i,j} \leq v_{\max,i}\,, \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J} \\ & \sum_{j\in\mathcal{J}} \rho_{i,j} \leq \bar{\rho}_i\,, \quad \forall i \in \mathcal{I} \\ & 0 \leq \beta_{i,j} \leq 1\,, \quad \forall (i,j) \in \mathcal{D} \times \mathcal{J} \end{array} \tag{P1}$$

where J represents either J_{NBS} or J_{AFM}.

What makes the above formulation of the centralized problem distinct from a formulation that does not incorporate individual airlines is that each link can now have $N_{\mathcal{J}}$ distinct velocity profiles and a separate cost function is defined for each airline. The expanded flow control possibilities result in a larger space of feasible solutions, and hence it is possible to exceed the performance of the original optimization technique presented.[2]

III. Centralized Optimization

Many of the constraints specified as part of the centralized problem (namely all the inequality constraints) can be dealt with using a standard barrier method (see Bertsekas[10] as a reference). This method has an advantage over penalty methods and Lagrange multiplier techniques in that it returns a feasible solution at each iteration, and hence the optimization algorithm can be stopped at any time and still return a feasible, albeit sub-optimal solution. The barrier method requires a feasible initial condition, which can be efficiently determined for the problem of interest. Let $M \in \mathbb{R}_+$ be the barrier parameter. The resulting constrained optimization problem is,

$$\begin{array}{llll} \text{minimize} & \tilde{J}(v,\beta,\rho,M) & = & J(\rho,v) - \frac{1}{M}\sum_{i\in\mathcal{I}}\int_0^T\int_0^{L_i}\log(\bar{\rho}_i - \sum_{j\in\mathcal{J}}\rho_{i,j})\,dx_i\,dt \\ & & & -\frac{1}{M}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{I}}\int_0^T\int_0^{L_i}\log(v_{\max,i}-v_{i,j})(v_{i,j}-v_{\min,i}))\,dx_i\,dt \\ & & & -\frac{1}{M}\sum_{j\in\mathcal{J}}\sum_{i\in\mathcal{D}}\int_0^T\log(\beta_{i,j})(1-\beta_{i,j})dt \\ \text{subject to} & & & (1) \text{ and } (2) \end{array} \tag{P2}$$

where $\tilde{J}(v,\beta,\rho,M)$ is referred as a centralized augmented cost function and the optimization variables, $\rho_{i,j}, v_{i,j}$ (resp. $\beta_{i,j}$) are understood to depend implicitly on time and space (resp. time). Note that summations of log functions have been converted to logs of function products where possible to simplify the notation.

This section proceeds with a presentation of an efficient solution of PDE constrained optimization programs using an approximate second order method using the AFM global cost function. A similar derivation can be followed for the NBS cost function, with only minor modifications.

A. Review of first order method

In our previous work,[2] the gradient of $\tilde{J}$ with respect to the control variables v and β has been derived as,

$$
\begin{aligned}
&\nabla_{v_{i,j}} \tilde{J} = \rho_{i,j} \tfrac{\partial p_{i,j}}{\partial x} + \tfrac{1}{M}\left(\tfrac{1}{v_{\max,i} - v_{i,j}} - \tfrac{1}{v_{i,j} - v_{\min,i}}\right), \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J} \\
&\nabla_{\beta_{i,j}} \tilde{J} = \rho_{i,j}(x_i = L_i) v_{i,j}(x_i = L_i)(p_{i_l,j}(x_{i_l} = 0) - p_{i_r,j}(x_{i_r} = 0)) + \tfrac{1}{M}\left(\tfrac{1}{1-\beta_{i,j}} - \tfrac{1}{\beta_{i,j}}\right), \quad \forall (i,j) \in \mathcal{I} \times \mathcal{J}
\end{aligned} \tag{7}
$$

in which $p_{i,j}(x_i, t) : [0, L_i] \times \mathbb{R}_+ \to \mathbb{R}_+$ is the adjoint variable and is the solution of the following PDE with terminal boundary conditions,

$$
\mathcal{N}^*_{i,j}(p_{i,j}, v_{i,j}) := -\frac{\partial p_{i,j}}{\partial t} - v_{i,j} \frac{\partial p_{i,j}}{\partial x} = \frac{1}{M(\bar{\rho}_i - \sum_{j \in \mathcal{J}} \rho_{i,j})} \tag{8}
$$

$$
\begin{cases}
p_{i,j}(x_i, T) = 0 & \forall i \in \mathcal{I},\ \forall j \in \mathcal{J},\ \forall x_i \in [0, L_i] \\
p_{i,j}(L_i, t) = -1 & \forall i \in \mathcal{F},\ \forall j \in \mathcal{J},\ \forall t \in [0, T] \\
p_{i,j}(L_i, t) = p_{k,j}(0, t) & \forall (i,k) \in \mathcal{M}_k \times \mathcal{I},\ \forall j \in \mathcal{J},\ \forall t \in [0, T] \\
p_{i,j}(L_i, t) = \beta_{l,j}(t) p_{l,j}(0, t) + (1 - \beta_{k,j}(t)) p_{k,j}(0, t) & \forall (i,l,k) \in \mathcal{I}^3 \,|\, i \in \mathcal{D}_{l\,l} \cap \mathcal{D}_{r\,k},\ \forall j \in \mathcal{J},\ \forall t \in [0, T]
\end{cases} \tag{9}
$$

$\mathcal{N}^*_{i,j}$ is called the adjoint operator. Based on this gradient, it is possible to run a gradient descent algorithm in order to minimize $\tilde{J}$. The procedure is the following, in which $u = (v, \beta)$ represents all global control variables of the problem, $\epsilon_a, \epsilon_b \in \mathbb{R}_+$ are stopping criteria and α is the gradient step size resulting from a line search.

Algorithm 1 Adjoint based gradient algorithm

1: Set $M = 1$ and guess an initial value for v and β.
2: **while** $\frac{1}{M}$ is greater than the stopping criteria ϵ_a, **do**
3:   **while** $||\nabla_u \tilde{J}||$ is greater than the stopping criteria ϵ_b, **do**
4:     Solve equations (1) and (2) for ρ, using the current control $u = (v, \beta)$.
5:     Solve the adjoint equation (8) and (9) for p, using the current u and ρ.
6:     Determine the gradient $\nabla_u \tilde{J} = (\nabla_v \tilde{J}, \nabla_\beta \tilde{J})$ according to equation (7).
7:     Perform a line search: compute $\alpha > 0$ so that $\tilde{J}(u - \alpha \nabla_u \tilde{J})$ is minimized.
8:     Update $u := u - \alpha \nabla_u \tilde{J}$.
9:   **end while**
10:   Increase M by letting $M := \mu M$, where typically, $\mu \in [2, 10]$.
11: **end while**
12: **RETURN** $u_{\text{optimal}} = u$.

In implementation, this optimization program consists of a gradient descent in a very large dimensional space and as such, does not converge quickly. In order to ameliorate the convergence properties of the optimization algorithm, second order methods are proposed.

B. Second order Newton method[11]

For any control input, u, the Newton step, Δu_{nt}, is defined as the input that should be added to u in order to set the first order approximation of the gradient to 0. Therefore, using $\widehat{\nabla \tilde{J}}(u + \delta u)$ to denote the first order approximation of $\nabla \tilde{J}(u + \delta u)$, we have:

$$
\widehat{\nabla \tilde{J}}(u + \Delta u_{\text{nt}}) = 0\,, \tag{10}
$$

In optimization programs in which the second derivative can be easily computed, $\widehat{\nabla \tilde{J}}(u + \delta u)$ is generally expressed as

$$
\widehat{\nabla \tilde{J}}(u + \delta u) = \nabla \tilde{J}(u) + \nabla^2 \tilde{J}(u) \delta u \tag{11}
$$

$\nabla^2 \tilde{J}(u)$ denotes the second order derivative, or Hessian, of $\tilde{J}$. We will denote the Hessian as $\mathcal{H}$. Therefore, the Newton step Δu_{nt}, if it exists, also solves the following differential equation, which is equivalent to (10):

$$\mathcal{H}\Delta u_{\text{nt}} = -\nabla \tilde{J}(u) \tag{12}$$

Now, in order to compute Δu_{nt}, we need to give an explicit expression for the Hessian, $\mathcal{H}$, or equivalently, give an explicit expression for $\widehat{\nabla \tilde{J}(u+\delta u)}$. For this purpose, we write the first order variation of the gradient, $\nabla \tilde{J}(u)$. Recall that the gradient is a function of the control variable, the state, and the costate: $\nabla \tilde{J}(u) = \mathcal{G}(u, \rho, p)$; thus

$$\begin{aligned} \nabla \tilde{J}(u+\delta u) - \nabla \tilde{J}(u) &= \mathcal{G}(u+\delta u, \rho+\delta\rho, p+\delta p) \\ &\quad -\mathcal{G}(u,\rho,p) \\ &= \nabla_u \mathcal{G}(u,\rho,p)\ \delta u \\ &\quad +\nabla_\rho \mathcal{G}(u,\rho,p)\ \delta\rho \\ &\quad +\nabla_p \mathcal{G}(u,\rho,p)\ \delta p + \mathcal{O}||\delta u||^2 \end{aligned} \tag{13}$$

Keeping only the first order terms, we obtain

$$\begin{aligned} \mathcal{H}\delta u &= \nabla_u \mathcal{G}(u,\rho,p)\ \delta u + \nabla_\rho \mathcal{G}(u,\rho,p)\ \delta\rho \\ &\quad +\nabla_p \mathcal{G}(u,\rho,p)\ \delta p \end{aligned} \tag{14}$$

where $\delta\rho$ and δp are the first order variation of the state and the costate generated by a first order variation of the input δu. Therefore, the first order variations, $\rho + \delta\rho$ and $p + \delta p$, solve the state and the costate PDEs:

$$\begin{aligned} &\mathcal{N}(u+\delta u, \rho+\delta\rho) = 0 \\ &\mathcal{N}^*(u+\delta u, \rho+\delta\rho, p+\delta p) = 0\,, \end{aligned} \tag{15}$$

whose approximations to the first order give the following linear PDEs satisfied by $\delta\rho$ and δp.

$$\begin{aligned} &\widehat{\mathcal{N}}(\delta u, \delta\rho) = 0 \\ &\widehat{\mathcal{N}^*}(\delta u, \delta\rho, \delta p) = 0 \end{aligned} \tag{16}$$

$\widehat{\mathcal{N}}$ and $\widehat{\mathcal{N}^*}$ represent the linearized form of $\mathcal{N}$ and $\mathcal{N}^*$, respectively. Finally, using (12), (14) and (16), the differential equation satisfied by the Newton step, Δu_{nt}, becomes

$$\begin{gathered} \nabla_u \mathcal{G}(u,\rho,p)\ \Delta u_{\text{nt}} + \nabla_\rho \mathcal{G}(u,\rho,p)\ \delta\rho + \nabla_p \mathcal{G}(u,\rho,p)\ \delta p = -\mathcal{G}(u,\rho,p) \\ \widehat{\mathcal{N}}(\Delta u_{\text{nt}}, \delta\rho) = 0 \\ \widehat{\mathcal{N}^*}(\Delta u_{\text{nt}}, \delta\rho, \delta p) = 0 \end{gathered} \tag{17}$$

This system is a *linear* system of PDEs and Δu_{nt} can be computed by solving the following auxiliary optimization program:

$$\boxed{\begin{aligned} \Delta u_{\text{nt}} \;=\; &\underset{\delta u}{\text{argmin}} && \{||\mathcal{H}\delta u + \nabla \tilde{J}(u)||^2\} \\ &\text{subject to} && \widehat{\mathcal{N}}(\delta u, \delta\rho) = 0 \\ & && \widehat{\mathcal{N}^*}(\delta u, \delta\rho, \delta p) = 0 \end{aligned} \qquad \text{(P3)}}$$

where $\mathcal{H}\delta u$ is given by (14).

In this optimization program, the Newton step Δu_{nt} is expressed as the minimizer of the first order approximation of the gradient. Mathematically, the constraints of this problem are homogeneous linear PDEs and the objective is a positive definite quadratic form. Therefore, the problem is a (convex) quadratic program, which certifies that a descent algorithm will return a global optimal solution, *i.e.* the Newton step.

C. Approximate second order algorithm

The Newton step provides a second order optimal descent direction, but unfortunately, its computation via the auxiliary optimization program (P3) may be slow. In this subsection, an approximation of the Newton step is presented which remains a good descent direction, and whose computation can be performed much more quickly.

We have shown that the Newton step can be determined by solving a system of PDEs, as in Equation (17). The complexity in the full Newton step method arises from computing the variation of the state $\delta\rho$ and the variation of the costate δp, which are themselves implicitly given by PDE constraints. A straightforward simplification can be made by ignoring these terms in the computation of Δu_{nt}, which results in a simplified direction finding algorithm while maintaining much of the nature of the second order Newton method. The result is an approximate Newton step, Δu_{ap}, based on the following equation:

$$\nabla_u \mathcal{G}(u,\rho,p)\ \Delta u_{ap} = -\mathcal{G}(u,\rho,p) \tag{18}$$

In the context of our air traffic flow problem,

$$\begin{array}{l} \nabla_v \mathcal{G}(u,\rho,p) = \frac{1}{M}\left(\frac{\delta v}{(v_{\max}-v)^2} + \frac{\delta v}{(v-v_{\min})^2}\right) \\ \nabla_\beta \mathcal{G}(u,\rho,p) = \frac{1}{M}\left(\frac{\delta\beta}{(1-\beta)^2} + \frac{\delta\beta}{\beta^2}\right) \end{array} \tag{19}$$

Therefore, the approximate Newton step update for v and β is

$$\begin{aligned} v_{ap} &= -\left(\frac{M}{\frac{1}{(v_{\max}-v)^2} + \frac{1}{(v-v_{\min})^2}}\right)\nabla_v J \\ \beta_{ap} &= -\left(\frac{M}{\frac{1}{\beta^2} + \frac{1}{(1-\beta)^2}}\right)\nabla_\beta J \end{aligned} \tag{20}$$

IV. Distributed Optimization

Of all the constraints imposed upon the optimization problem, only the maximum link density constraint couples the airline solutions together. As will be seen, it is possible to distribute the computation amongst the airlines. The key advantages to distributing the network flow optimization process are threefold:

- *Increased flexibility for the airlines.* Each link's value can be updated at each time the process is executed, allowing aircraft load factors, connecting passengers, and mechanical problems to influence flow decisions.
- *Distribution of computational burden.* A distributed approach enables division of the workload amongst the parties involved, which in the motivating scenario allows the FAA to distribute the computation to the airlines.
- *Privacy of information.* The airlines need not reveal their preferences to all other agents in the system.

In order to distribute the multi-airline Eulerian network flow problem, it is necessary to decouple the problem formulation (P1) into separate optimization problems for each airline. In the centralized form, both costs and constraints can impose interdependence between the optimization variables of the airlines. The next two sections develop a distributed optimization algorithm that is flexible enough to apply to any of the above centralized problem formulations.

A. Decoupled Costs

For Pareto optimal solutions as described is Section II, the cost function is decoupled as the sum of all airline cost functions. In the case of the NBS, the cost function defined in Section II can also be decoupled using a technique from geometric programming as described by Boyd and Vandenberghe.[12] A technical assumption is needed with regard to the disagreement point in order to proceed.

Assumption 2. *The disagreement point shall be strictly worse than any feasible solution. That is, for some fixed* $\epsilon > 0$, $d_j - \max J_j(\rho_j, v_j) > \epsilon$, *and as a result, for all feasible solutions,* $d_j - J_j(\rho_j, v_j) > 0$.

This assumption ensures that each agent always strictly improves on the disagreement point, which can be ensured by adding additional constraints to the problem.

The NBS cost function can now be decoupled. Since the following two optimization problems are equivalent,

$$\begin{array}{ll} \text{minimize} & J_{NBS}(\rho, v) \\ \text{subject to} & (1) \text{ and } (2) \\ & \sum\limits_{j\in\mathcal{J}} \rho_j \le \bar{\rho} \\ & v_{\min} \le v \le v_{\max} \\ & 0 \le \beta \le 1 \end{array} \quad \Longleftrightarrow \quad \begin{array}{ll} \text{minimize} & -\log(-J_{NBS}(\rho, v)) \\ \text{subject to} & (1) \text{ and } (2) \\ & \sum\limits_{j\in\mathcal{J}} \rho_j \le \bar{\rho} \\ & v_{\min} \le v \le v_{\max} \\ & 0 \le \beta \le 1 \end{array} \tag{21}$$

and since,

$$\begin{aligned} -\log(-J_{NBS}) &= -\log\left(\prod_{j\in\mathcal{J}} (d_j - J_j(\rho_j, v_j))\right) \\ &= \sum_{j\in\mathcal{J}} (-\log(d_j - J_j(\rho_j, v_j))) \\ &= \sum_{j\in\mathcal{J}} J_{j,NBS} \end{aligned} \tag{22}$$

the original formulation is equivalent to

$$\boxed{\begin{array}{ll} \text{minimize} & \sum\limits_{j\in\mathcal{J}} J_{j,NBS}(\rho_j, v_j) \\ \text{subject to} & (1) \text{ and } (2) \\ & \sum\limits_{j\in\mathcal{J}} \rho_j \le \bar{\rho} \\ & v_{\min} \le v \le v_{\max} \\ & 0 \le \beta \le 1 \end{array} \qquad \text{(P4)}}$$

where $J_{j,NBS} = -\log(d_j - J_j(\rho_j, v_j))$ is defined as the decoupled local cost function to be maximized by each agent in the system. Due to Assumption (2), $J_{j,NBS}$ is well-defined. With the NBS cost function separated between the agents, there remains only the need to treat interconnecting constraints between the agents in order to complete the presentation of the distributed algorithm.

B. Distributed Constraints

For the motivating example of this paper, only the maximum aggregate link density constraints cause interconnection between the airlines. Dual decomposition techniques using slack variables, as implemented by Raffard, Tomlin and Boyd[13] and described by Boyd and Vandenberghe,[12] are used to decouple the barrier costs functions of Problem (P2) from this interconnected constraint. Ultimately, each airline is assigned its own optimization problem, subject to coordination constraints that are updated by a centralized coordinator (ie. the FAA). From Problem (P2),

$$\tilde{J} = \sum_{j\in\mathcal{J}} \tilde{J}_j(\rho_j, v_j, \beta_j) - \frac{1}{M} \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \log(\bar{\rho}_i - \sum_{j\in\mathcal{J}} \rho_{i,j})\, dx_i\, dt \tag{23}$$

where $\tilde{J}_j$ are defined here to combine airline-specific cost functions, J_j or $J_{j,NBS}$ (denoted $J_{j,*}$), with the barrier costs on airline-specific constraints as follows,

$$\begin{aligned} \tilde{J}_j(\rho_j, v_j, \beta_j) = \quad & J_{j,*}(\rho_j, v_j) - \tfrac{1}{M} \textstyle\sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \log(v_{\max,i} - v_{i,j})(v_{i,j} - v_{\min,i})\, dx_i\, dt \\ & -\tfrac{1}{M} \textstyle\sum_{i\in\mathcal{D}} \int_0^T \log(\beta_{i,j})(1 - \beta_{i,j}) dt \end{aligned} \tag{24}$$

Introducing slack variables, $\tilde{\rho}_{i,j}$, for each variable in the interconnected constraint, the optimization becomes

$$\begin{array}{lrl} \text{minimize} & \tilde{J} = & \sum_{j\in\mathcal{J}} \tilde{J}_j(\rho_j, v_j, \beta_j) - \frac{1}{M}\sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \log(\bar{\rho}_i - \sum_{j\in\mathcal{J}} \tilde{\rho}_{i,j})\, dx_i\, dt \\ \text{subject to} & & (1) \text{ and } (2) \\ & & \rho_{i,j} = \tilde{\rho}_{i,j}\,, \quad \forall\{i,j\} \in \mathcal{I}\times J \end{array} \tag{25}$$

The Lagrangian can now be formed,

$$\begin{array}{rcl} \mathcal{L} & = & \sum_{j\in\mathcal{J}} \left(\tilde{J}_j(\rho_j, v_j, \beta_j) + \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \lambda_{i,j}(\rho_{i,j} - \tilde{\rho}_{i,j})\, dx_i\, dt \right) \\ & & -\frac{1}{M} \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \left(\log\left(\bar{\rho}_i - \sum_{j\in\mathcal{J}} \tilde{\rho}_{i,j} \right) \right) dx_i\, dt \end{array} \tag{26}$$

The dual function can be written as a function of $N_{\mathcal{J}}+1$ components, an individual PDE optimization for each airline plus the remaining terms that deal with the coupling of constraints.

$$\begin{array}{rcll} g & = & \inf\limits_{\rho_j, v_j, \beta_j} & \left[\sum_{j\in\mathcal{J}} \left(\tilde{J}_j(\rho_j, v_j, \beta_j) + \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \lambda_{i,j}\rho_{i,j}\, dx_i\, dt \right) \right. \\ & & & \left. -\frac{1}{M} \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \log\left(\bar{\rho}_i - \sum_{j\in\mathcal{J}} \tilde{\rho}_{i,j} \right) dx_i\, dt - \sum_{i\in\mathcal{I}} \sum_{j\in\mathcal{J}} \int_0^T \int_0^{L_i} \lambda_{i,j}\tilde{\rho}_{i,j}\, dx_i\, dt \right], \\ & & \text{subject to} & (1), (2) \text{ and } \tilde{\rho} \ge 0 \end{array} \tag{27}$$

which can be divided into individual optimizations for each airline and a centralized infimum calculation to enforce that link density constraints have to be met, resulting in safe operation of the airspace,

$$g = g_0 + \sum_{j\in\mathcal{J}} g_j\,, \quad \text{subject to} \;\; (1), (2) \text{ and } \tilde{\rho} \ge 0 \tag{28}$$

where,

$$\begin{array}{rcl} g_0 & = & \inf\limits_{\tilde{\rho}} [\mathcal{L}_0] = \inf\limits_{\tilde{\rho}} \left[\sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \left(-\frac{1}{M} \log\left(\bar{\rho}_i - \sum_{j\in\mathcal{J}} \tilde{\rho}_{i,j} \right) - \sum_{j\in\mathcal{J}} \lambda_{i,j}\tilde{\rho}_{i,j} \right) dx_i\, dt \right] \\ g_j & = & \inf\limits_{\rho_j, v_j, \beta_j} [\mathcal{L}_j] = \inf\limits_{\rho_j, v_j, \beta_j} \left[\tilde{J}_j(\rho_j, v_j, \beta_j) + \sum_{i\in\mathcal{I}} \int_0^T \int_0^{L_i} \lambda_{i,j}\rho_{i,j}\, dx_i\, dt \right] \end{array} \tag{29}$$

An analytic solution is available for g_0. Indeed, the gradient of $\mathcal{L}_0$ is

$$\nabla_{\tilde{\rho}_{i,j}} \mathcal{L}_0 = \frac{1}{M} \frac{1}{\bar{\rho}_i - \sum_{j\in\mathcal{J}} \tilde{\rho}_{i,j}} - \lambda_{i,j}\,, \tag{30}$$

and the minimum of $\mathcal{L}_0$ is achieved either at the point where the gradient vanishes or at the boundary of the domain of $\mathcal{L}_0$. Define $\lambda_i = \max_j \lambda_{i,j}$, then the resulting pointwise (in time and space) constrained infimum solution h_{0i} defined by $g_0 = \sum_{i\in\mathcal{I}} \int_0^{L_i} \int_0^T h_{0i}\, dx_i\, dt$ is

$$\begin{array}{ll} \text{Case 1:} \quad \lambda_i \le \frac{1}{\bar{\rho}_i M} & \\ \quad \tilde{\rho}^*_{i,j} = 0\,, \quad \forall j \in \mathcal{J} & h_{0i} = \frac{-1}{M}\log(\bar{\rho}_i) \\ \text{Case 2:} \quad \lambda_i > \frac{1}{\bar{\rho}_i M} & \\ \text{Let } \mathcal{K}_i = \{k \,|\, \lambda_{i,k} = \lambda_i\}, \tilde{\rho}^*_i \text{ is any Pareto Optimal solution that satisfies} & \\ \quad \sum_{k\in\mathcal{K}_i} \tilde{\rho}^*_{i,k} = \bar{\rho}_i - \frac{1}{\lambda_i M}\,, \text{ and } \quad \rho^*_{i,j} = 0\,, \quad \forall j \notin \mathcal{K}_i & h_{0i} = \frac{1}{M}(\log(\lambda_i M) - 1) - \lambda_i\bar{\rho}_i \end{array} \tag{31}$$

in which $\rho^*_i = \underset{\tilde{\rho}_i \ge 0}{\operatorname{argmin}}[h_{0i}]$.

Now the algorithm consists of maximizing the dual function g. For this purpose, subgradients of g can be extracted as follows

$$\frac{\partial g}{\partial \lambda_{i,j}} = \rho^*_{i,j} - \tilde{\rho}^*_{i,j} \tag{32}$$

Algorithm 2 Cooperative, Distributed Optimization

The centralized coordinator sets the barrier parameter $M = 1$ and sets the dual variables $\lambda_{i,j}$ to 0.
while $\frac{1}{M} > \epsilon_a$ and time permitting, **do**
    while Subgradient ascent on g has not converged **do**
        The airlines solve their dual problems to determine their optimal density fields $\rho^*_{i,j}$.
        The airlines transmit their optimal density fields to the centralized coordinator.
        The centralized coordinator updates values of the dual variables in the direction of the gradient $\frac{\partial g}{\partial \lambda_{i,j}}$.
        The centralized coordinator broadcasts the updated dual variables to all airlines.
    end while
end while

in which $\rho^*_j = \underset{\rho_j, v_j, \beta_j}{\operatorname{argmin}} \left[\tilde{J}_j(\rho_j, v_j, \beta_j) + \sum_{i \in \mathcal{I}} \int_0^T \int_0^{L_i} \lambda_{i,j} \rho_{i,j} \, dx_i \, dt \right]$, subject to (1) and (2).

The distributed algorithm may now proceed as follows:

This algorithm can be seen to be working in two directions, the outer loop controls the size of the neighborhood in which the variables are optimized, while the inner dual optimization pushes the optimization variables toward this neighborhood. As the barrier parameter M is increased, the neighborhood increases accordingly, and as such it is possible to achieve infeasible solutions. Infeasibility arises when the optimal density ρ^*, not constrained by $\bar{\rho}$, is divergent from the slack variable $\tilde{\rho}^*$, which is constrained by $\bar{\rho}$. As the dual optimization converges, however, the density and slack variables will approach each other, and ultimately, feasibility will be restored. This allows a coordinator to abort the distributed optimization process (by breaking the outer loop) prior to completion if need be, a useful feature for time-critical optimization processes.

V. Simulation Results

The algorithms presented in Section III and IV were applied to an air traffic flow problem as originally depicted in Figure 2. A standard scenario was selected for which the cost functions and optimization methods were varied in order to contrast them accurately.

A. Scenario

Two airlines wish to minimize their cost functions over the Boston, New York, Chicago network. Airline 1 has no preference with regard to which route its aircraft take. Therefore, $\omega_3 = \omega_4 = 1$. Airline 2, however, has a preference for the North route. Therefore $\omega_3 = 2$ and $\omega_4 = 1$. Initially, aircraft density was set to be uniform along all links, at a level one eighth of the maximum density, which represents one aircraft every eighty miles for each airline. Additionally, only flow along link 1 was permitted to split between links 3 and 4. A nominal initial velocity profile was assumed, and inflow into links 1 and 2 was set to be constant for both airlines. The simulation covers a 42 minute period (2500 seconds), and, as mentioned in Section II, the cost functions for the airlines capture a situation where links are to be closed at the end of the simulation. As a result, the goal of the airlines is to get as many of their aircraft to land prior to link closure as possible.

B. Characteristic Solution

Figures 3 and 4 show the resulting density of aircraft along all four links at various times $t = 0, T/11, \ldots, 10T/11, T$. Initially, the density is uniform over the network, but as the cutoff time approaches, a region of increased density can be seen to form at the point in the flow where aircraft speeds are increasing in order to arrive ahead of the cutoff. Additionally, airline 1 routes can be seen to have divided its aircraft almost equally between the southern and the the northern links while airline 2, which has a preference for the northern route, sends most of its aircraft toward link 3. Since link 4 is longer than link 3, a surge in density is seen along link 4 prior to the surge in density along link 3, in essence, clearing the way on link 3 for the wave to come. This effect is reflected in the flow split parameter, β_1, of Figures 7, 8 and 9, where an increase in flow to link 4 is seen prior to a diversion of almost all aircraft to link

3. Furthermore, note that any aircraft on the links after the final time would have to be diverted, according to the scenario described, and as such, do not contribute to the overall cost.

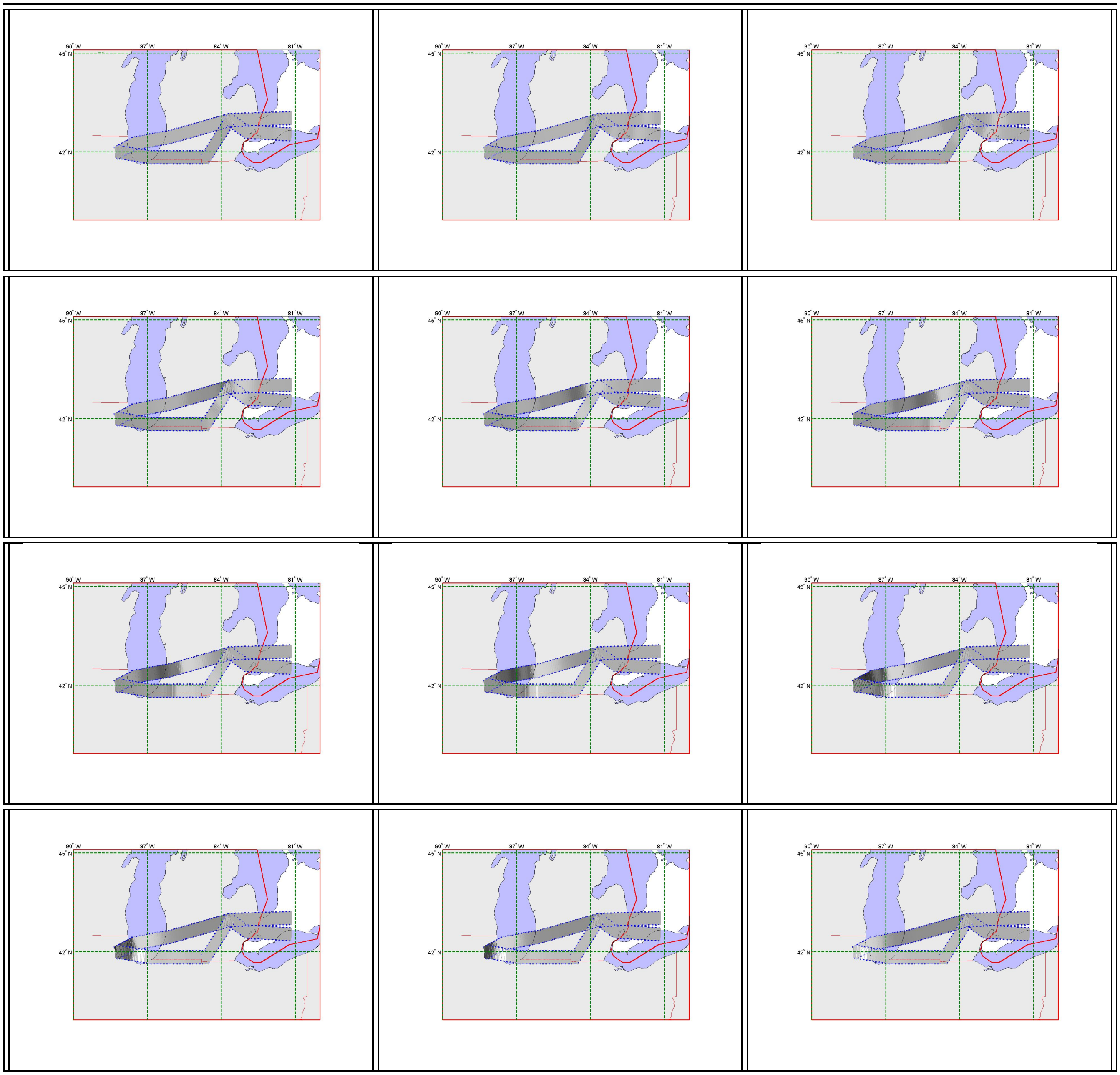

Figure 3. Scenario: Aircraft density for airline 1 in the network around Chicago with velocity assignment control. The density of the links is depicted by its shading, with a scale of white for zero density, and black for maximum density. Airline 1 has almost equally routed its aircraft between the southern and the northern links.

C. Complexity

1. Centralized Algorithm

For the centralized optimization algorithm, both the gradient descent and the approximate second order optimization techniques were applied. The gradient descent method scales linearly with the number of airlines, due to the fact that each airline introduces the same number of optimization variables over the network and the gradient calculation scales linearly with the number of optimization variables. Furthermore, the approximate second order method also scales linearly with the number of optimization variables, and yet, since it does capture some of the advantage of a second order descent direction, its convergence rate presents a significant improvement over the gradient descent method, as seen in Figure 5. The second order method is therefore used throughout the remaining simulations as the centralized solver of choice.

2. Distributed Algorithm

In order to analyze the overall complexity of the distributed algorithm proposed, it in necessary to investigate both the global algorithm requirements and the underlying subproblem solver used by each airline. The PDE constrained subproblem, solved by each airline is equivalent to a single airline centralized solution. The distributed algorithm requires increased computation, as the bargaining process between airlines and the central coordinator must also converge. The number of iterations necessary for the 2 airline bargaining process is about 50. Therefore, in theory, the distributed algorithm would run $50/2 = 25$ times as slow as the centralized one – the division by 2 takes into account the distribution of the computational burden between airlines. In practice, at each iteration of the distributed algorithm, it is possible to use the last optimal solution as a guess for the solution of the next iteration. As a consequence, the amount of required descent iterations reduces by a factor of 2 or 3. Therefore the distributed algorithm typically runs $25/2.5 = 10$ times as slow as the centralized algorithm.

Finally, if the computation were extended to more than 2 airlines, then it has to be expected that the difference between the distributed and the centralized algorithm would inverse. Indeed, the running time would increase linearly for the centralized algorithm, whereas we expect the running time to increase logarithmically[13] for the distributed algorithm.

D. Distributed Feasibility

An essential validation for the distributed algorithm is to assess that it returns a feasible solution. As described in Section IV, the solution is very likely to be feasible at the end of each dual optimization process as long as the barrier parameter is not too large. For the present simulation, $1/M$ has been reduced to 10^{-5} and the solution has remained feasible.

Figure 6 shows the total optimal density over link 3 (the most crowded link) for 4 different times. For these 4 times, the optimal density always remains below the maximum allowed density. Although it is not shown in the figure, we have verified that this was true for all time.

E. Comparison of Solutions

Table 1 presents a summary of the cumulative flux, namely the total number of aircraft landing in Chicago, for each airline using three separate methods: Centralized Solution with AFM costs, Centralized Solution with NBS costs, and Distributed Solution with NBS costs. It is interesting to note that with different weights on links 3 and 4 for airline 2, the AFM cost function returns a solution that is biased toward that airline, while the NBS manages to maintain a fair balance in total aggregate flux. Despite enforcing fairness, the NBS solution has a greater total value. Inspection of the flow splitting parameter β of Figures 7 and 8 shows that the airline with the greater preference for Link 3 has a greatest portion of flux on the preferred link. Both figures also reveal the two waves of aircraft from both airlines described earlier, that are caused by the terminal time boundary condition.

The distributed optimization method resulted in improved performance over the centralized methods regardless of cost function used, which is not surprising considering the fact that the centralized technique relies entirely on interior point methods that can get trapped in local minima, whereas the distributed method searches infeasible densities while solving the dual problem. In order to get a sense on the control policy returned by the distributed algorithm, the β routing parameter has been plotted in Figure 9 for each airline.

At the beginning of the simulation (from $t = 0$ to $t = 0.18T$), airline 2 routes its aircraft mostly toward link 3, and airline 1 accepts to route a majority of its aircraft toward link 4. Note that no matter which velocity control is applied, these aircraft will easily arrive to Chicago by time T, therefore, airline 1 just route its aircraft to the link which is less crowded (link 4). From time $t = 0.3T$ to time $t = 0.42T$, both airlines send their aircraft uniquely

	Central AFM		Central NBS		Distributed NBS	
	Airline 1	Airline 2	Airline 1	Airline 2	Airline 1	Airline 2
Link 3	3.677	4.110	3.816	4.024	4.455	4.846
Link 4	2.719	2.597	2.748	2.621	2.992	2.531
Sum per Airline	6.396	6.707	6.564	6.645	7.447	7.377
Total Sum	13.103		13.209		14.824	

Table 1. Cumulative flux breakdown oreturned by the specified algorithms. All elements refer to number of aircraft landing in Chicago.

toward link 3. The reason is that, since link 4 is longer that link 3, the aircraft positioned in link 1 at that time would not have the time to arrive at Chicago by time T if they take the longer route (link 4). Between time $t = 0.18T$ and time $t = 0.3T$, airlines have come to an interesting agreement, in which, airline 2, tends to send some of its aircraft to link 4 in order to relieve congestion in link 3 and therefore clear the way for the upcoming aircraft (those arriving at the FNT junction between $t = 0.3T$ and $t = 0.42T$). After $t = 0.42T$, no matter what policy airline use, the aircraft will not be able to land on time, therefore, both airlines route half of their aircraft to link 3 and half of their aircraft to link 4.

VI. Conclusion

The network flow optimization problem is a complex, non-linear and multi-dimensional problem. This paper presents three key advances to the continuous flow optimization problem in order to facilitate a futuristic vision of air traffic control. The approximate second order techniques presented defines a marked improvement in convergence rate over the gradient descent approach and allows for the implementation of real-time optimization schemes that could handle weather disruptions for the NAS. Adding airline preferences to the flow optimization problem enables greater customization of the optimization process, and may ultimately allow airlines to operate more efficiently. Finally, the distributed computational approach allows oversight from the FAA without the need to perform all calculations centrally. Although the distributed technique requires greater computational resources in total for small problems, this deficiency dissolves as the number of airlines increases. One can only hope that efficiency gains in today's air traffic flow management ultimately get passed on to the consumer and provide for more affordable travel in the years to come.

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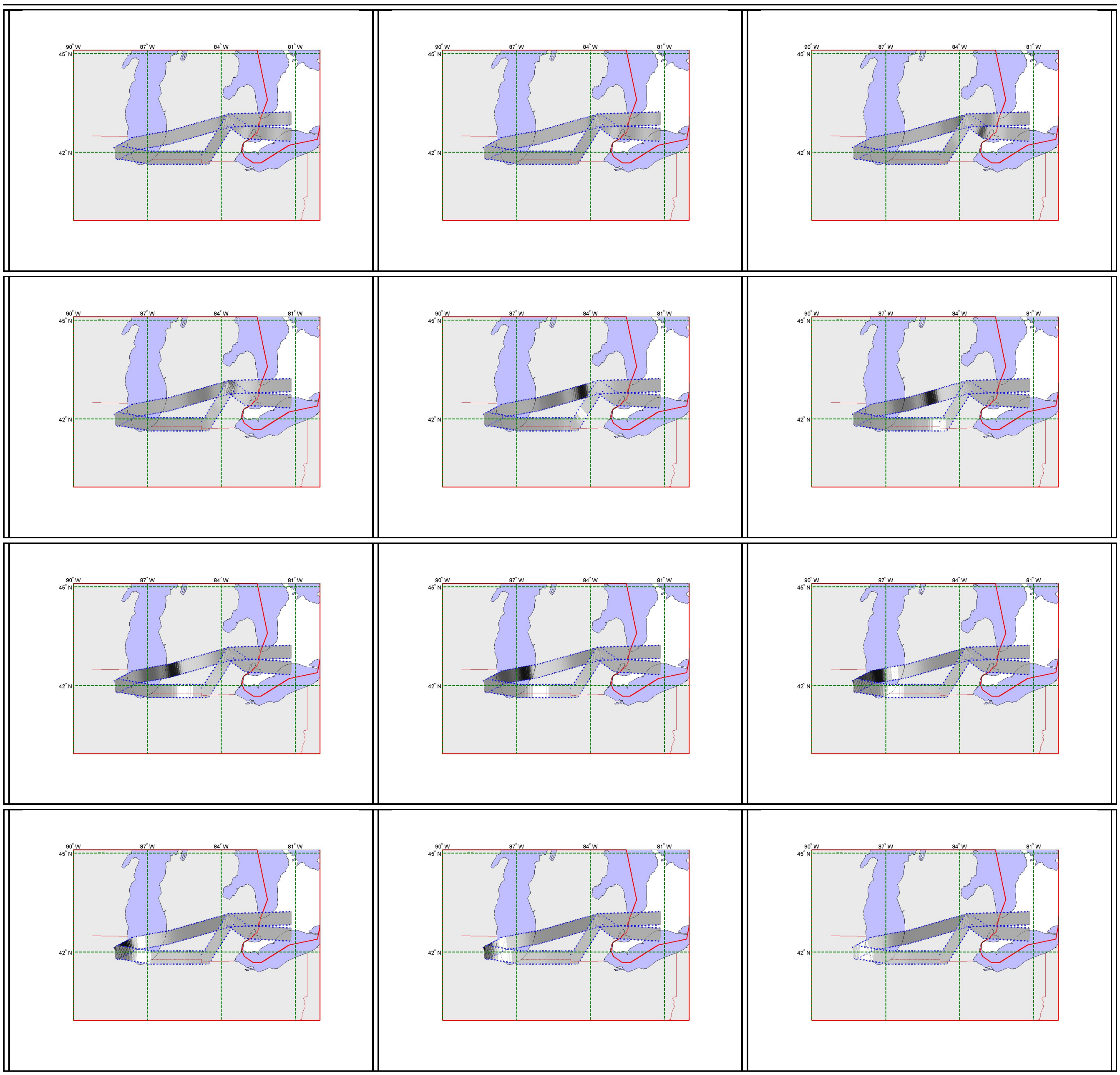

Figure 4. Scenario: Aircraft density for airline 2 in the network around Chicago with velocity assignment control. The density of the links is depicted by its shading, with a scale of white for zero density, and black for maximum density. Airline 2 has sent most of its aircraft toward the southern link.

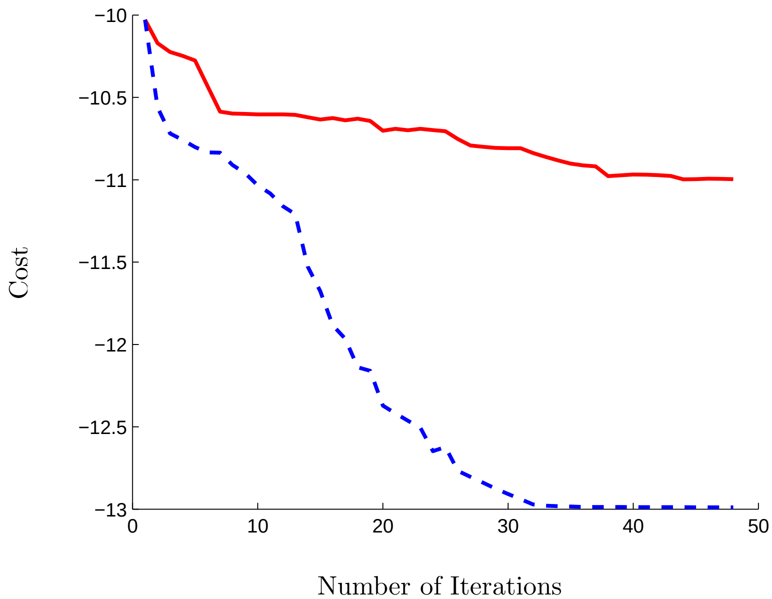

Figure 5. Cost convergence of gradient descent (solid) and approximate second order (dashed) methods for a centralized solution using the AFM cost.

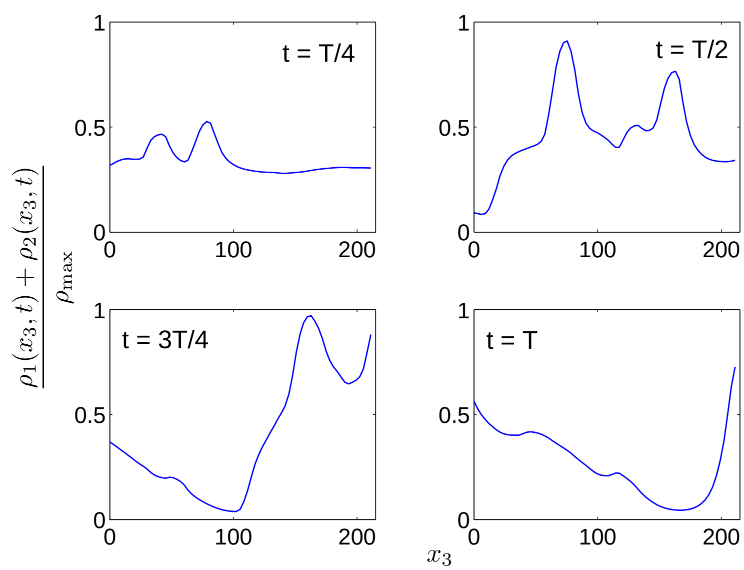

Figure 6. Total density along link 3 at various times for the distributed optimization method. The total density never exceeds the maximum allowed density.

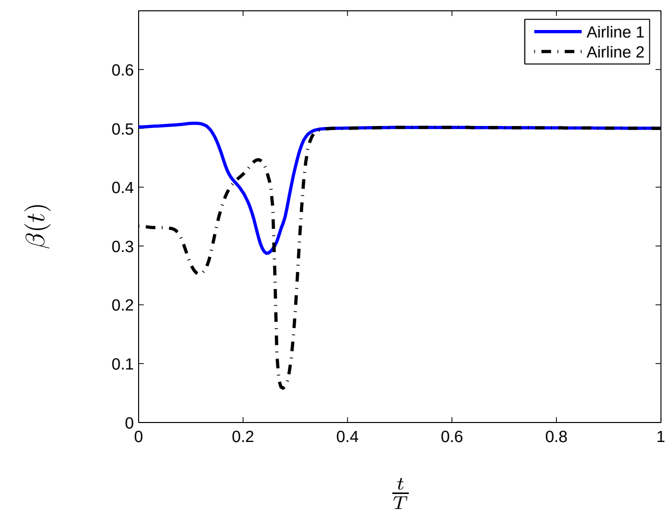

Figure 7. Portion of the flow routed to the South link (Link 4) in centralized case with AFM cost.

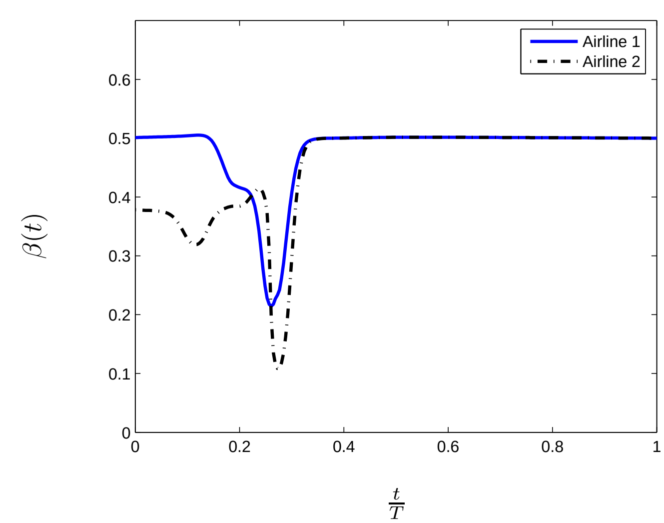

Figure 8. Portion of the flow routed to the South link (Link 4) in centralized case with NBS cost.

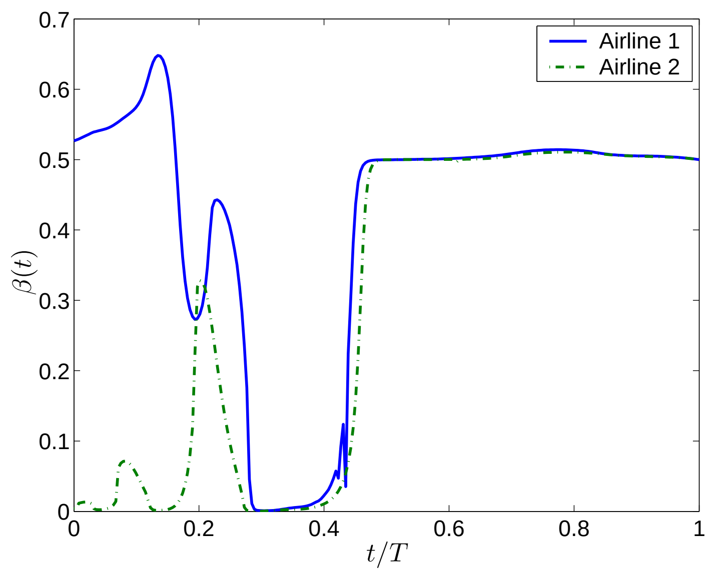

Figure 9. Portion of the flow routed to the South link (Link 4) in distributed case with NBS cost.