University of California Berkeley

CEE 291F

FROM PDEs TO SIMULATION GAMES WITH CELLULAR AUTOMATA

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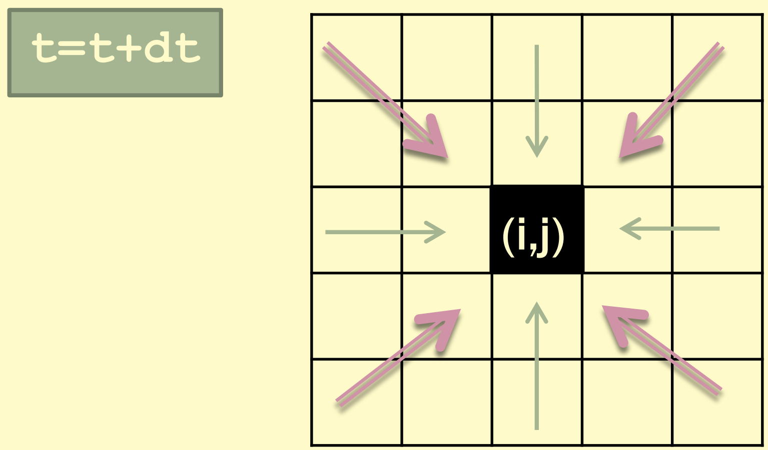
WHAT IS A CELLULAR AUTOMATON?

Discretization, *time* and *space* wise, of a system



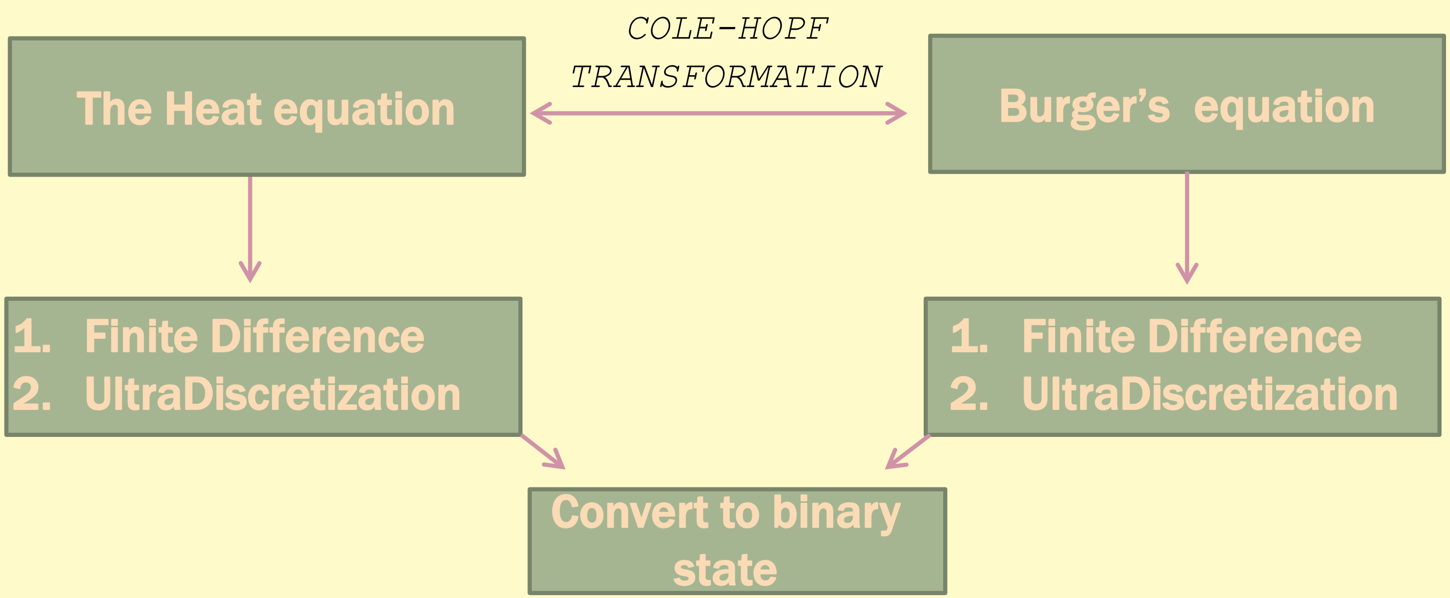
At each time step, through a time-evolution rule the next state of a cell is determined by:

- ***Its present state***

- ***The states of its local neighbors***

OBJECTIVE & MOTIVATION

Determine *Cellular Automata* for selected *PDEs*



CAs have the benefit of:

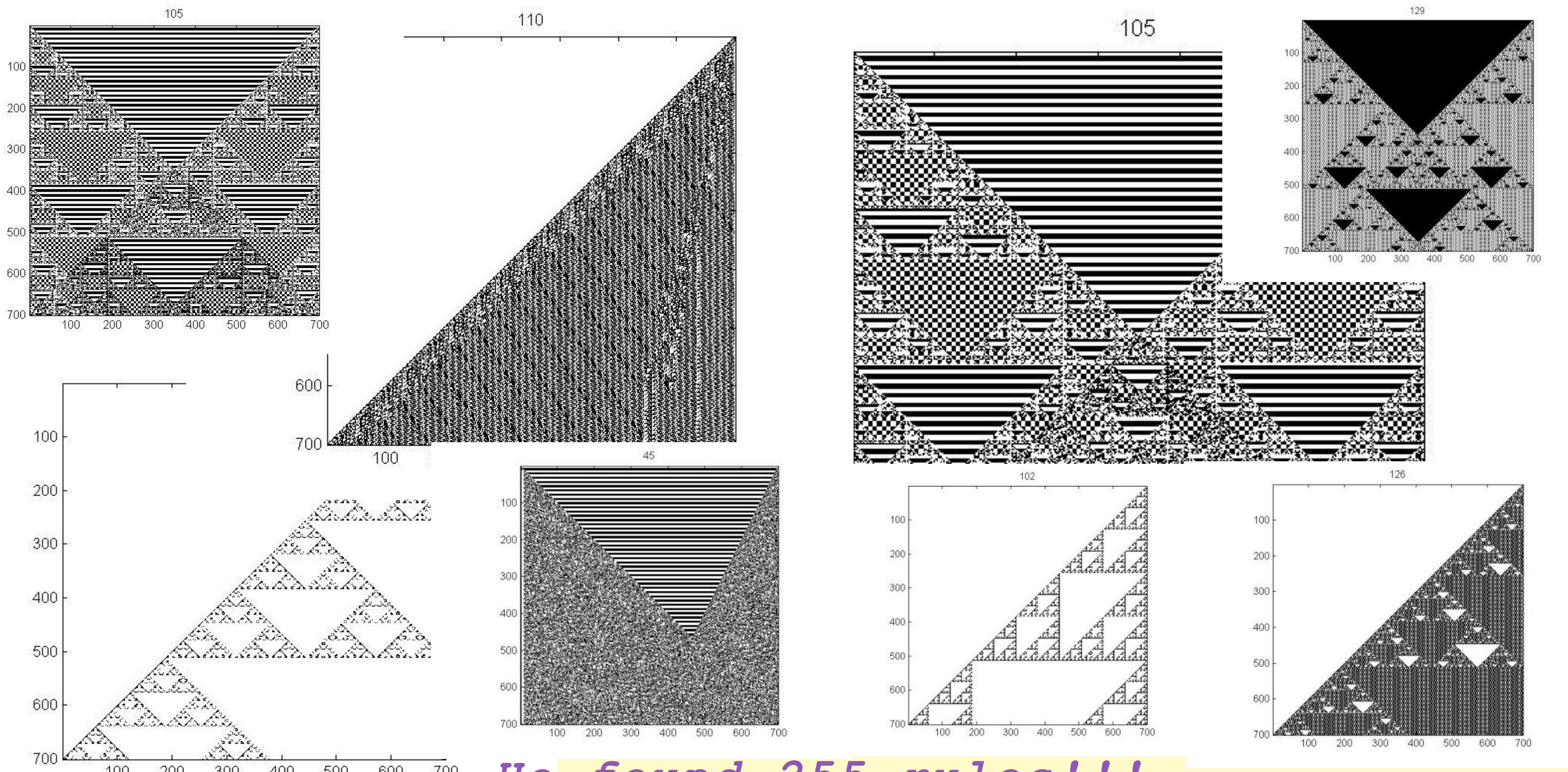
- Using very simple mathematical rules

- Complex results can be easily simulated

- Change the perspective of the model to a local view

STEVEN WOLFRAM CAs

Wolfram researched cellular automata models for several decades and has linked cellular automata with differential equations (*New Kind of Science*)

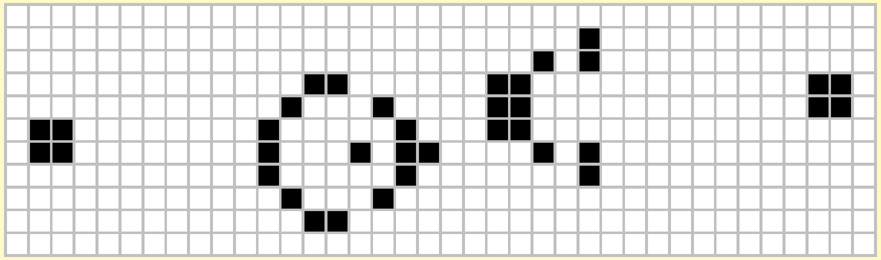


He found 255 rules!!!

JOHN CONWAY'S GAME OF LIFE

Each cell has 8 neighboring cells:

- 4 orthogonally
- 4 diagonally



CONWAY'S GENETIC LAWS

SURVIVAL
counter w/2-3 neighbors

DEATH
counter w/4+ or 1- neighbors

BIRTH
empty cell w/3 neighbors

Possible situations at the end of the «life history»:

- The society reaches a steady-state
- The society dies out
- The society oscillates forever

THE 2D HEAT EQUATION: NUMERICS

$$T_t = k\left(T_{xx} + T_{yy}\right)$$

FINITE DIFFERENCE

$$\frac{T_{i,j}^{t+1} - T_{i,j}^{t}}{\Delta t} = \kappa\left(\frac{T_{i-1,j}^{t} - 2T_{i,j}^{t} + T_{i+1,j}^{t}}{(\Delta x)^2} + \frac{T_{i,j-1}^{t} - 2T_{i,j}^{t} + T_{i,j+1}^{t}}{(\Delta y)^2}\right)$$

$\Delta x = \Delta y = \Delta t$

$$T_{i,j}^{t+1} = \frac{T_{i-1,j}^{t} + T_{i,j-1}^{t} + (M-4)T_{i,j}^{t} + T_{i+1,j}^{t} + T_{i,j+1}^{t}}{M}$$

Stability condition:

$M \geq 4$

Implementation in MATLAB

THE 2D HEAT EQUATION: IMPLEMENTATION & SIMULATION

GOAL : **Represent the PDE in terms of a game**

HOW:

1) Dicretize in time and space
2) Set the ICs and BCs
3) Set the binary life/death rules

ATTEMPTS:

Middle Source

Multiple sources

Life/death rules 1

Life/death rules 2

Problem Set 1: CE291F – ME 236 – EE291

Professor Alexandre Bayen

Posted: January 21st, 2013

Problem 1. The goal of the first problem is to go through a brief ODE review. Solve the following ODE problems in the indicated range of the independent variable x. Be brief, but sprinkle a few words in with your maths to explain what you are doing.

Question 1. $y' = 2x^2$ with the initial condition $y(0) = 1$ for $x \geq 0$.

Question 2. $y' = 3y$ with the initial condition $y(0) = 1$ for $x \geq 0$.

Question 3. $y'' - \alpha^2 y = 0$ with the boundary conditions $y(0) = 1$ and $y'(1) = 0$, where $\alpha \in \mathbb{R}$ is a real number.

Question 4. $y'' + \alpha^2 y = 0$ with the boundary conditions $y(0) = 0$ and $y'(1) = 0$, and with the normalization condition $y'(0) = 1$, for $x \in [0, 1]$, where $\alpha \in \mathbb{R}$ is a real number. Find the smallest positive α for which a solution exists.

Question 5. $y'' + xy = x$ with the boundary conditions $y(0) = 0$ and $y'(0) = 1$, for $x \geq 0$. Give the solution as a power series in x or in terms of known special functions if you know about them.

Problem 2. Derivation of the wave equation for cables. The goal of this problem is to derive a partial differential equation from simple physical modeling principles, to gain familiarity with modeling. We consider a cable, with tension F uniform in norm. We call x the coordinate along the cable. We describe the vertical displacement of the cable by a function $y(x, t)$ (see Figure 1). We call $\alpha(x, t)$ the local angle of the cable with the horizontal axis, and we denote by μ the lineic mass of the cable, i.e. the mass of the cable per unit length.

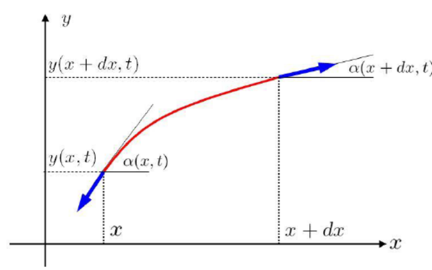


Figure 1: Illustration of the forces on the cable chunk of length dx. The arrows denote the tension, which is supposed to be uniform in norm along the cable.

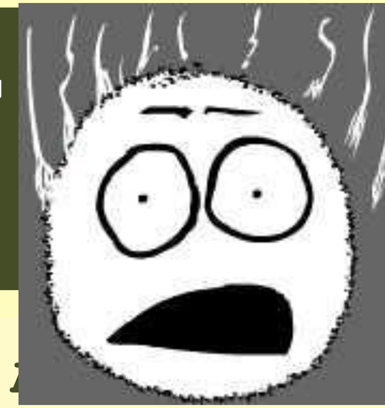
i) Write Newton's law along the vertical coordinate of the system (in the y direction) for an infinitesimal chunk of the cable of length dx. Use the angle $\alpha(x, t)$ in order to compute the projection of the forces along the vertical axis.

ii) For small angles, use the approximation $\alpha(x,t) \sim \sin \alpha(x,t) \sim \tan \alpha(x,t) \sim \frac{dy(x,t)}{dx}$. Express Newton's law in terms

F.

iii) Take the limit dx -

Problem 3. Cole-Hop
of variables rules and n
to transform partial d
dimensional heat equat
Prove that the functio



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RACT

game-rules from

ues

HOPF

RMATION

Problem 3. Cole-Hopf transformation. The goal of this problem is to apply some simple change of variables rules and methods (derived in the notes) to gain some familiarity with techniques used to transform partial differential equations. Let us consider $\phi(x, t)$, a function satisfying the one dimensional heat equation: $\phi_t = \nu \phi_{xx}$. Let us consider the function $u(x, t)$ defined by $u = -2\nu \frac{1}{\phi} \frac{\partial \phi}{\partial x}$. Prove that the function u satisfies the *viscous Burgers equation*: $u_t + uu_x = \nu u_{xx}$.

FROM HEAT TO BURGER'S EQUATION WITH ULTRA-DISCRETIZATION

$$g_t = g_{xx}$$

COLE-HOPF TRANS.

$$v = (\log g)_x$$

$$v_t = 2vv_x + v_{xx}$$

$$f^{n+1}{}_j = \frac{1}{2}(f^n{}_{j+1} + f^n{}_{j-1})$$

DIFFERENCE

COLE-HOPF

$$u_j{}^n = \frac{1}{\Delta x}(\log f_{j+1}{}^n - \log f_j{}^n)$$

$$u_j{}^{n+1} = u_j{}^n + \frac{1}{\Delta x}\left[\log\left(e^{-\Delta x u_j{}^n} + e^{-\Delta x u_{j+1}{}^n}\right)\right.$$

$$\lim_{\varepsilon \to +0} \varepsilon \log(e^{A/\varepsilon} + e^{B/\varepsilon} + \cdots) = \max(A, B, \cdots),$$

$$F^{n+1}{}_j = max\ (F^n{}_{j+1}, F^n{}_{j-1})$$

UD

COLE-HOPF

$$U^n{}_j = F^n{}_{j+1} - F^n{}_j - 1/2$$

$$U_j{}^{n+1} = U_j{}^n + min\left(U_{j-1}{}^n, 1 - U_j{}^n\right) - min(U_j{}^n, 1$$

It's a most efficient way to discretize from 0/1 to 0/1

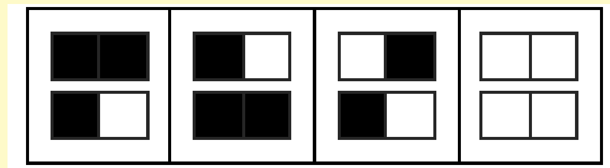
1D BURGER'S EQUATION AS A CA

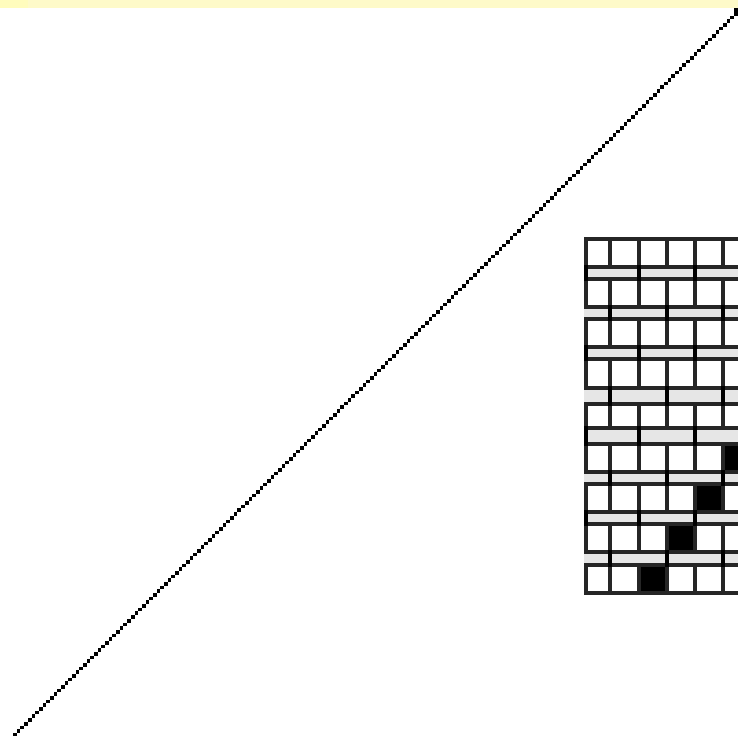
Assuming initial U's are all 0 or 1, we can easily show U's at any time also become 0 or 1

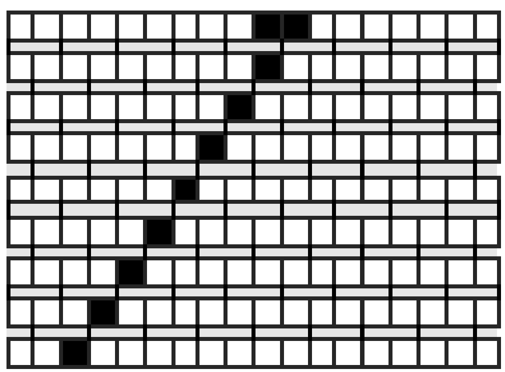
The Burger's equation is a cellular automaton (CA) which follows the time evolution rule:

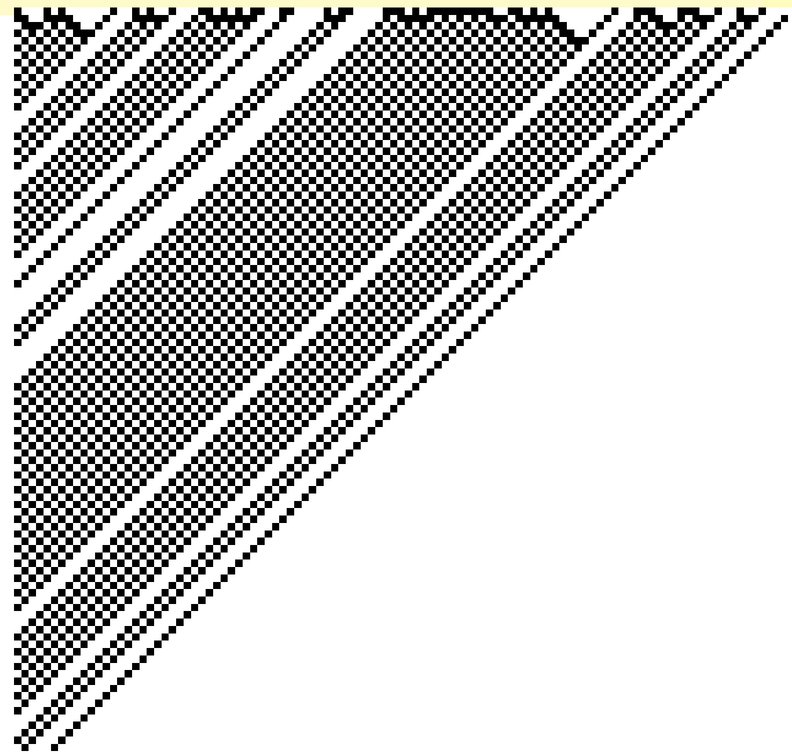
$$\frac{U_{j-1}{}^{n}U_{j}{}^{n}U_{j+1}{}^{n}}{U_{j}{}^{n+1}} = \frac{\textit{current state}}{\textit{new state for center cell}} : \quad \frac{000}{0} \quad \frac{001}{0} \quad \frac{010}{0} \quad \frac{011}{1} \quad \frac{100}{1} \quad \frac{101}{1} \quad \frac{110}{0} \quad \frac{111}{1}$$

STEVEN WOLFRAM: RULE 184

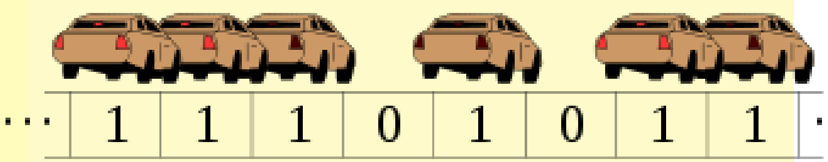


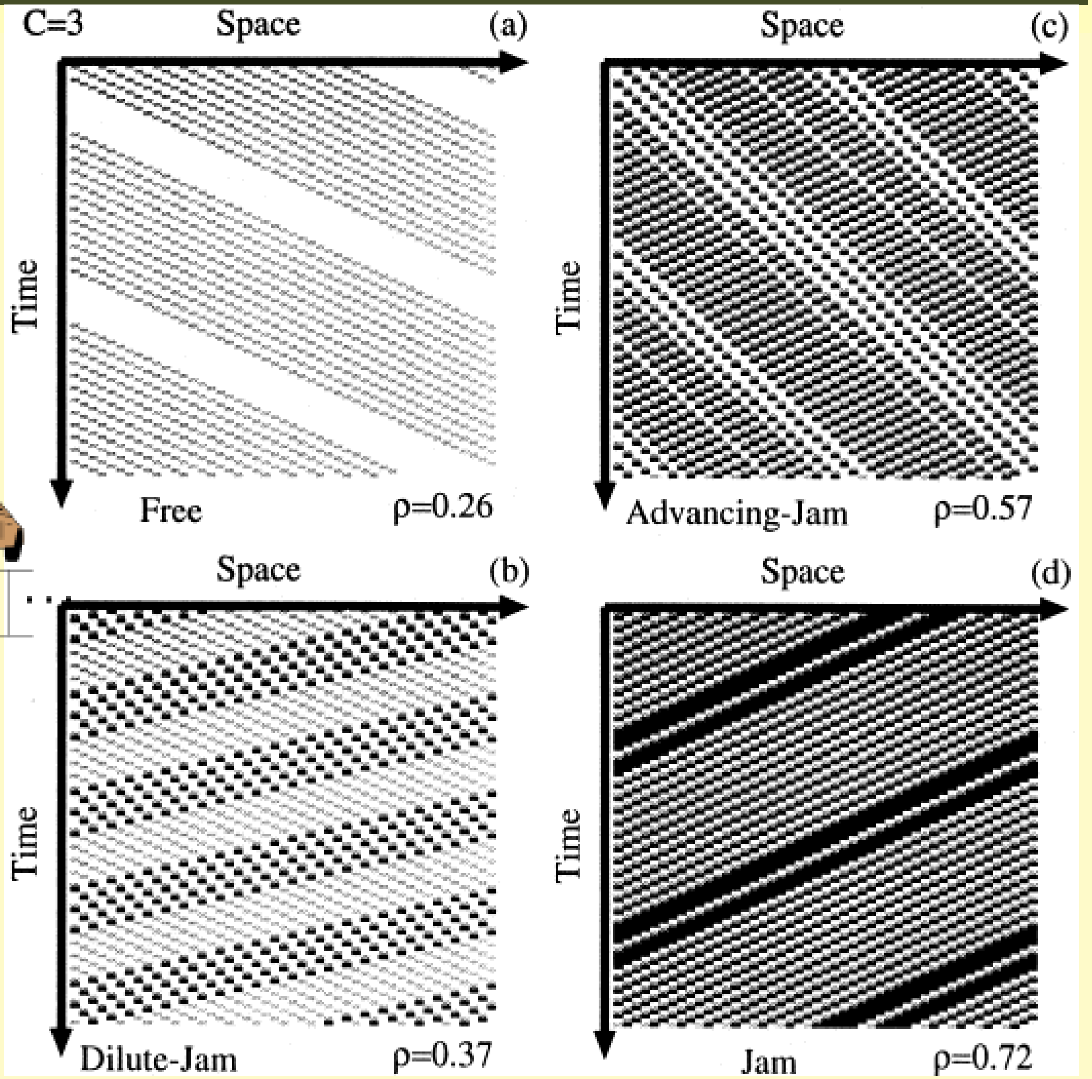






CORRELATION TO TRAFFIC FLOW





CONCLUSIONS & FURTHER WORK

Transforming continuous-valued PDEs to CAs is not an easy task

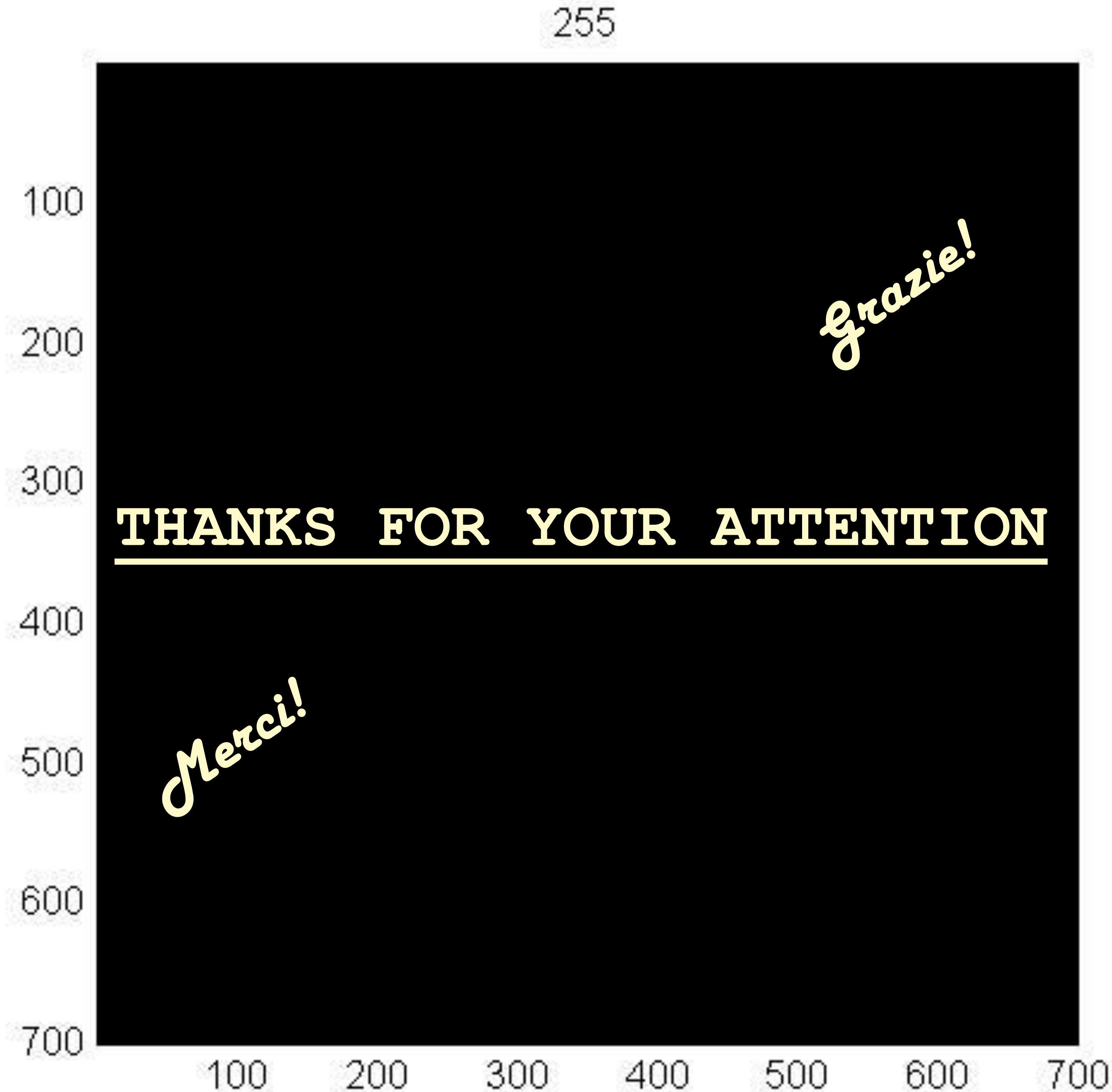
Through the UD method, it was easy to convert Burger's equation into a rule-based CA

Rule 184 has a correspondance to the real world

FURTHER WORK:

1. Investigate different ICs and BCs in order to obtain traffic-flow situations

2. Try to extend the 1D-Burgers' equation to 2D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Grazie!
THANKS FOR YOUR ATTENTION
Merci!
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