EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 7 – Steady-State Errors

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Lecture abstract

Topics covered in this presentation

- Steady-state error sources
- Test waveform inputs for evaluation
- Sensitivity

Chapter outline

Some definitions & test waveforms, [1, p. 340]

Steady-state error

- The difference between the input and the output for a prescribed test input as $t \to \infty$.

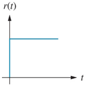
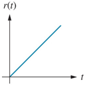
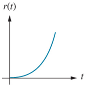
Waveform	Name	Physical interpretation	Time function	Laplace transform
	Step	Constant position	1	$\frac{1}{s}$
	Ramp	Constant velocity	t	$\frac{1}{s^2}$
	Parabola	Constant acceleration	$\frac{1}{2}t^2$	$\frac{1}{s^3}$

Table: Test waveforms for evaluating steady-state errors of position control systems

Steady-state error in terms of CL TF, $T(s)$, [1, p. 344]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

$$C(s) = R(s)T(s)$$

$$E(s) = R(s)[1 - T(s)]$$

Applying the final value theorem

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)]$$

Steady-state error in terms of $G(s)$, [1, p. 345]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Applying the final value theorem

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static error constants, [1, p. 350]

For a step input, $u(t)$

$$e_{step}(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

Position constant

$$K_p = \lim_{s \to 0} G(s)$$

For a ramp input, $tu(t)$

$$e_{ramp}(\infty) = \frac{1}{\lim_{s \to 0} sG(s)}$$

Velocity constant

$$K_v = \lim_{s \to 0} sG(s)$$

For a parabolic input, $\frac{1}{2}t^2u(t)$

$$e_{step}(\infty) = \frac{1}{\lim_{s \to 0} s^2G(s)}$$

Acceleration constant

$$K_a = \lim_{s \to 0} s^2G(s)$$

Steady-state error for disturbances, [1, p. 356]

Output

$$C(s) = E(s)G_1(s)G_2(s)+D(s)G_2(s)$$

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

After substitution

$$E(s) = \frac{1}{1+G_1(s)G_2(s)}R(s) - \frac{G_2(s)}{1+G_1(s)G_2(s)}D(s)$$

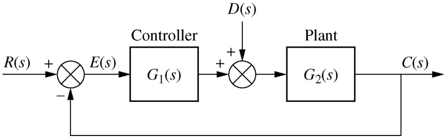


Figure: Feedback control system showing disturbance

Steady-state error for disturbances, [1, p. 356]

Final value theorem

$$e(\infty) = \lim_{s\to 0} sE(s) = e_R(\infty) + e_D(\infty)$$

Steady-state error due to reference

$$e_R(\infty) = \lim_{s\to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s)$$

Steady-state error due to disturbance

$$e_D(\infty) = -\lim_{s\to 0} \frac{s\, G_2(s)}{1 + G_1(s)G_2(s)} D(s)$$

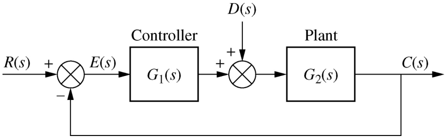


Figure: Feedback control system showing disturbance

Steady-state error due to step disturbance, [1, p. 356]

Steady-state error due to *step* disturbance

$$e_D(\infty) = \frac{1}{\lim\limits_{s\to 0} \dfrac{1}{G_2(s)} + \lim\limits_{s\to 0} G_1(s)}$$

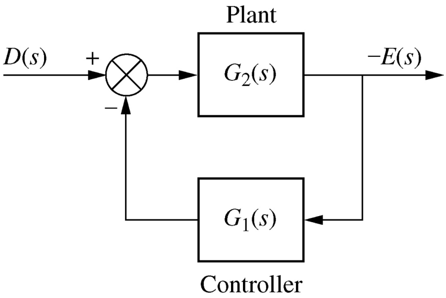


Figure: System rearranged to show disturbance as input and error as output, with $R(s) = 0$

Sensitivity, [1, p. 362]

Sensitivity: The ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero

$$S_{F:P} = \frac{P}{F}\frac{\delta F}{\delta P}$$

Analysis via final value theorem, [1, p. 364]

CL system represented in SS

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Error

$$E(s) = R(s) - Y(s)$$

Output

$$Y(s) = R(s)T(s)$$

Substitution

$$E(s) = R(s)[1 - C(sI - A)^{-1}B]$$

Final value theorem

$$\lim_{s\to 0} sE(s) = \lim_{s\to 0} sR(s)[1 - C(sI - A)^{-1}B]$$

Analysis via step input substitution, [1, p. 366]

Steady-state unit step input

$$u = 1$$

Steady-state solution

$$x_{ss} = V$$

$$\dot{x}_{ss} = 0$$

CL system represented in SS

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Substitution

$$0 = AV + B$$
$$y_{ss} = CV$$

Solving

$$V = -A^{-1}B$$

Steady-state error

$$\begin{aligned} e(\infty) &= 1 - y_{ss} \\ &= 1 - CV \\ &= 1 + CA^{-1}B \end{aligned}$$

Analysis via ramp input substitution, [1, p. 366]

Steady-state unit step input

$$u = t$$

Steady-state solution

$$x_{ss} = Vt + W$$

$$\dot{x}_{ss} = V$$

CL system represented in SS

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

Substitution

$$V = A(Vt + W) + B$$
$$y_{ss} = C(Vt + W)$$

Solving

$$V = -A^{-1}B$$

$$W = A^{-1}V$$

Steady-state error

$$e(\infty) = \lim_{t \to \infty} t - y_{ss}$$

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.