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Topics covered in this presentation

- Steady-state error sources
- Test waveform inputs for evaluation
- Sensitivity
7 Steady-State Errors

- 7.1 Introduction
- 7.2 Steady-state error for unity feedback systems
- 7.3 Static error constants and system type
- 7.4 Steady-state error specifications
- 7.5 Steady-state error for disturbances
- 7.6 Steady-state error for non-unity feedback systems
- 7.7 Sensitivity
- 7.8 Steady-state error for systems in state space
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Some definitions & test waveforms, [1, p. 340]

### Steady-state error

- The difference between the input and the output for a prescribed test input as \( t \to \infty \).

#### Table: Test waveforms for evaluating steady-state errors of position control systems

<table>
<thead>
<tr>
<th>Waveform</th>
<th>Name</th>
<th>Physical interpretation</th>
<th>Time function</th>
<th>Laplace transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>( r(t) )</td>
<td>Step</td>
<td>Constant position</td>
<td>1</td>
<td>( \frac{1}{s} )</td>
</tr>
<tr>
<td>( r(t) )</td>
<td>Ramp</td>
<td>Constant velocity</td>
<td>( t )</td>
<td>( \frac{1}{s^2} )</td>
</tr>
<tr>
<td>( r(t) )</td>
<td>Parabola</td>
<td>Constant acceleration</td>
<td>( \frac{1}{2}t^2 )</td>
<td>( \frac{1}{s^3} )</td>
</tr>
</tbody>
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Steady-state error in terms of CL TF, $T(s)$, [1, p. 344]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

$$C(s) = R(s)T(s)$$

$$E(s) = R(s)[1 - T(s)]$$

Applying the final value theorem

$$e(\infty) = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

$$e(\infty) = \lim_{s \to 0} sR(s)[1 - T(s)]$$
Steady-state error in terms of $G(s)$, [1, p. 345]

Error between the input, $R(s)$, and the output $C(s)$

$$E(s) = R(s) - C(s)$$

$$C(s) = E(s)G(s)$$

$$E(s) = \frac{R(s)}{1 + G(s)}$$

Applying the final value theorem

$$e(\infty) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$
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Static error constants, [1, p. 350]

For a step input, $u(t)$

$$e_{\text{step}}(\infty) = \frac{1}{1 + \lim_{s \to 0} G(s)}$$

For a ramp input, $tu(t)$

$$e_{\text{ramp}}(\infty) = \frac{1}{\lim_{s \to 0} sG(s)}$$

For a parabolic input, $\frac{1}{2}t^2u(t)$

$$e_{\text{step}}(\infty) = \frac{1}{\lim_{s \to 0} s^2G(s)}$$

**Position constant**

$$K_p = \lim_{s \to 0} G(s)$$

**Velocity constant**

$$K_v = \lim_{s \to 0} sG(s)$$

**Acceleration constant**

$$K_a = \lim_{s \to 0} s^2G(s)$$
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Steady-state error for disturbances, [1, p. 356]

Output

\[ C(s) = E(s)G_1(s)G_2(s) + D(s)G_2(s) \]

Error between the input, \( R(s) \), and the output \( C(s) \)

\[ E(s) = R(s) - C(s) \]

After substitution

\[ E(s) = \frac{1}{1 + G_1(s)G_2(s)} R(s) \]

\[ - \frac{G_2(s)}{1 + G_1(s)G_2(s)} D(s) \]

**Figure:** Feedback control system showing disturbance
Steady-state error for disturbances, [1, p. 356]

Final value theorem

\[ e(\infty) = \lim_{s \to 0} sE(s) = e_R(\infty) + e_D(\infty) \]

Steady-state error due to reference

\[ e_R(\infty) = \lim_{s \to 0} \frac{s}{1 + G_1(s)G_2(s)} R(s) \]

Steady-state error due to disturbance

\[ e_D(\infty) = -\lim_{s \to 0} \frac{sG_2(s)}{1 + G_1(s)G_2(s)} D(s) \]

Figure: Feedback control system showing disturbance
Steady-state error due to step disturbance, [1, p. 356]

Steady-state error due to *step* disturbance

\[ e_D(\infty) = \lim_{s \to 0} \frac{1}{G_2(s)} + \lim_{s \to 0} G_1(s) \]

**Figure:** System rearranged to show disturbance as input and error as output, with \( R(s) = 0 \)
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**Sensitivity:** The ratio of the fractional change in the function to the fractional change in the parameter as the fractional change of the parameter approaches zero

\[
S_{F:P} = \frac{P \delta F}{F \delta P}
\]
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Analysis via final value theorem, [1, p. 364]

CL system represented in SS

\[
\dot{x} = Ax + Bu \\
y = Cx
\]

Error

\[
E(s) = R(s) - Y(s)
\]

Output

\[
Y(s) = R(s)T(s)
\]

Substitution

\[
E(s) = R(s)[1 - C(sI - A)^{-1}B]
\]

Final value theorem

\[
\lim_{s \to 0} sE(s) = \lim_{s \to 0} sR(s)[1 - C(sI - A)^{-1}B]
\]
Analysis via step input substitution, [1, p. 366]

Steady-state unit step input

\[ u = 1 \]

Steady-state solution

\[ x_{ss} = V \]
\[ \dot{x}_{ss} = 0 \]

CL system represented in SS

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Substitution

\[ 0 = AV + B \]
\[ y_{ss} = CV \]

Solving

\[ V = -A^{-1}B \]

Steady-state error

\[ e(\infty) = 1 - y_{ss} \]
\[ = 1 - CV \]
\[ = 1 + CA^{-1}B \]
Analysis via ramp input substitution, [1, p. 366]

Steady-state unit step input

\[ u = t \]

Steady-state solution

\[ x_{ss} = Vt + W \]
\[ \dot{x}_{ss} = V \]

CL system represented in SS

\[ \dot{x} = Ax + Bu \]
\[ y = Cx \]

Substitution

\[ V = A(Vt + W) + B \]
\[ y_{ss} = C(Vt + W) \]

Solving

\[ V = -A^{-1}B \]
\[ W = A^{-1}V \]

Steady-state error

\[ e(\infty) = \lim_{t \to \infty} t - y_{ss} \]
Bibliography