EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 6 – Stability

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Lecture abstract

Topics covered in this presentation

- Stable, marginally stable, & unstable linear systems
- Relationship between pole locations and stability
- Routh-Hurwitz criterion
- Relationship between stability and eigenvalues

Chapter outline

Stability for LTI systems, [1, p. 302]

Total response of a system

$$c(t) = c_{\text{forced}}(t) + c_{\text{natural}}(t)$$

Stability for LTI systems

- Natural response as $t \to \infty$
  - *Stable:* $\to 0$
  - *Unstable:* Grows without bound
  - *Marginally stable:* Neither decays nor grows but remains constant
- Total response (BIBO)
  - *Stable:* Every bounded input yields a bounded output
  - *Unstable:* Any bounded input yields an unbounded output
    - *Marginally stable:* Some bounded inputs yield unstable outputs
- Stability $\implies$ only the forced response remains

Stability for LTI systems, [1, p. 302]

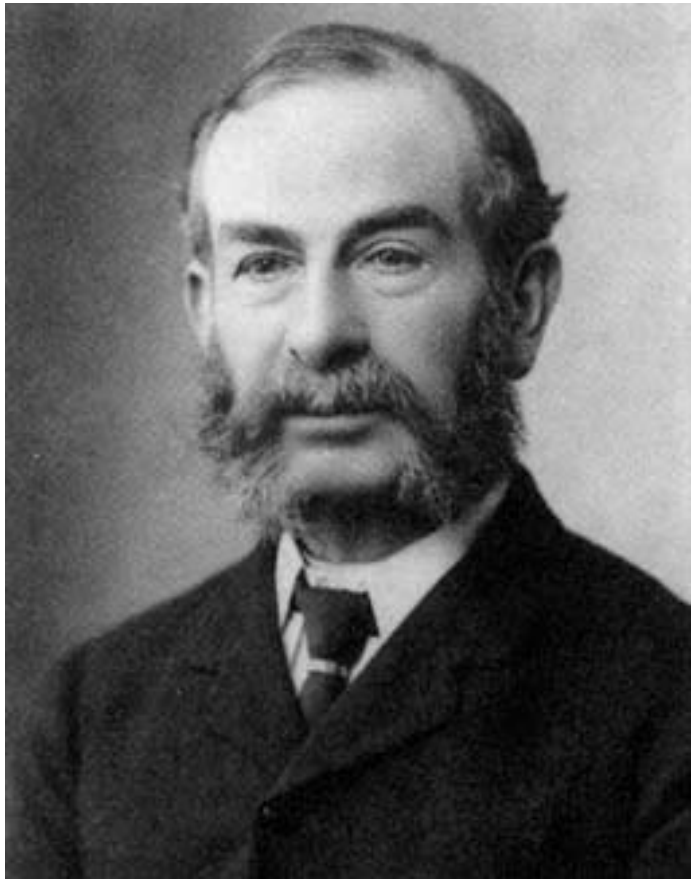
Stability for LTI systems in terms of pole locations

- Closed-loop TF poles
  - *Stable:* Only in LHP
  - *Unstable:* At least 1 in RHP and/or multiplicity greater than 1 on the imaginary axis
  - *Marginally stable:* Only imaginary axis poles of multiplicity 1 and poles in the LHP

History interlude

Edward John Routh

- 1831 – 1907
- English mathematician
- 1876 – Proposed what became the *Routh-Hurwitz stability criterion*



History interlude

Adolf Hurwitz

- 1859 – 1919
- German mathematician
- 1895 – Determined the *Routh-Hurwitz stability criterion*

Some definitions, [1, p. 305]

Routh-Hurwitz stability criterion

- Stability information without the need to solve for the CL system poles
- How many CL system poles are in the LHP, RHP, and on the imaginary axis

2 steps

1. Generate Routh table
2. Interpret the Routh table

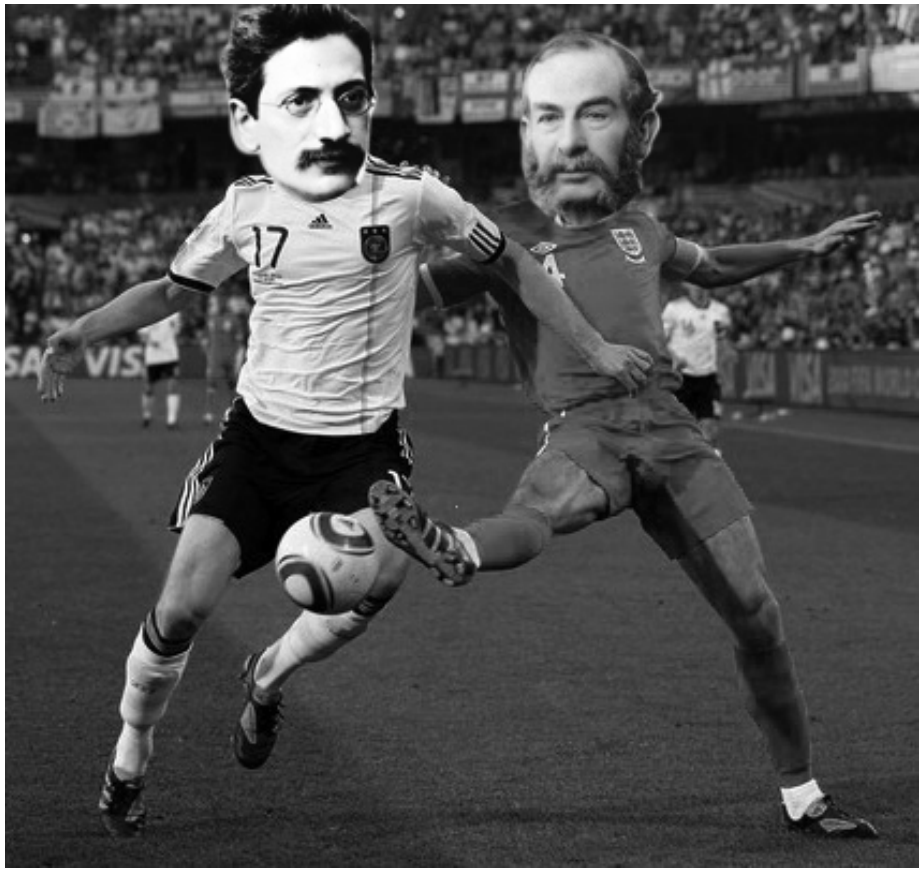


Figure : 1905 FIFA World Cup – Germany vs. England

Generating a basic Routh table, [1, p. 306]

Procedure

1. Label rows with powers of s from the highest power of the denominator of the CLTF down to s^0
2. In the 1st row, horizontally list every other coefficient starting with the coefficient of the highest power of s
3. In the 2nd row, horizontally list every other coefficient starting with the coefficient of the next highest power of s

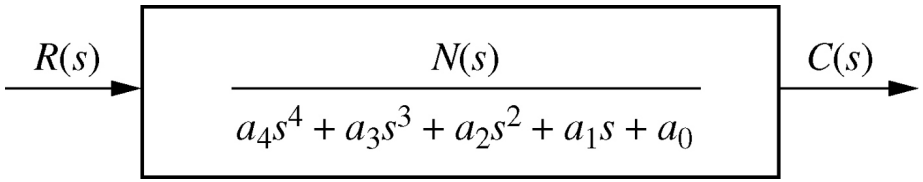


Figure : Equivalent CL TF

Table : Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2			
s^1			
s^0			

Generating a basic Routh table, [1, p. 306]

Procedure

4. Remaining row entries are filled with the negative determinant of entries in the previous 2 rows divided by entry in the 1st column directly above the calculated row. The left-hand column of the determinant is always the 1st column of the previous 2 rows, and the right-hand column is the elements of the column above and to the right.

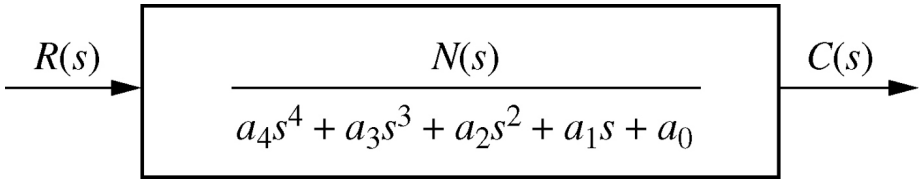


Figure : Equivalent CL TF

Table : Routh table

s^4	a_4	a_2	a_0
s^3	a_3	a_1	0
s^2	$\frac{-\begin{vmatrix} a_4 & a_2 \\ a_3 & a_1 \end{vmatrix}}{a_3} = b_1$	$\frac{-\begin{vmatrix} a_4 & a_0 \\ a_3 & 0 \end{vmatrix}}{a_3} = b_2$	$\frac{-\begin{vmatrix} a_4 & 0 \\ a_3 & 0 \end{vmatrix}}{a_3} = 0$
s^1	$\frac{-\begin{vmatrix} a_3 & a_1 \\ b_1 & b_2 \end{vmatrix}}{b_1} = c_1$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$	$\frac{-\begin{vmatrix} a_3 & 0 \\ b_1 & 0 \end{vmatrix}}{b_1} = 0$
s^0	$\frac{-\begin{vmatrix} b_1 & b_2 \\ c_1 & 0 \end{vmatrix}}{c_1} = d_1$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$	$\frac{-\begin{vmatrix} b_1 & 0 \\ c_1 & 0 \end{vmatrix}}{c_1} = 0$

Interpreting a basic Routh table, [1, p. 307]

- The number of roots of the polynomial that are in the RHP is equal to the number of signs changes in the 1st column of a Routh table
- A system is stable if there are no sign changes in the first column of the Routh table

Special cases, [1, p. 308]

2 special cases

1. Zero only in the 1st column
   - If the 1st element of a row is a zero, division by zero would be required to form the next row
2. Entire row of zeros
   - Result of there being a purely even polynomial that is a factor of the original polynomial

Zero only in the 1st column, [1, p. 308]

2 procedures

1. Epsilon procedure
   - To avoid this phenomenon, an epsilon, ϵ is assigned to replace zero in the 1st column
   - The value ϵ is then allowed to approach zero from either the positive or the negative side, after which the signs of the entries in the 1st column can be determined
2. Reciprocal roots procedure
   - A polynomial that has the reciprocal roots of the original polynomial has its roots distributed the same–RHP, LHP, or imaginary axis–because taking the reciprocal of the root value does not move it to another region
   - The polynomial that has the reciprocal roots of the original may not have a zero in the 1st column
   - Replacing s with $\frac{1}{d}$ results in the original polynomial with its coefficients written in reverse order

Entire row of zeros, [1, p. 311]

- *Purely even polynomials:* Only have roots that are symmetrical and real
  - Root positions to generate even polynomials (symmetrical about the origin)
    1. Symmetrical and real
    2. Symmetrical and imaginary
    3. Quadrantal
- Even polynomial appears in the row directly above the row of zeros

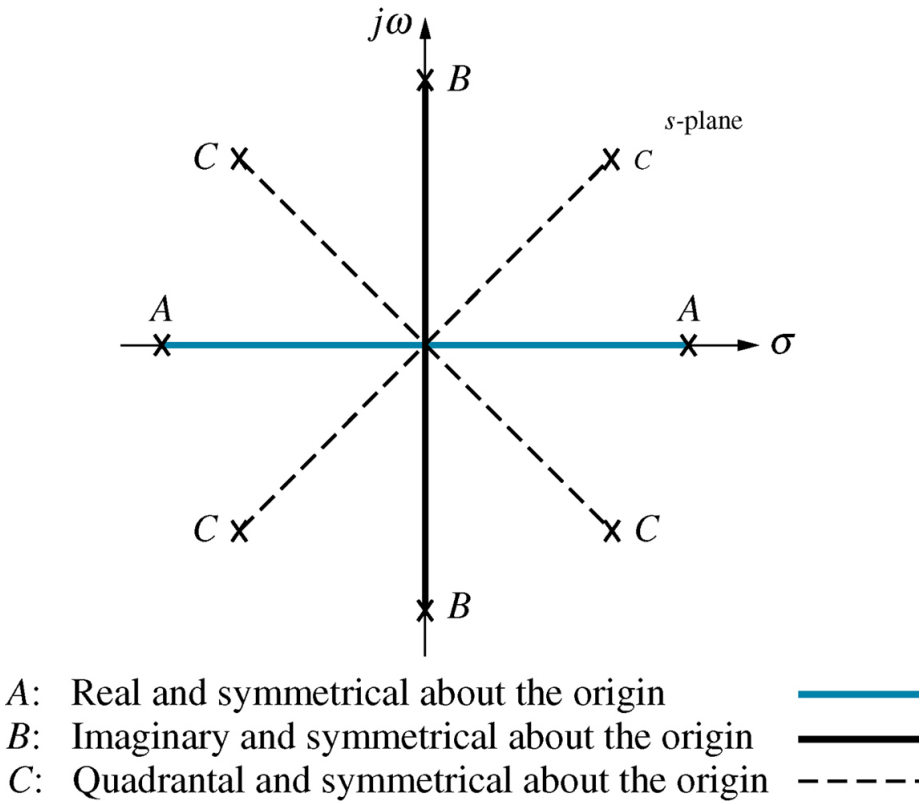


Figure : Root position to generate even polynomials: A, B, C, or any combination

Entire row of zeros, [1, p. 313]

- Every entry in the table from the even polynomial's row to the end of the chart applies only to the even polynomial
- Number of sign changes from the even polynomial to the end of the table equals the number of RHP roots of the even polynomial
- Even polynomial must have the same number of LHP roots as it does RHP roots
- Remaining poles must be on the imaginary axis
- The number of sign changes, from the beginning of the table down to the even polynomial, equals the number of RHP roots
- Remaining roots are LHP roots
- The other polynomial can contain no roots on the imaginary axis

Some examples, [1, p. 314]

Example (Standard)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

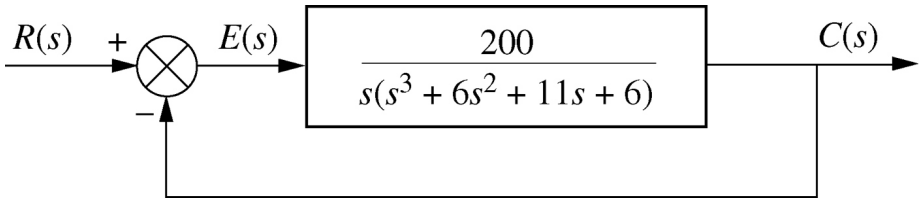


Figure : FB control system

Some examples, [1, p. 314]

Example (Standard)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
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  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

Table : Routh table

s^4		1		11	200
s^3	$\cancel{6}$	1	$\cancel{6}$	1	
s^2	$\cancel{10}$	1	$\cancel{200}$	20	
s^1		-19			
s^0		20			

$$T(s) = \frac{200}{s^4 + 6s^3 + 11s^2 + 6s + 200}$$

$$CE(s) = s^4 + 6s^3 + 11s^2 + 6s + 200$$

Some examples, [1, p. 314]

Example (Zero in 1st column)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - *Epsilon method*
  - Interpret Routh table

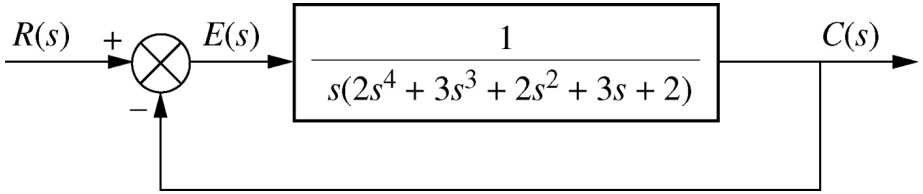

Figure : FB control system

Some examples, [1, p. 314]

Example (Zero in 1st column)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - *Epsilon method*
  - Interpret Routh table

Table : Routh table

s^5	2	2	2
s^4	3	3	1
s^3	$\cancel{0}$ ϵ	$\frac{4}{3}$	
s^2	$\frac{3\epsilon - 4}{\epsilon}$	1	
s^1	$\frac{12\epsilon - 16 - 3\epsilon^2}{9\epsilon - 12}$		
s^0	1		

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

$$CE(s) = 2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1$$

Some examples, [1, p. 314]

Example (Zero in 1st column)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - *Reciprocal root method*
  - *Epsilon method*
  - Interpret Routh table

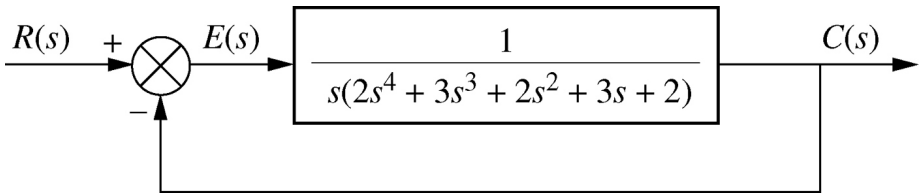

Figure : FB control system

Some examples, [1, p. 314]

Example (Zero in 1st column)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - *Reciprocal root method*
  - *Epsilon method*
  - Interpret Routh table

Table : Routh table

s^5	1	3	3
s^4	2	2	2
s^3	2	2	
s^2	$\cancel{0}\ \ \epsilon$	2	
s^1	$\frac{2\epsilon - 4}{\epsilon}$		
s^0	2		

$$T(s) = \frac{1}{2s^5 + 3s^4 + 2s^3 + 3s^2 + 2s + 1}$$

$$RCCE(s) = s^5 + 2s^4 + 3s^3 + 2s^2 + 3s + 2$$

Some examples, [1, p. 314]

Example (Row of zeros)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

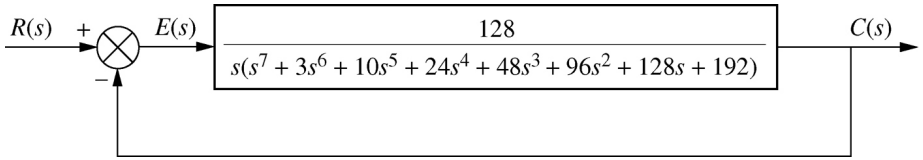

Figure : FB control system

Some examples, [1, p. 314]

Example (Row of zeros)

- *Problem:* Find the number of poles in the LHP, RHP, and on the imaginary axis
- *Solution:* On board
  - CLTF
  - Characteristic equation
  - Generate Routh table
  - Interpret Routh table

Table : Routh table

s^8			1			10			48			128		128	
s^7		$\cancel{3}$	1		$\cancel{24}$	8		$\cancel{96}$	32		$\cancel{192}$	64			
s^6		$\cancel{2}$	1		$\cancel{16}$	8		$\cancel{64}$	32		$\cancel{128}$	64			
s^5	$\cancel{0}$	$\cancel{6}$	3	$\cancel{0}$	$\cancel{32}$	16	$\cancel{0}$	$\cancel{64}$	32	$\cancel{0}$	$\cancel{0}$	0			
s^4		$\cancel{\frac{8}{3}}$	1		$\cancel{\frac{64}{3}}$	8		$\cancel{64}$	24						
s^3		$\cancel{-8}$	-1		$\cancel{-40}$	-5									
s^2		$\cancel{3}$	1		$\cancel{24}$	8									
s^1			3												
s^0			8												

$$T(s) = \frac{12}{s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128}$$

$$CE(s) = s^8 + 3s^7 + 10s^6 + 24s^5 + 48s^4 + 96s^3 + 128s^2 + 192s + 128$$

Some definitions, [1, p. 320]

Definition (eigenvalues, eigenvectors, & characteristic equation)

Eigenvalues, λ, of system matrix, A

- System poles
- Values that permit a nontrivial solution (other than 0) for *eigenvectors*, x, in the equation

$$\begin{aligned} Ax &= \lambda x \\ x &= (\lambda I - A)^{-1} 0 \\ &= \frac{\operatorname{adj}(\lambda I - A)}{\det(\lambda I - A)} 0 \end{aligned}$$

- All solutions will be the null vector except for the occurrence of zero in the denominator
- This is the only condition where elements of x will be $0/0$ or indeterminate, it is the only case where a nonzero solution is possible
- Solutions of the *characteristic equation* $\det(sI - A) = 0$, a polynomial

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.