EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 2 – Modeling in the Frequency Domain

Alexandre Bayen

Department of Electrical Engineering & Computer Science
University of California Berkeley

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Lecture abstract

Topics covered in this presentation

- Laplace transform
- Transfer function
- Conversion between systems in time-, frequency-domain, and transfer function representations
- Electrical, translational-, and rotational-mechanical systems in time-, frequency-domain, and transfer function representations
- Nonlinearities
- Linearization of nonlinear systems in time-, frequency-domain, and transfer function representations

Chapter outline

- 2 Modeling in the frequency domain
  - 2.1 Introduction
  - 2.2 Laplace transform review
  - 2.3 The transfer function
  - 2.4 Electrical network transfer functions
  - 2.5 Translational mechanical system transfer functions
  - 2.6 Rotational mechanical system transfer functions
  - 2.7 Transfer functions for systems with gears
  - 2.8 Electromechanical system transfer functions
  - 2.9 Electric circuit analogs
  - 2.10 Nonlinearities
  - 2.11 Linearization

History interlude

Pierre-Simon Laplace

- 1749 – 1827
- French mathematician and astronomer
- Pioneered the *Laplace transform*
- AKA French Newton
- "...all the effects of nature are only mathematical results of a small number of immutable laws."
- "What we know is little, and what we are ignorant of is immense."

The Laplace transform definitions, [1, p. 35]

Laplace transform

$$\mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st}dt$$

Inverse Laplace transform

$$\begin{aligned}\mathcal{L}^{-1}[F(s)] &= \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st}ds \\ &= f(t)u(t)\end{aligned}$$

where

$$s = \sigma + j\omega$$

Laplace transform table, [1, p. 36]

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
$tu(t)$	$\frac{1}{s^2}$
$t^n u(t)$	$\frac{n!}{s^{n+1}}$
$e^{-at}u(t)$	$\frac{1}{s+a}$
$\sin(\omega t)u(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)u(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin(\omega t)u(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos(\omega t)u(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$

Laplace transform theorems, [1, p. 37]

Some basic algebraic operations, such as multiplication by exponential functions or shifts have simple counterparts in the Laplace domain

Theorem (Frequency shift)

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

Theorem (Time shift)

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

Laplace transform theorems, [1, p. 37]

Theorem (Linearity)

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(s) + c_2 F_2(s)$$

Theorem (Scaling)

$$\mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

Laplace transform theorems, [1, p. 37]

Theorem (Differentiation)

$$\mathcal{L}\left[\frac{d^n f}{dt^n}\right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} f}{dt^{k-1}}(0-)$$

Examples

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0-)$$

Laplace transform theorems, [1, p. 37]

Theorem (Integration)

$$\mathcal{L}\left[\int_{0-}^{t} f(\tau)d\tau\right] = \frac{F(s)}{s}$$

Laplace transform theorems, [1, p. 37]

Theorem (Final value)

$$\mathcal{L}[f(\infty)] = \lim_{s \to 0} sF(s)$$

To yield correct finite results, all roots of the denominator of $F(s)$ must have negative real parts, and no more than one can be at the origin.

Laplace transform theorems, [1, p. 37]

Theorem (Initial value)

$$\mathcal{L}[f(0+)] = \lim_{s\to\infty} sF(s)$$

To be valid, $f(t)$ *must be continuous or have a step discontinuity at* $t = 0$*, i.e., no impulses or their derivatives at* $t = 0$*.*

Partial fraction expansion, [1, p. 37]

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term

$$F(s) = \frac{N(s)}{D(s)}$$

How $F(s)$ can be expanded is governed by the relative order between $N(s)$ and $D(s)$

1. $\mathcal{O}(N(s)) < \mathcal{O}(D(s))$
2. $\mathcal{O}(N(s)) \geq \mathcal{O}(D(s))$

and the type of roots of $D(s)$

1. Real and distinct
2. Real and repeated
3. Complex or imaginary

The transfer function, [1, p. 44]

General n-th order, linear, time-invariant differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + ... + a_0 c(t) =$$
$$b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + ... + b_0 r(t)$$

Under the assumption that *all initial conditions are zero* the transfer function (TF) from input, $c(t)$, to output, $r(t)$, i.e., the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$ is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0}$$

Also, the output transform, $C(s)$ can be written as

$$C(s) = R(s)G(s)$$

Electrical network TFs, [1, p. 47]

Table: Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

Component	Voltage-current	Current-voltage	Voltage-charge	Impedance $Z(s) = V(s)/I(s)$	Admittance $Y(s) = I(s)/V(s)$
 Capacitor	$v(t) = \frac{1}{C}\int_0^1 i(\tau)d\tau$	$i(t) = C\frac{dv(t)}{dt}$	$v(t) = \frac{1}{C}q(t)$	$\frac{1}{Cs}$	Cs
 Resistor	$v(t) = Ri(t)$	$i(t) = \frac{1}{R}v(t)$	$v(t) = R\frac{dq(t)}{dt}$	R	$\frac{1}{R} = G$
 Inductor	$v(t) = L\frac{di(t)}{dt}$	$i(t) = \frac{1}{L}\int_0^1 v(\tau)d\tau$	$v(t) = L\frac{d^2q(t)}{dt^2}$	Ls	$\frac{1}{Ls}$

Electrical network TFs, [1, p. 48]

Example (Resistor-inductor-capacitor (RLC) system)

- *Problem:* Find the TF relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$
- *Solution:* On board

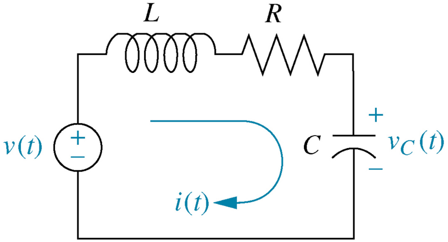


Figure: RLC system

Electrical network TFs, [1, p. 59]

Example (Inverting operational amplifier system)

- *Problem:* Find the TF relating the output voltage, $V_o(s)$, to the input voltage $V_i(s)$
- *Solution:* On board

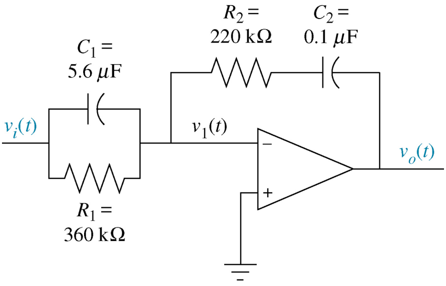
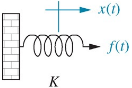
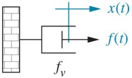
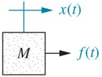


Figure: Inverting operational amplifier system

Translational mechanical system TFs, [1, p. 61]

Table: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

Component	**Force-velocity**	**Force-displacement**	**Impedence** $Z_M(s) = F(s)/X(s)$
Spring  K	$f(t) = K\int_0^t v(\tau)d\tau$	$f(t) = Kx(t)$	K
Viscous damper  f_v	$f(t) = f_v v(t)$	$f(t) = f_v \frac{dx(t)}{dt}$	$f_v s$
Mass 	$f(t) = M\frac{dv(t)}{dt}$	$f(t) = M\frac{d^2x(t)}{dt^2}$	Ms^2

Translational mechanical system TFs, [1, p. 63]

Example (Translational inertia-spring-damper system)

- *Problem:* Find the TF relating the position, $X(s)$, to the input force, $F(s)$
- *Solution:* On board

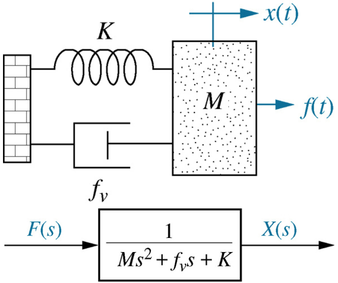
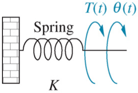
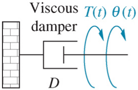
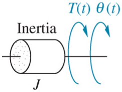


Figure: Physical system; block diagram

Rotational mechanical system TFs, [1, p. 69]

Table: Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

Component	**Torque-angular velocity**	**Torque-angular displacement**	**Impedence** $Z_M(s) = T(s)/\theta(s)$
	$T(t) = K\int_0^t \omega(\tau)d\tau$	$T(t) = K\theta(t)$	K
	$T(t) = D\omega(t)$	$T(t) = D\dfrac{d\theta(t)}{dt}$	Ds
	$T(t) = J\dfrac{d\omega(t)}{dt}$	$T(t) = J\dfrac{d^2\theta(t)}{dt^2}$	Js^2

Rotational mechanical system TFs, [1, p. 63]

Example (Rotational inertia-spring-damper system)

- *Problem:* Find the TF relating the position, $\Theta_2(s)$, to the input torque, $T(s)$
- *Solution:* On board

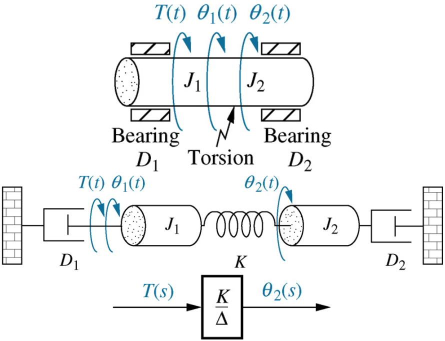


Figure: Physical system; schematic; block diagram

Nonlinearities, [1, p. 88]

Common physical nonlinearities found in nonlinear (NL) systems

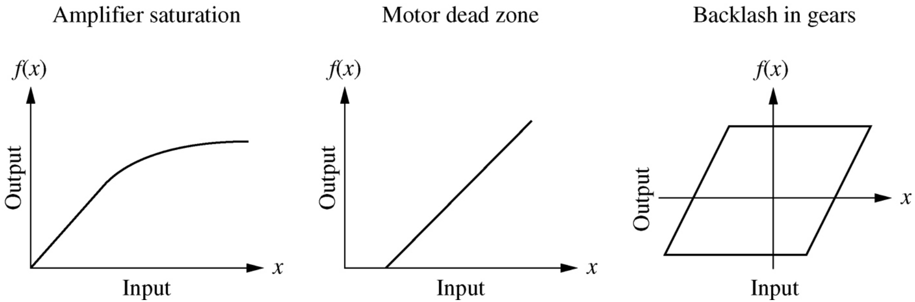


Figure: Some physical nonlinearities

Linearization, [1, p. 89]

Motivation

- Must linearize a NL system into a LTI DE before we can find a TF

Linearization procedure

1. Recognize the NL component and write the NL DE
2. Linearize the NL DE into an LTI DE
3. Laplace transform of LTI DE assuming zero initial conditions
4. Separate input and output variables
5. Form the TF

Linearization, [1, p. 89]

1^{st}-order linearization

- Output, $f(x)$
- Input, x
- Operating at point A, $[x_0, f(x_0)]$
- Small changes in the input can be related to changes in the output about the point by way of the slope of the curve, m_a, at point A

$$
\begin{aligned}
[f(x) - f(x_0)] &\approx m_a(x - x_0) \\
\delta f(x) &\approx m_a \delta x \\
f(x) &\approx f(x_0) + m_a \delta x
\end{aligned}
$$

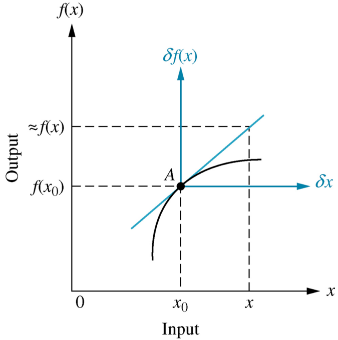


Figure: Linearization about point A

Linearization, [1, p. 89]

General linearization via Taylor series expansion

$$f(x) = f(x_0) + \frac{df}{dx}|_{x=x_0} \frac{(x - x_0)}{1!} + \frac{d^2 f}{dx^2}|_{x=x_0} \frac{(x - x_0)^2}{2!} + \dots$$

For small excursions of x from x_0, we can neglect higher-order terms. The resulting approximation yields a straight-line relationship between the change in $f(x)$ and the excursion away from x_0. Neglecting higher-order terms yields

$$f(x) - f(x_0) \approx \frac{df}{dx}|_{x=x_0}(x - x_0) \quad \text{or} \quad \bigtriangledown f = \frac{\delta f}{\delta x} \approx m|_{x=x_0}$$

which is a linear relationship between $\delta f(x)$ and δx for small excursions away from x_0.

Linearization, [1, p. 92]

Example (NL electrical system)

- *Problem:* Find the TF relating the inductor voltage, $V_L(s)$, to the input voltage, $V(s)$. The NL resistor voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where i_r and v_r are the resistor current and voltage, respectively. Also the input voltage, v, is a small-signal source.
- *Solution:* On board

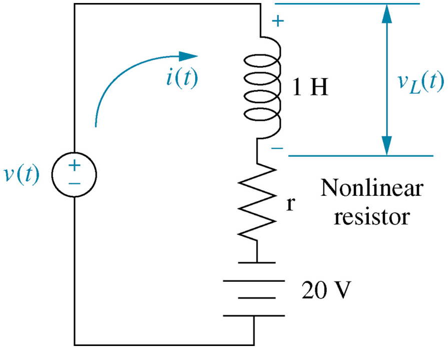


Figure: NL electrical system

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.