EE C128 / ME C134 – Feedback Control Systems
Lecture – Chapter 2 – Modeling in the Frequency Domain

Alexandre Bayen

Department of Electrical Engineering & Computer Science
University of California Berkeley

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Lecture abstract

Topics covered in this presentation

- Laplace transform
- Transfer function
- Conversion between systems in time-, frequency-domain, and transfer function representations
- Electrical, translational-, and rotational-mechanical systems in time-, frequency-domain, and transfer function representations
- Nonlinearities
- Linearization of nonlinear systems in time-, frequency-domain, and transfer function representations
Chapter outline

2 Modeling in the frequency domain
   2.1 Introduction
   2.2 Laplace transform review
   2.3 The transfer function
   2.4 Electrical network transfer functions
   2.5 Translational mechanical system transfer functions
   2.6 Rotational mechanical system transfer functions
   2.7 Transfer functions for systems with gears
   2.8 Electromechanical system transfer functions
   2.9 Electric circuit analogs
   2.10 Nonlinearities
   2.11 Linearization
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History interlude

Pierre-Simon Laplace

- 1749 – 1827
- French mathematician and astronomer
- Pioneered the *Laplace transform*
- AKA French Newton
- “…all the effects of nature are only mathematical results of a small number of immutable laws.”
- “What we know is little, and what we are ignorant of is immense.”
The Laplace transform definitions, [1, p. 35]

**Laplace transform**

\[ \mathcal{L}[f(t)] = F(s) = \int_{0-}^{\infty} f(t)e^{-st} \, dt \]

**Inverse Laplace transform**

\[ \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(s)e^{st} \, ds \]

\[ = f(t)u(t) \]

where

\[ s = \sigma + j\omega \]
### Laplace transform table, [1, p. 36]

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
</tr>
<tr>
<td>$t^n u(t)$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
</tr>
<tr>
<td>$e^{-at} u(t)$</td>
<td>$\frac{1}{s+a}$</td>
</tr>
<tr>
<td>$\sin(\omega t) u(t)$</td>
<td>$\frac{\omega}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$\cos(\omega t) u(t)$</td>
<td>$\frac{s}{s^2 + \omega^2}$</td>
</tr>
<tr>
<td>$e^{-at} \sin(\omega t) u(t)$</td>
<td>$\frac{\omega}{(s+a)^2 + \omega^2}$</td>
</tr>
<tr>
<td>$e^{-at} \cos(\omega t) u(t)$</td>
<td>$\frac{s+a}{(s+a)^2 + \omega^2}$</td>
</tr>
</tbody>
</table>
Laplace transform theorems, [1, p. 37]

Some basic algebraic operations, such as multiplication by exponential functions or shifts have simple counterparts in the Laplace domain.

**Theorem (Frequency shift)**

\[ \mathcal{L}[e^{-at}f(t)] = F(s + a) \]

**Theorem (Time shift)**

\[ \mathcal{L}[f(t - T)] = e^{-sT}F(s) \]
Laplace transform theorems, [1, p. 37]

**Theorem (Linearity)**

\[ \mathcal{L}[c_1f_1(t) + c_2f_2(t)] = c_1F_1(s) + c_2F_2(s) \]

**Theorem (Scaling)**

\[ \mathcal{L}[f(at)] = \frac{1}{a} F \left( \frac{s}{a} \right) \]
Laplace transform theorems, [1, p. 37]

**Theorem (Differentiation)**

\[ \mathcal{L} \left[ \frac{d^n f}{dt^n} \right] = s^n F(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1}f}{dt^{k-1}}(0-) \]

**Examples**

\[ \mathcal{L} \left[ \frac{df}{dt} \right] = sF(s) - f(0-) \]
Laplace transform theorems, [1, p. 37]

**Theorem (Integration)**

\[ \mathcal{L} \left[ \int_{0^-}^{t} f(\tau) d\tau \right] = \frac{F(s)}{s} \]
Laplace transform theorems, [1, p. 37]

Theorem (Final value)

\[ \mathcal{L}[f(\infty)] = \lim_{s \to 0} s F(s) \]

To yield correct finite results, all roots of the denominator of \( F(s) \) must have negative real parts, and no more than one can be at the origin.
Laplace transform theorems, [1, p. 37]

**Theorem (Initial value)**

\[ \mathcal{L}[f(0+)] = \lim_{s \to \infty} sF(s) \]

To be valid, \( f(t) \) must be continuous or have a step discontinuity at \( t = 0 \), i.e., no impulses or their derivatives at \( t = 0 \).
Partial fraction expansion, [1, p. 37]

To find the inverse Laplace transform of a complicated function, we can convert the function to a sum of simpler terms for which we know the Laplace transform of each term

\[ F(s) = \frac{N(s)}{D(s)} \]

How \( F(s) \) can be expanded is governed by the relative order between \( N(s) \) and \( D(s) \)

1. \( \mathcal{O}(N(s)) < \mathcal{O}(D(s)) \)
2. \( \mathcal{O}(N(s)) \geq \mathcal{O}(D(s)) \)

and the type of roots of \( D(s) \)

1. Real and distinct
2. Real and repeated
3. Complex or imaginary
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The transfer function, [1, p. 44]

General $n$-th order, linear, time-invariant differential equation

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \ldots + a_0 c(t) =$$

$$b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \ldots + b_0 r(t)$$

Under the assumption that all initial conditions are zero the transfer function (TF) from input, $c(t)$, to output, $r(t)$, i.e., the ratio of the output transform, $C(s)$, divided by the input transform, $R(s)$ is given by

$$G(s) = \frac{C(s)}{R(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_0}$$

Also, the output transform, $C(s)$ can be written as

$$C(s) = R(s)G(s)$$
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### Table: Voltage-current, voltage-charge, and impedance relationships for capacitors, resistors, and inductors

<table>
<thead>
<tr>
<th>Component</th>
<th>Voltage-current</th>
<th>Current-voltage</th>
<th>Voltage-charge</th>
<th>Impedance $Z(s) = V(s)/I(s)$</th>
<th>Admittance $Y(s) = I(s)/V(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Capacitor</td>
<td>$v(t) = \frac{1}{C} \int_0^1 i(\tau)d\tau$</td>
<td>$i(t) = C \frac{dv(t)}{dt}$</td>
<td>$v(t) = \frac{1}{C} q(t)$</td>
<td>$\frac{1}{Cs}$</td>
<td>$Cs$</td>
</tr>
<tr>
<td>Resistor</td>
<td>$v(t) = Ri(t)$</td>
<td>$i(t) = \frac{1}{R} v(t)$</td>
<td>$v(t) = R \frac{dq(t)}{dt}$</td>
<td>$R$</td>
<td>$\frac{1}{R} = G$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$v(t) = L \frac{di(t)}{dt}$</td>
<td>$i(t) = \frac{1}{L} \int_0^1 v(\tau)d\tau$</td>
<td>$v(t) = L \frac{d^2q(t)}{dt^2}$</td>
<td>$Ls$</td>
<td>$\frac{1}{Ls}$</td>
</tr>
</tbody>
</table>
Example (Resistor-inductor-capacitor (RLC) system)

- **Problem:** Find the TF relating the capacitor voltage, $V_C(s)$, to the input voltage, $V(s)$
- **Solution:** On board
Example (Inverting operational amplifier system)

- **Problem**: Find the TF relating the output voltage, \( V_o(s) \), to the input voltage \( V_i(s) \)

- **Solution**: On board

**Figure**: Inverting operational amplifier system

\[ C_1 = 5.6 \, \mu F \]
\[ R_1 = 360 \, k\Omega \]
\[ R_2 = 220 \, k\Omega \]
\[ C_2 = 0.1 \, \mu F \]
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Translational mechanical system TFs, [1, p. 61]

Table: Force-velocity, force-displacement, and impedance translational relationships for springs, viscous dampers, and mass

<table>
<thead>
<tr>
<th>Component</th>
<th>Force-velocity</th>
<th>Force-displacement</th>
<th>Impedance $Z_M(s) = F(s)/X(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>$f(t) = K \int_0^t v(\tau)d\tau$</td>
<td>$f(t) = Kx(t)$</td>
<td>$K$</td>
</tr>
<tr>
<td>Viscous damper</td>
<td>$f(t) = f_v v(t)$</td>
<td>$f(t) = f_v \frac{dx(t)}{dt}$</td>
<td>$f_v s$</td>
</tr>
<tr>
<td>Mass</td>
<td>$f(t) = M \frac{dv(t)}{dt}$</td>
<td>$f(t) = M \frac{d^2x(t)}{dt^2}$</td>
<td>$Ms^2$</td>
</tr>
</tbody>
</table>
Example (Translational inertia-spring-damper system)

- **Problem**: Find the TF relating the position, $X(s)$, to the input force, $F(s)$
- **Solution**: On board

**Figure**: Physical system; block diagram
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Table: Torque-angular velocity, torque-angular displacement, and impedance rotational relationships for springs, viscous dampers, and inertia

<table>
<thead>
<tr>
<th>Component</th>
<th>Torque-angular velocity</th>
<th>Torque-angular displacement</th>
<th>Impedance $Z_M(s) = T(s)/\theta(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spring</td>
<td>$T(t) = K \int_0^t \omega(\tau)d\tau$</td>
<td>$T(t) = K\theta(t)$</td>
<td>$K$</td>
</tr>
<tr>
<td>Viscous damper</td>
<td>$T(t) = D\omega(t)$</td>
<td>$T(t) = D\frac{d\theta(t)}{dt}$</td>
<td>$Ds$</td>
</tr>
<tr>
<td>Inertia</td>
<td>$T(t) = J\frac{d\omega(t)}{dt}$</td>
<td>$T(t) = J\frac{d^2\theta(t)}{dt^2}$</td>
<td>$Js^2$</td>
</tr>
</tbody>
</table>

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Rotational mechanical system TFs, [1, p. 63]

Example (Rotational inertia-spring-damper system)

- **Problem**: Find the TF relating the position, $\Theta_2(s)$, to the input torque, $T(s)$
- **Solution**: On board

**Figure**: Physical system; schematic; block diagram
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Nonlinearities, [1, p. 88]

Common physical nonlinearities found in nonlinear (NL) systems

- Amplifier saturation
- Motor dead zone
- Backlash in gears

*Figure:* Some physical nonlinearities
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Linearization, [1, p. 89]

**Motivation**

- Must linearize a NL system into a LTI DE before we can find a TF

**Linearization procedure**

1. Recognize the NL component and write the NL DE
2. Linearize the NL DE into an LTI DE
3. Laplace transform of LTI DE assuming zero initial conditions
4. Separate input and output variables
5. Form the TF
Linearization, [1, p. 89]

1st-order linearization

- Output, $f(x)$
- Input, $x$
- Operating at point $A$, $[x_0, f(x_0)]$
- Small changes in the input can be related to changes in the output about the point by way of the slope of the curve, $m_a$, at point $A$

$$[f(x) - f(x_0)] \approx m_a(x - x_0)$$
$$\delta f(x) \approx m_a \delta x$$
$$f(x) \approx f(x_0) + m_a \delta x$$

Figure: Linearization about point $A$
Linearization, [1, p. 89]

**General linearization via Taylor series expansion**

\[ f(x) = f(x_0) + \frac{df}{dx}igg|_{x=x_0} \frac{(x-x_0)}{1!} + \frac{d^2f}{dx^2}igg|_{x=x_0} \frac{(x-x_0)^2}{2!} + \ldots \]

For small excursions of \( x \) from \( x_0 \), we can neglect higher-order terms. The resulting approximation yields a straight-line relationship between the change in \( f(x) \) and the excursion away from \( x_0 \). Neglecting higher-order terms yields

\[ f(x) - f(x_0) \approx \frac{df}{dx}igg|_{x=x_0} (x-x_0) \quad \text{or} \quad \nabla f = \frac{\delta f}{\delta x} \approx m \bigg|_{x=x_0} \]

which is a linear relationship between \( \delta f(x) \) and \( \delta x \) for small excursions away from \( x_0 \).
Example (NL electrical system)

Problem: Find the TF relating the inductor voltage, $V_L(s)$, to the input voltage, $V(s)$. The NL resistor voltage-current relationship is defined by $i_r = 2e^{0.1v_r}$, where $i_r$ and $v_r$ are the resistor current and voltage, respectively. Also the input voltage, $v$, is a small-signal source.

Solution: On board
Bibliography