Topics covered in this presentation

▶ Important considerations in CT-to-DT conversion that yield errors
▶ $z$-transform & its inverse
▶ 2 CT-to-DT conversion methods
▶ DT region of stability
▶ DT RL
▶ DT TR characteristics
▶ Designing DT compensators in CT
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- 13.2 Modeling the digital computer
- 13.3 The $z$-transform
- 13.4 Transfer functions
- 13.5 Block diagram reduction
- 13.6 Stability
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13.1 Intro

Intro, [1, p. 724]

Concept

- Only frequency-domain analysis & design
- Not state-space techniques
- TFs built with analog components → digital computer that performs calculations that emulate the physical compensator

Figure: a. analog; b. digital
Definitions, [1, p. 724]

- **Analog**: CT, dynamic variables retain a particular value for only an infinitesimally short amount of time

- **Digital**: DT, dynamic variables evolve in between computer measurements of outputs & control inputs, but the computer remain unchanged throughout each non-zero period of sampling time
Digital control implementation, [1, p. 725]

**Concept**

- **Digital computer**
  - Control of multiple loops at the same time
  - Signals are sampled at specified intervals & held
    - $\Delta$system performance $\propto \Delta \omega_{sample}$
- **A/D converter**: Measured outputs sampler
  - 2-step process
    1. Analog signal $\rightarrow$ sampled signal
    2. Sampled signal $\rightarrow$ sequence of binary numbers
  - Not instantaneous, i.e., there is a delay
  - $\omega_{sample} > \omega_{Nyquist} = 2\omega_{BW}$
- **D/A converter**: Control inputs zero-order hold
  - Instantaneous

**Figure**: Digital computer with A/D & D/A converters
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Modeling the sampler, [1, p. 728]

Concept

- **z-transform**: Laplace transform replacement for sampled signals
- Sampled waveform

\[ f_{T_W}^*(t) = f(t) s(t) \]

\[ = f(t) \sum_{k=-\infty}^{\infty} [u(t - kT) - u(t - kT - T_W)] \]

- Integer, \( k \in [-\infty, \infty] \)
- Period of pulse train, \( T \)
- Period of pulse width, \( T_W \)

**Figure**: Views of uniform-rate sampling
Modeling the sampler, [1, p. 729]

**Concept**

- ...simplification & Laplace transform...
- Sampled waveform portion not dependent upon the sampling waveform characteristics

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(kT) \delta(t - kT)$$

**Figure:** Modeling of sampling with a uniform rectangular pulse train
Concept

- **Zero-order hold (ZOH):** Hold the last sampled value of $f(t)$ until the next sample
  - Staircase approximation to $f(t)$
  - Sequence of step functions whose amplitude is $f(t)$ at the sampling instant, or $f(kT)$
  - TF of the step that starts at $t = 0$ & ends at $t = T$

\[
G_h(s) = \frac{1 - e^{-Ts}}{s}
\]

**Figure:** Ideal sampling & ZOH
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**Intro, [1, p. 730]**

**Concept**

- Stability & TR of a sampled-data system depend upon sampling rate
- Laplace transform of the sampled time waveform

\[
F^*(s) = \sum_{k=0}^{\infty} f(kT) e^{-kTs}
\]

- Letting

\[
z = e^{Ts}
\]

- \(z\)-transform

\[
F(z) = \sum_{k=0}^{\infty} f(kT) z^{-k}
\]

\[f(kT) \leftrightarrow F(z)\]
### $z$- & $s$-transforms, [1, p. 732]

<table>
<thead>
<tr>
<th>$f(t)$</th>
<th>$F(s)$</th>
<th>$F(z)$</th>
<th>$f(kT)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\delta(t)$</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$u(t)$</td>
<td>$\frac{1}{s}$</td>
<td>$\frac{z}{z-1}$</td>
<td>$u(kT)$</td>
</tr>
<tr>
<td>$tu(t)$</td>
<td>$\frac{1}{s^2}$</td>
<td>$\frac{Tz}{(z-1)^2}$</td>
<td>$kT$</td>
</tr>
<tr>
<td>$t^n u(t)$</td>
<td>$\frac{n!}{s^{n+1}}$</td>
<td>$\lim_{a \to 0} (-1)^n \frac{d^n}{da^n} \left[ \frac{z}{z-e^{-aT}} \right]$</td>
<td>$(kT)^n$</td>
</tr>
<tr>
<td>$e^{-at} u(t)$</td>
<td>$\frac{1}{s+a}$</td>
<td>$\frac{z}{z-e^{-aT}}$</td>
<td>$e^{-akT}$</td>
</tr>
<tr>
<td>$t^n e^{-at} u(t)$</td>
<td>$( -1 )^n \frac{d^n}{da^n} \left[ \frac{z}{z-e^{-aT}} \right]$</td>
<td>$(kT)^n e^{-akT}$</td>
<td></td>
</tr>
<tr>
<td>$\sin(\omega t) u(t)$</td>
<td>$\frac{\omega}{s^2+\omega^2}$</td>
<td>$\frac{z \sin(\omega T)}{z^2-2z \cos(\omega T)+1}$</td>
<td>$\sin(\omega kT)$</td>
</tr>
<tr>
<td>$\cos(\omega t) u(t)$</td>
<td>$\frac{s}{s^2+\omega^2}$</td>
<td>$\frac{z(\cos(\omega T))}{z^2-2z \cos(\omega T)+1}$</td>
<td>$\cos(\omega kT)$</td>
</tr>
<tr>
<td>$e^{-at} \sin(\omega t) u(t)$</td>
<td>$\frac{\omega}{(s+a)^2+\omega^2}$</td>
<td>$\frac{ze^{-aT} \sin(\omega T)}{z^2-2ze^{-aT} \cos(\omega T)+e^{-2aT}}$</td>
<td>$e^{-akT} \sin(\omega kT)$</td>
</tr>
<tr>
<td>$e^{-at} \cos(\omega t) u(t)$</td>
<td>$\frac{s+a}{(s+a)^2+\omega^2}$</td>
<td>$\frac{z^2 ze^{-aT} \cos(\omega T)}{z^2-2ze^{-aT} \cos(\omega T)+e^{-2aT}}$</td>
<td>$e^{-akT} \cos(\omega kT)$</td>
</tr>
</tbody>
</table>
### z-transform theorems, [1, p. 733]

<table>
<thead>
<tr>
<th>Name</th>
<th>Theorem</th>
</tr>
</thead>
<tbody>
<tr>
<td>Linearity</td>
<td>( \mathcal{Z}[c_1 f_1(t) + c_2 f_2(t)] = c_1 F_1(z) + c_2 F_2(z) )</td>
</tr>
<tr>
<td>Frequency shift</td>
<td>( \mathcal{Z}[e^{-aT} f(t)] = F(e^{aT} z) )</td>
</tr>
<tr>
<td>Time shift</td>
<td>( \mathcal{Z}[f(t - nT)] = z^{-n} F(z) )</td>
</tr>
<tr>
<td>Complex scale</td>
<td>( \mathcal{Z}[c^{-k} f(t)] = F(c z) ) where ( c \in \mathbb{C} )</td>
</tr>
<tr>
<td>Complex differentiation</td>
<td>( \mathcal{Z}[tf(t)] = -T z \frac{dF(z)}{dz} )</td>
</tr>
<tr>
<td>Real convolution</td>
<td>( \mathcal{Z} \left[ \sum_{k=-\infty}^{\infty} f_1(kT) f_2(nT - kT) \right] = F_1(z) F_2(z) )</td>
</tr>
<tr>
<td>Initial value theorem</td>
<td>( f(0) = \lim_{z \to \infty} F(z) )</td>
</tr>
<tr>
<td>Final value theorem</td>
<td>( f(\infty) = \lim_{z \to 1} (z - 1) F(z) )</td>
</tr>
</tbody>
</table>

**Note:** \( t \) may be substituted for \( kT \) in the table.
Definitions, [1, p. 733]

- **Inverse $z$-transform**: Sampled time function from its $z$-transform
  - Only yields the values of the time function at the sampling instants
  - Results in closed-form time functions that are *only valid at sampling instants*
  - 2 approaches
    - Partial-fraction expansion PFE
    - Power series
Approach – PFE, [1, p. 733]

Procedure

1. Sampled exponential time functions are related to their $z$-transforms

   \[ e^{-akT} \leftrightarrow \frac{z}{z - e^{-aT}} \]

2. Predict that a PFE should be of the following form

   \[ F(z) = \frac{A}{z - z_1} + \frac{B}{z - z_2} + \ldots \]

3. PFE of $F(s)$ did not contain terms with $s$ in numerator of partial fractions

4. Form $\frac{F(z)}{z}$ to eliminate $z$ terms in numerator

5. Perform a PFE of $\frac{F(z)}{z}$

6. Multiply the result by $z$ to replace the $zs$ in the numerator
Example (Inverse $z$-transform via PFE)

- **Problem**: Find the sampled time function

$$F(z) = \frac{0.5z}{(z - 0.5)(z - 0.7)}$$

- **Solution**: On the board
Approach – power series, [1, p. 734]

Procedure

- Values of the sampled time function found directly from $F(z)$
- Does not yield closed-form expressions for $f(kT)$

1. Indicated division yields a power series for $F(z)$
2. Transform power series for $F(z)$ into $F^*(s)$ and $f^*(t)$
Example (Inverse $z$-transform via power series)

Problem: Find the sampled time function

$$F(z) = \frac{0.5z}{(z - 0.5)(z - 0.7)}$$

Solution: On the board
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TFs of sampled-data systems, [1, p. 736]

**Concept**
- The output is conceptually sampled in synchronization with the input by a phantom sampler.

**Figure:** Sampled-data systems: a. CT; b. sampled input; c. input & output.

- **Note:** Phantom sampler is shown in color.
**Pulse TF, [1, p. 736]**

**Concept**

- Sampled input is a sum of impulses

\[
r^*(t) = \sum_{n=0}^{\infty} r(nT) \delta(t - nT)
\]

- Output

\[
c(t) = \sum_{n=0}^{\infty} r(nT) g(t - nT)
\]

- Sampled output

\[
C(z) = \sum_{k=0}^{\infty} c(kT) z^{-k}
\]

...substitution...

- \( t = kT \) and \( m = k - n \)

\[
C(z) = G(z) R(z) = \sum_{m=0}^{\infty} g(mT) z^{-m} \sum_{n=0}^{\infty} r(nT) z^{-n}
\]
Example (Converting $G(s)$ in cascade with ZOH to $G(z)$)

- **Problem**: Given a ZOH in cascade with the OL TF, $G(s)$, find the sampled-data TF, $G(z)$, if the sampling time, $T = 0.5$ seconds

\[
\text{ZOH} = \frac{1 - e^{-Ts}}{s}, \quad G(s) = \frac{s + 2}{s + 1}
\]

- **Solution**: On the board
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Methodology, [1, p. 739]

Procedure

- Find the CL sampled-data TF of an arrangement of subsystems
- Be careful!
  - E.g. $z[G_1(s)G_2(s)] \neq G_1(z)G_2(z)$

1. Multiply $s$-domain functions before taking $z$-transform
2. Place a phantom sampler at the output of any subsystem that has a sampled input
   - Justification is that the output of a sampled-data system can only be found at the sampling instants, and the signal is not an input to any other block
3. Add phantom samplers at the input to summing junctions whose outputs are sampled
   - Justification is that the sampled sum is equivalent to the sum of the sampled inputs, and that all samples are synchronized
4. Use block diagram manipulations to yield isolated TFs with input and output samplers
Figure: Sampled-data systems and their $z$-transforms
Example (Pulse TF of a FB system)

- **Problem:** Find the $z$-transform

- **Solution:** On the board

**Figure:** Sampled-data system
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Digital system stability via the $z$-plane, [1, p. 742]

**Concept**

- Sampling rate changes TR & stability
- Relate $s$-plane stability to $z$-plane stability
- Substitution of $z = e^{Ts}$ & $s = \alpha + j\omega$

$$
z = e^{T(\alpha+j\omega)} = e^{\alpha T} e^{j\omega T} = e^{\alpha T}(\cos(\omega T) + j \sin(\omega T)) = e^{\alpha T} \angle \omega T
$$

- $s$-plane regions $\rightarrow$ $z$-plane regions
  - RHP $\rightarrow$ region outside unit circle
  - $j\omega$-axis $\rightarrow$ unit circle
  - LHP $\rightarrow$ region inside unit circle

**Figure:** Mapping regions of $s$-plane onto $z$-plane
Digital system stability via the $z$-plane, [1, p. 743]

**Concept**

- Digital control system is
  - **Stable:** All CL poles are inside the unit circle
  - **Unstable:** Any pole is outside the unit circle and/or there are poles of multiplicity greater than 1 on the unit circle
  - **Marginally stable:** Poles of multiplicity 1 are on the unit circle and all other poles are inside the unit circle

- Tabular methods for determining stability, e.g., Routh-Hurwitz stability criterion, exist for sampled-data system
  - Raible’s tabular method
  - Jury’s stability test
  - Bilinear transformations $\rightarrow$ Routh-Hurwitz stability criterion
Example (Range of $T$ for stability)

- **Problem:** Determine the ranges of the sampling interval, $T$, that will make the system stable and unstable

**Figure:** Digital system

- **Solution:** On the board
Bilinear transformations, [1, p. 746]

Concept

- **Exact transformations:**

  \[ z = e^{Ts} \quad \text{and} \quad s = \frac{\ln(z)}{T} \]

- **Bilinear transformations:** Mappings from the complex plane where one point, \( s \), is mapped into another point, \( z \), of the form

  \[ z = \frac{as + b}{cs + d} \quad \text{and} \quad s = \frac{-dz + b}{cz - a} \]

- Allow application of \( s \)-plane analysis & design to digital systems
- Yield linear arguments when transforming in both directions through direct substitution and without the complicated \( z \)-transform
- Different values of \( a, b, c, \) & \( d \) have been derived for particular applications & yield various degrees of accuracy when comparing properties of continuous & sampled functions
Digital system stability via the $s$-plane, [1, p. 747]

**Concept**

- **Stability bilinear transformations**: Used to obtain stability information about the digital system by working in $s$-plane. The resulting TR of CT system, $G(s)$, is not same as that of DT system, $G(z)$.

\[
s = \frac{z + 1}{z - 1} \quad \text{and} \quad z = \frac{s + 1}{s - 1}
\]

- $s$-plane regions $\rightarrow$ $z$-plane regions
  - $j\omega$-axis $\rightarrow$ points on unit-circle
  - RHP $\rightarrow$ points outside unit circle
  - LHP $\rightarrow$ points inside unit circle

- Transforms the denominator of the pulsed TF, $D(z)$, to the denominator of a CT TF, $D(s)$, allowing the use of Routh-Hurwitz stability criterion.
Example (Stability via Routh-Hurwitz stability criterion)

- **Problem:** Given the CL CE, $D(z)$, i.e. the denominator of the CL TF, $T(z)$, use Routh-Hurwitz stability criterion to find the number of $z$-plane poles $T(z)$ inside, outside, and on the unit circle. Is the system stable?

$$D(z) = z^3 - z^2 - 0.2z + 0.1$$

- **Solution:** On the board
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Concept

- For digital systems, the placement of the sampler changes the OL TF. Assume the typical placement of the sampler after the error and in the position of the cascade controller.

- Sampled error

\[
E^*(s) = E(z) = \frac{R(z)}{1 + G(z)}
\]

- Final value theorem for discrete signals

\[
e^*(\infty) = \lim_{z \to 1} \left( \frac{z-1}{z} \right) E(z) = \lim_{z \to 1} \left( \frac{z-1}{z} \right) \frac{R(z)}{1 + G(z)}
\]
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13.7 Steady-state error

**Intro, [1, p. 750]**

**Figure:** a. Digital FB control system for evaluation of steady-state errors; b. phantom samplers added

**Figure:** c. pushing $G(s)$ and its samplers to the right past the pickoff point; d. z-transform equivalent system

Note: Phantom samplers are shown in color.
### Common inputs, [1, p. 750]

**Table: Sampled steady-state error**

<table>
<thead>
<tr>
<th>Input</th>
<th>$R(s)$</th>
<th>$R(z)$</th>
<th>Static error constant</th>
<th>$e^*(\infty)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Unit step</td>
<td>$\frac{1}{s}$</td>
<td>$\frac{z}{z-1}$</td>
<td>$K_p = \lim_{z \to 1} G(z)$</td>
<td>$\frac{1}{1+K_p}$</td>
</tr>
<tr>
<td>Unit ramp</td>
<td>$\frac{1}{s^2}$</td>
<td>$\frac{Tz}{(z-1)^2}$</td>
<td>$K_v = \frac{1}{T} \lim_{z \to 1} (z - 1)G(z)$</td>
<td>$\frac{1}{K_v}$</td>
</tr>
<tr>
<td>Unit parabolic</td>
<td>$\frac{2}{s^3}$</td>
<td>$\frac{T^2 z(z+1)}{2(z-1)^3}$</td>
<td>$K_a = \frac{1}{T^2} \lim_{z \to 1} (z - 1)^2G(z)$</td>
<td>$\frac{1}{K_a}$</td>
</tr>
</tbody>
</table>

Note: Multiple pole placement at $z = 1$ reduces the steady-state error to zero.
Example (Finding steady-state error)

- **Problem:** Find the steady-state error for step, ramp, and parabolic inputs

\[ G(s) = \frac{10}{s(s + 1)} \]

- **Solution:** On the board
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Visual interpretation of the $z$-plane, [1, p. 753]

$s$-plane $\rightarrow$ $z$-plane

- **Constant $T_s$**
  - Constant real part, $\sigma = -\frac{4}{T_s}$
  - Vert. lines $\rightarrow$ conc. circles
  - $s = \sigma + j\omega \rightarrow z = re^{j\omega T}$

- **Constant $T_p$**
  - Constant im. part, $\omega = \frac{\pi}{T_p}$
  - Hori. lines $\rightarrow$ radial lines
  - $s = \sigma + j\omega \rightarrow z = e^{\sigma T}e^{j\theta}$

- **Constant $\%OS$**
  - Constant $\zeta$, $\frac{\sigma}{\omega} = -\tan(\sin^{-1}(\zeta)) = -\frac{\zeta}{\sqrt{1-\zeta^2}}$
  - Radial lines $\rightarrow$ spiral lines
  - $s = \sigma + j\omega \rightarrow$
  - $z = e^{-\omega T\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)} \angle \omega T$

**Figure:** Constant $\zeta$, normalized $T_s$, & normalized $T_p$ plots on the $z$-plane
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Digital system RL, [1, p. 755]

**Concept**

- Plot RL & determine gain for stability & TR requirement
- Same CT RL rules
- Stability divided by unit circle rather than imaginary axis
- Superimpose TR curves on $z$-plane
- Same drawback with CT RL, limited to simple gain adjustment to accomplish design objective
- Limitations solved with compensation

![Generic digital FB control system](image)
Example (Stability design via RL)

- **Problem:** Sketch the RL and determine the range of gain, $K$, for stability from the RL plot

\[
\begin{align*}
R(z) & \quad + \quad K(z + 1) \quad \frac{1}{(z - 1)(z - 0.5)} \quad C(z) \\
& \quad - \\
\end{align*}
\]

**Figure:** Digital FB control

- **Solution:** On the board
Example, [1, p. 756]

**Example (Stability design via RL)**

- **Problem:** Find the value of gain, $K$, to yield
  - $\zeta = 0.7$

![Digital FB control diagram](image)

**Figure:** Digital FB control

- **Solution:** On the board
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Intro, [1, p. 758]

Concept

- No design directly in the $z$-domain
  1. Design on the $s$-plane
  2. $s$-plane design $\rightarrow$ digital implementation, i.e., bilinear transformation
  3. Apply cascade compensator

- **Tustin transformation**: Bilinear transformation that can be performed with hand calculations & yields a DT TF whose output response at sampling instants is approximately same as equivalent CT TF

\[
s = \frac{2(z - 1)}{T(z + 1)} \quad \text{and} \quad z = -\frac{s + \frac{2}{T}}{s - \frac{2}{T}} = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s}
\]

- $\downarrow T \rightarrow$ DT compensator $\approx$ CT compensator
- $\uparrow T \rightarrow$ discrepancy higher frequencies
Selecting the sampling interval, $T$

- *Astrom & Wittenmark, 1984*

\[
T \approx \frac{0.15}{\omega_{\Phi M}} \text{ to } \frac{0.5}{\omega_{\Phi M}}
\]

- Where $\omega_{\Phi M}$ is the 0 dB frequency (rad/s) of the magnitude frequency response curve for the cascaded CT compensator & plant

- *Rule of thumb*

\[
T \approx \frac{10}{\omega_{BW}} \text{ to } \frac{20}{\omega_{BW}}
\]

- Where $\omega_{BW}$ is the frequency at which the magnitude frequency response is $-3$ dB below the magnitude at 0 frequency
Example, [1, p. 760]

Example (Digital cascade compensator design)

- **Problem**: Design a digital lead compensator in the $s$-domain and transform the compensator to the $z$-domain

\[ G_p(s) = \frac{1}{s(s + 6)(s + 10)} \]

- %\(\text{OS}\) = 20%
- \(T_s = 1.1\) seconds

- **Solution**: On the board
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- 13.9 Gain design on the $z$-plane
- 13.10 Cascade compensation via the $s$-plane
- 13.11 Implementing the digital compensator
Algorithm to emulate the compensator, [1, p. 762]

**Concept**

- 2\textsuperscript{nd}-order example
  - Compensator
    
    \[ G_c(z) = \frac{X(z)}{E(z)} = \frac{a_3 z^3 + a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0} \]
  
  - ...cross multiply, solve for \( X(s) \), inverse z-transform...
    
    \[ x^*(t) = \frac{a_3}{b_2} e^*(t + T) + \frac{a_2}{b_2} e^*(t) + \frac{a_1}{b_2} e^*(t - T) + \frac{a_0}{b_2} e^*(t - 2T) \]
    
    \[ -\frac{b_1}{b_2} x^*(t - T) - \frac{b_0}{b_2} x^*(t - 2T) \]

- To be physically realizable

  - Present sample of compensator output, \( x^*(t) \), cannot be a function of future sample of error, \( e^*(t + T) \) → \( a_3 = 0 \)
  
  - Compensator TF numerator order ≤ denominator order
The output is a weighted linear combination of several successive values of the input & output.

Figure: Flowchart for a 2nd-order DT compensator
Example (Digital cascade compensator implementation)

- **Problem:** Develop a block diagram for the digital compensator

\[
G_c(z) = \frac{X(z)}{E(z)} = \frac{z + 0.5}{z^2 - 0.5z + 0.7}
\]

- **Solution:** On the board