

EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 12 – Design via State Space

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Topics covered in this presentation

- ▶ Controller design
- ▶ Controllability
- ▶ Observer design
- ▶ Observability

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 - 12.1 Introduction
 - 12.2 Controller design
 - 12.3 Controllability
 - 12.4 Alternative approaches to controller design
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Intro, [1, p. 664]

Concept

- ▶ Design via state space
 - ▶ Can be applied to a wider class of systems than transform methods
 - ▶ Systems with nonlinearities
 - ▶ Multiple-input, multiple output (MIMO) systems
 - ▶ Systems of higher order than 2
 - ▶ We will focus on the application to linear systems
 - ▶ Specify all CL poles
 - ▶ Parameters for each CL pole
 - ▶ Technique for finding these parameter values
 - ▶ Cannot specify CL zero locations
 - ▶ Sensitive to parameter changes
 - ▶ Wide range of computational support
 - ▶ Loss of graphic insight into a design problem

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State-variable FB control, [1, p. 665]

Concept

- ▶ n^{th} -order CL characteristic equation (CE)

$$\det(sI - A) = s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$$

- ▶ There are n coefficients whose values determine the n CL poles
- ▶ Introduce n adjustable parameters into the system and relate them to the n coefficients, so that we can place the n CL poles

State-variable FB control, [1, p. 666]

Concept

- ▶ Before, output-variable FB, now, state-variable FB
 - ▶ Each state variable is fed back to the control, u , through a gain, k_i
 - ▶ *State-variable FB gain*: $-K$
- ▶ CL system is plant with state-variable FB

$$\begin{aligned}\dot{x} &= Ax + Bu \\ &= Ax + B(-Kx + r) \\ &= (A - BK)x + Br \\ y &= Cx\end{aligned}$$

State-variable FB control, [1, p. 666]

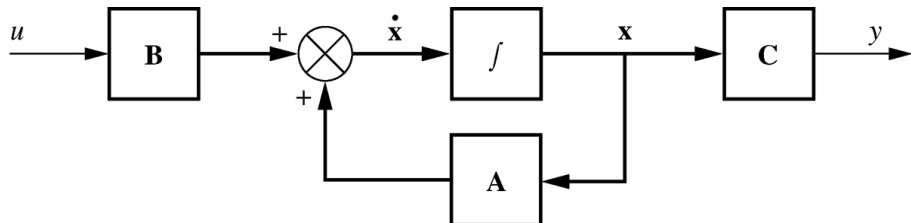


Figure: Plant with state-variable FB

State-variable FB control, [1, p. 666]

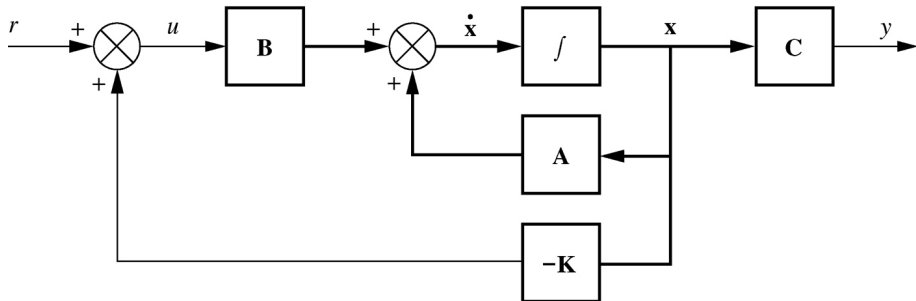


Figure: State-space representation of a plant

Methodology, [1, p. 668]

Procedure

- ▶ Pole placement for plants in phase-variable (PV) form
 1. Represent the plant in PV form
 2. FB each PV to the input of the plant through a gain, k_i
 3. Find the CE for the CL system
 4. Decide upon all CL pole locations and determine equivalent CE
 5. Equate like coefficients of the CE and solve for k_i

State-variable FB control in PV form, [1, p. 668]

Concept

► Plant

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix};$$
$$C = [c_1 \quad c_2 \quad \dots \quad c_n]$$

► Plant CE

$$\det(sI - A) =$$
$$s^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0 = 0$$

State-variable FB control in PV form, [1, p. 668]

Concept

- ▶ State-variable FB

$$u = -Kx; \quad K = [k_1 \quad k_2 \quad \dots \quad k_n]$$

- ▶ CL system

$$A - BK =$$

$$\begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ -(a_0 + k_1) & -(a_1 + k_2) & -(a_2 + k_3) & \dots & -(a_{n-1} + k_n) \end{bmatrix}$$

- ▶ CL system CE

$$\det(sI - (A - BK)) =$$

$$s^n + (a_{n-1} + k_n)s^{n-1} + \dots + (a_1 + k_2)s + (a_0 + k_1) = 0$$

State-variable FB control in PV form, [1, p. 669]

Concept

- ▶ Desired CL system CE

$$\det(sI - (A - BK)) =$$
$$s^n + d_{n-1}s^{n-1} + \dots + d_1s + d_0 = 0$$

$$d_i = a_i + k_{i+1}; \quad i = 0, 1, 2, \dots, n - 1$$

- ▶ CL system TF
 - ▶ Denominator polynomial: the CE
 - ▶ Numerator polynomial: formed from the coefficients of the output coupling matrix, C , for systems written in PV form
 - ▶ Same for plant and CL system

Example, [1, p. 669]

Example (Controller design for PV form)

- ▶ **Problem:** Design the PV FB gains to yield
 - ▶ $\%OS = 9.5\%$
 - ▶ $T_s = 0.74$ seconds
- ▶ **Solution:** On the board

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Definitions, [1, p. 672]

- ▶ **Controllability:** If an input to a system can be found that takes every state variable from a desired initial state to a desired final state, the system is said to be controllable; otherwise the system is uncontrollable.
 - ▶ Control variable, u , can be used to control the behavior of each state variable in x
 - ▶ Poles of the control system can be placed where we desire
 - ▶ Determine, a priori, whether pole placement is a viable design technique for a controller

Controllability by inspection, [1, p. 673]

Concept

- ▶ When the system matrix, A , is in diagonal or parallel form, it is apparent whether or not the system is controllable
 - ▶ A system with distinct (no repeat) eigenvalues and a diagonal system matrix, A , is controllable if the input coupling matrix, B , does not have any rows that are zero

The controllability matrix, [1, p. 674]

- ▶ In other forms, the existence of paths from the input to the state variables is not a criterion for controllability since the equations are not decoupled
 - ▶ n^{th} -order plant

$$\dot{x} = Ax + Bu$$

is completely controllable if the matrix

$$C_M = [B \quad AB \quad \dots \quad A^{n-1}B]$$

is of rank n , where C_M is called the controllability matrix

Example, [1, p. 675]

Example (Controllability via the controllability matrix)

- ▶ *Problem:* Determine if the system is controllable

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

- ▶ *Solution:* On the board

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Alternative approaches to controller design, [1, p. 676]

Concept

- ▶ For systems not represented in PV form
- ▶ 2 approaches
 - ▶ *Matching coefficients*: Matching the coefficients of

$$\det(sI - (A - BK))$$

with coefficients of the desired CE

- ▶ Same method used for systems in PV representation
- ▶ *Transformation*: Transforming the system to PV form, designing the control FB gain, & transforming the designed system back to its original state-variable representation

Approach – matching coefficients, [1, p. 677]

Concept

- ▶ Matching the coefficients of

$$\det(sI - (A - BK))$$

with coefficients of the desired CE

- ▶ Leads to difficult calculations of the control gains, especially for higher-order systems not represented with PVs

Example, [1, p. 677]

Example (Controller design by matching coefficients)

- ▶ **Problem:** Design state-variable control FB gain for the plant to yield
 - ▶ %OS = 15%
 - ▶ $T_s = 0.5$ second

$$G(s) = \frac{10}{(s+1)(s+2)}$$

- ▶ **Solution:** On the board

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation
2. Design the state-variable control FB gain
3. Transform the system in PV representation back to the original representation

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation
 - ▶ Plant not in PV representation

$$\dot{z} = Az + Bu$$

$$y = Cz$$

with controllability matrix

$$C_{M_z} = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B]$$

Approach – transformation, [1, p. 678]

Procedure

1. Transform the system to PV representation

- ▶ Assume the system can be transformed into the PV representation with the transformation

$$z = Px$$

Transformed plant

$$\dot{x} = P^{-1}APx + P^{-1}Bu$$

$$y = CPx$$

with controllability matrix

$$C_{M_x} = P^{-1}C_{M_z}$$

- ▶ Solving for P

$$P = C_{M_z} C_{M_x}^{-1}$$

Approach – transformation, [1, p. 678]

Procedure

2. Design the control FB

- ▶ Include both state-variable control FB and input

$$u = -K_x x + r$$

- ▶ Transformed plant with state-variable control FB

$$\dot{x} = (P^{-1}AP - P^{-1}BK_x)x + P^{-1}Br$$

$$y = CPx$$

- ▶ Zeros of this CL system are determined from the polynomial formed from the elements of CP

Approach – transformation, [1, p. 679]

Procedure

3. Transform the system in PV representation back to the original representation

- ▶ Plant not in PV representation with state-variable control FB

$$\begin{aligned}\dot{z} &= (A - BK_x P^{-1})z + Br \\ y &= Cz\end{aligned}$$

- ▶ State-variable control FB gain

$$K_z = K_x P^{-1}$$

- ▶ Zeros of the CL TF are the same as the zeros of the uncompensated plant

Example, [1, p. 679]

Example (Controller design by transformation)

- ▶ **Problem:** Design state-variable control FB gain for the plant to yield
 - ▶ %OS = 20.8%
 - ▶ $T_s = 4$ seconds

$$G(s) = \frac{s + 4}{(s + 1)(s + 2)(s + 5)}$$

- ▶ **Solution:** On the board

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Observer motivation, [1, p. 682]

Concept

- ▶ Controller design relies upon access to the state variables for FB through adjustable gains
- ▶ Estimate states can be fed to the controller
- ▶ **Observer:** Estimator used to calculate state variables that are not accessible from the plant

- ▶ Plant

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

- ▶ Observer

$$\dot{\hat{x}} = A\hat{x} + Bu$$

$$\hat{y} = C\hat{x}$$

- ▶ Observer error

$$\begin{aligned} \dot{e}_x &= \dot{x} - \dot{\hat{x}} &= A(x - \hat{x}) \\ y - \hat{y} &= C(x - \hat{x}) &= Ce_x \end{aligned}$$

Observer motivation, [1, p. 683]

Concept

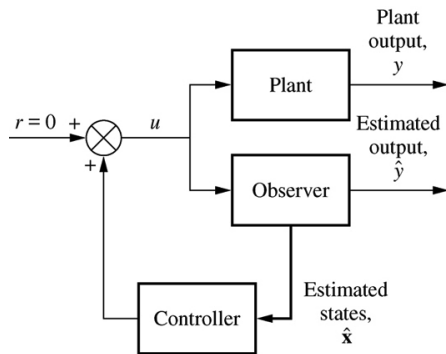


Figure: State-FB design using an OL observer

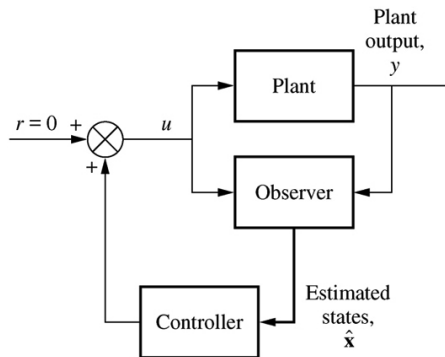


Figure: State-FB design using a CL observer

Observer motivation, [1, p. 683]

Concept

- ▶ Dynamics of the difference between the actual & estimated states is unforced, & if the plant is stable, this difference, due to the differences in initial state vectors, $\rightarrow 0$
- ▶ Speed of convergence between the actual state & the estimated state is the same as the TR of the plant since the CE of the observer error is the same as the plant
- ▶ Convergence is too slow, \rightarrow speed up the observer and make its response time much faster than that of the controlled CL system, \rightarrow the controller will receive the estimated states instantaneously
 - ▶ Error between the outputs of the plant and the observer is fed back to the derivatives of the observer's states
 - ▶ The system corrects to drive this error $\rightarrow 0$
 - ▶ Design a desired TR into the observer that is much quicker than that of the plant or controlled CL system

Observer motivation, [1, p. 683]

Concept

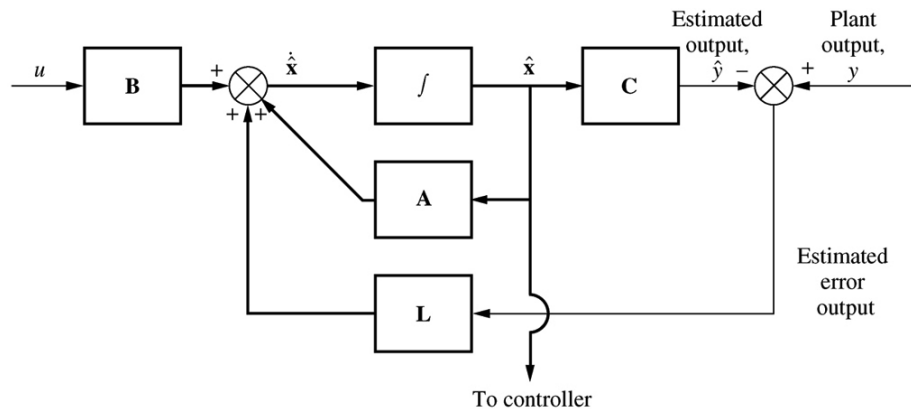


Figure: State-FB design using a CL observer exploded view showing FB arrangement to reduce state-variable estimation error

Observer motivation, [1, p. 683]

Concept

- ▶ Observer canonical form yields an easy solution for the observer FB gain
- ▶ *Observer FB gain, L* : Ensures the TR of the observer is faster than the response of the controlled loop in order to yield a rapidly updated estimate of the state vector

Design methodology, [1, p. 684]

Procedure

1. Find error system, i.e., state equations for error between actual state vector & estimated state vector, $x - \hat{x}$

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y})$$

$$\hat{y} = C\hat{x}$$

2. Find CE for error system
3. Evaluate required observer FB gain, L , to meet rapid TR for observer
4. Select eigenvalues of observer to yield stability & desired TR that is faster than controlled CL response

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Definitions, [1, p. 690]

- ▶ **Observability:** If the initial-state vector, $x(t_0)$, can be found from inputs, $u(t)$, and measurements, $y(t)$, over a finite interval of time from t_0 , the system is said to be observable; otherwise the system is said to be unobservable.
 - ▶ Knowledge of measured output variables, y , and control inputs, $u(t)$, can be used to observe the behavior of each state variable in x
 - ▶ Poles of the observer system can be placed where we desire
 - ▶ Determine, a priori, whether pole placement is a viable design technique for an observer

Observability by inspection, [1, p. 690]

Concept

- ▶ When the system matrix, A , is in diagonal or parallel form, it is apparent whether or not the system is observable
 - ▶ A system with distinct (no repeat) eigenvalues and a diagonal system matrix, A , is observable if the output coupling matrix, C , does not have any columns that are zero

The observability matrix, [1, p. 691]

- ▶ In other forms, the existence of paths from the output to the state variables is not a criterion for observability since the equations are not decoupled
 - ▶ n^{th} -order plant

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

is completely observable if the matrix

$$O_M = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

is of rank n , where O_M is called the observability matrix

Example, [1, p. 691]

Example (Observability via the observability matrix)

- ▶ *Problem:* Determine if the system is observable

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -3 & -2 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad C = [0 \quad 5 \quad 1]$$

- ▶ *Solution:* On the board

Example, [1, p. 692]

Example (Unobservability via the observability matrix)

- ▶ *Problem:* Determine if the system is observable

$$A = \begin{bmatrix} 0 & 1 \\ -5 & -\frac{21}{4} \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad C = [5 \quad 4]$$

- ▶ *Solution:* On the board

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Alternative approaches to controller design, [1, p. 676]

Concept

- ▶ For systems not represented in observer canonical form
- ▶ 2 approaches
 - ▶ *Matching coefficients*: Matching the coefficients of

$$\det(sI - (A - LC))$$

with coefficients of the desired CE

- ▶ Same method used for systems in PV representation
- ▶ *Transformation*: Transforming the system to observer canonical form, designing the observer FB gain, & transforming the designed system back to its original state-variable representation

Approach - matching coefficients, [1, p. 693]

Concept

- ▶ Matching the coefficients of

$$\det(sI - (A - LC))$$

with coefficients of the desired CE

- ▶ Leads to difficult calculations of the observer FB gain, especially for higher-order systems not in PV representation

Example, [1, p. 698]

Example (Observer design by matching coefficients)

- ▶ **Problem:** Design an observer FB gain for the system in PV representation with a TR described by
 - ▶ $\zeta = 0.7$
 - ▶ $\omega_n = 100$

$$G(s) = \frac{407(s + 0.916)}{(s + 1.27)(s + 2.69)}$$

- ▶ **Solution:** On the board

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
2. Design the observer FB gain
3. Transform the system in PV representation back to the original representation

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation
 - ▶ Plant not in PV representation

$$\dot{z} = Az + Bu$$

$$y = Cz$$

with observability matrix

$$O_{M_z} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Approach – transformation, [1, p. 695]

Procedure

1. Transform the system to PV representation

- ▶ Assume the system can be transformed into the PV, x , representation with the transformation

$$z = Px$$

Transformed plant

$$\dot{x} = P^{-1}APx + P^{-1}Bu$$

$$y = CPx$$

with observability matrix

$$O_{M_x} = O_{M_z}P$$

- ▶ Solving for P

$$P = O_{M_z}^{-1}O_{M_x}$$

Approach – transformation, [1, p. 695]

Procedure

2. Design the observer FB

- ▶ Transformed plant with FB gains

$$\begin{aligned}\dot{e}_x &= (P^{-1}AP - L_xCP)e_x \\ y - \hat{y} &= CPe_x\end{aligned}$$

Approach – transformation, [1, p. 695]

Procedure

3. Transform the system in PV representation back to the original representation
 - ▶ Plant not in PV representation with observer FB gain

$$\begin{aligned}\dot{e}_z &= (A - PL_x C)e_z \\ y - \hat{y} &= Ce_z\end{aligned}$$

- ▶ Observer FB gain

$$L_z = PL_x$$

Example, [1, p. 695]

Example (Observer design by transformation)

- ▶ *Problem:* Design an observer in cascade form

$$G(s) = \frac{1}{(s+1)(s+2)(s+5)}$$

- ▶ *Solution:* On the board

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Concepts, [1, p. 700]

Concept

- ▶ Design systems in state-space representation for steady-state error
- ▶ Error is fed forward to the controlled plant via an integrator
 - ▶ Additional state variable

$$\dot{x}_N = r - Cx$$

- ▶ Plant

$$\dot{x} = Ax + Bu$$

$$\dot{x}_N = -Cx + r$$

$$y = Cx$$

- ▶ Control FB

$$\begin{aligned} u &= -Kx + K_e x_N \\ &= - \begin{bmatrix} K & -K_e \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} \end{aligned}$$

Concepts, [1, p. 701]

Concept

- ▶ Error is fed forward to the controlled plant via an integrator
 - ▶ Augmented representation

$$\begin{bmatrix} \dot{x} \\ \dot{x}_N \end{bmatrix} = \begin{bmatrix} A - BK & BK_e \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ x_N \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$
$$y = [C \quad 0] \begin{bmatrix} x \\ x_N \end{bmatrix}$$

- ▶ K and K_e can be selected to yield the desired TR
- ▶ We have an additional pole to place
- ▶ Keep an eye on the CL zeros and their effect on TR

Example, [1, p. 701]

Example (Design of integral control)

▶ *Problem:*

a. Design a controller without integral control to yield

▶ $\%OS = 10\%$

▶ $T_s = 0.5$ second

Evaluate the steady-state error for a unit step

b. Repeat the design using integral control. Evaluate the steady-state error for a unit step input.

▶ *Solution:* On the board

Bibliography



Norman S. Nise. *Control Systems Engineering*, 2011.