EE C128 / ME C134 – Feedback Control Systems

Lecture – Chapter 10 – Frequency Response Techniques

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Lecture abstract

Topics covered in this presentation

- Advantages of FR techniques over RL
- Define FR
- Define Bode & Nyquist plots
- Relation between poles & zeros to Bode plots (slope, etc.)
- Features of 1^{st}- & 2^{nd}-order system Bode plots
- Define Nyquist criterion
- Method of dealing with OL poles & zeros on imaginary axis
- Simple method of dealing with OL stable & unstable systems
- Determining gain & phase margins from Bode & Nyquist plots
- Define static error constants
- Determining static error constants from Bode & Nyquist plots
- Determining TF from experimental FR data

Chapter outline

Advantages of frequency response (FR) methods, [1, p. 534]

In the following situations

- When modeling TFs from physical data
- When designing lead compensators to meet a steady-state error requirements
- When finding the stability of NL systems
- In settling ambiguities when sketching a root locus

The concept of FR, [1, p. 535]

- At steady-state, sinusoidal inputs to a linear system generate sinusoidal responses of the same frequency with different amplitudes and phase angle from the input, each of which are a function of frequency.
- *Phasor* – complex representation of a sinusoid
  - $||G(\omega)||$ – amplitude
  - $\angle G(\omega)$ – phase angle
  - $M\cos(\omega t + \phi)$ ... $M\angle\phi$

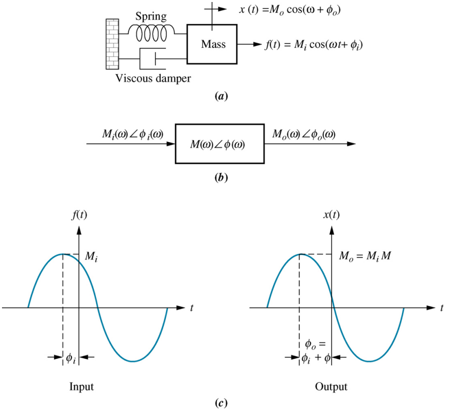

Figure: Sinusoidal FR: a. system; b. TF; c. IO waveforms

The concept of FR, [1, p. 535]

- Steady-state output sinusoid

$$M_o(\omega)\angle\phi_o(\omega)$$
$$= M_i(\omega)M(\omega)\angle(\phi_i(\omega)+\phi(\omega))$$

- *Magnitude FR*

$$M(\omega)=\frac{M_o(\omega)}{M_i(\omega)}$$

- *Phase FR*

$$\phi(\omega)=\phi_o(\omega)-\phi_i(\omega)$$

- *FR*

$$M(\omega)\angle\phi(\omega)$$

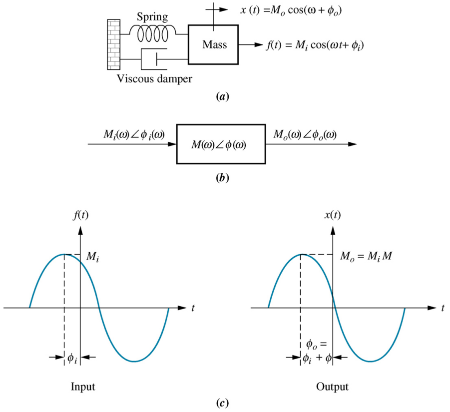

Figure: Sinusoidal FR: a. system; b. TF; c. IO waveforms

Analytical expressions for FR, [1, p. 536]

- General input sinusoid

$$\begin{aligned} r(t) &= A\cos(\omega t) + B\sin(\omega t) \\ &= \sqrt{A^2+B^2}\cos\left(\omega t - \tan^{-1}\left(\tfrac{B}{A}\right)\right) \end{aligned}$$

- Input phasor forms
  - Polar, $M_i \angle \phi_i$

$$M_i = \sqrt{A^2+B^2}$$

$$\phi_i = -\tan^{-1}\left(\tfrac{B}{A}\right)$$

  - Rectangular, $A - jB$
  - Euler's, $M_i e^{j\phi_i}$

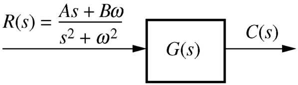

Figure: System with sinusoidal input

Analytical expressions for FR, [1, p. 536]

- Forced response

$$C(s) = \frac{As + B\omega}{s^2 + \omega^2} G(s)$$

- Steady-state forced response after partial fraction expansion

$$C_{ss}(s) = \frac{\frac{M_i M_G}{2} e^{-j(\phi_i - \phi_G)}}{s + j\omega} + \frac{\frac{M_i M_G}{2} e^{j(\phi_i - \phi_G)}}{s - j\omega}$$

where $M_G = ||G(j\omega)||$ and $\phi_G = \angle G(j\omega)$

- Time-domain response

$$c(t) = M_i M_G \cos(\omega t + \phi_i + \phi_G)$$

- Time-domain response in phasor form

$$M_o \angle \phi_o = (M_i \angle \phi_i)(M_G \angle \phi_G)$$

- FR of system

$$G(j\omega) = G(s)|_{s \to j\omega}$$

10 Frequency response techniques

History interlude

History (Hendrik Wade Bode)

- 1905 – 1982
- American engineer
- 1930s – Inventor of *Bode plots, gain margin, & phase margin*
- 1944 – WWII anti-aircraft (including V-1 flying bombs) systems
- 1947 – Cold War anti-ballistic missiles
- 1957 – Served on NACA (now NASA) with Wernher von Braun (inventor of V-1 flying bombs & V-2 rockets)

Figure: Hendrik Wade Bode

General Bode plots, [1, p. 540]

$$G(j\omega) = M_G(\omega)\angle\phi_G(\omega)$$

- Separate magnitude and phase plots as a function of frequency
  - Magnitude – decibels (dB) vs. $\log(\omega)$, where $dB = 20\log(M)$
  - Phase – phase angle vs. $\log(\omega)$

Bode plots approximations, [1, p. 542]

- TF

$$G(s) = s + a$$

- Low frequencies

$$G(j\omega) \approx a\angle 0^\circ$$

- High frequencies

$$G(j\omega) \approx \omega\angle 90^\circ$$

- *Asymptotes* – straight-line approximations
  - *Low-frequency*
  - *Break frequency*
  - *High-frequency*

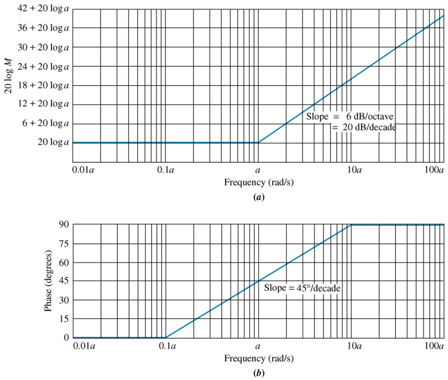

Figure: Bode plots of $s + a$: a. magnitude plot; b. phase plot

Simple Bode plots, [1, p. 542]

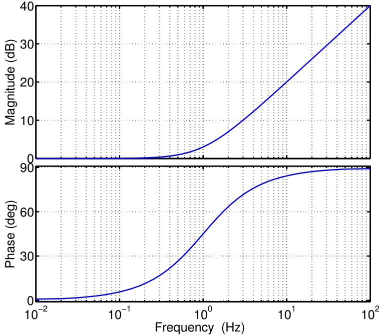

Figure: Bode plot of $\frac{s+a}{a}$

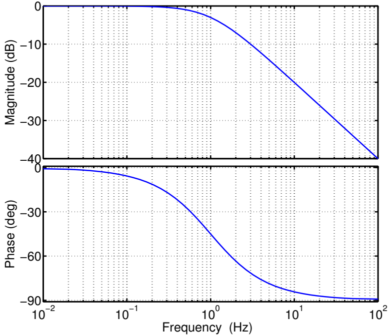

Figure: Bode plot of $\frac{a}{s+a}$

Simple Bode plots, [1, p. 545]

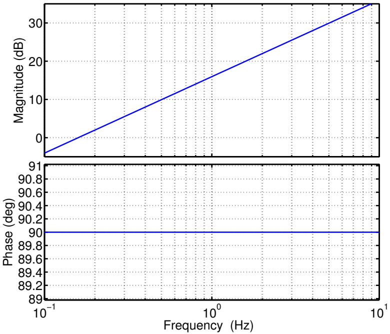

Figure: Bode plot of s

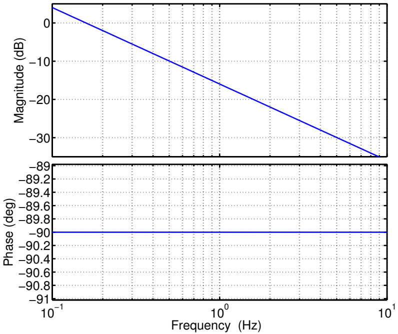

Figure: Bode plot of $\frac{1}{s}$

Simple Bode plots, [1, p. 549]

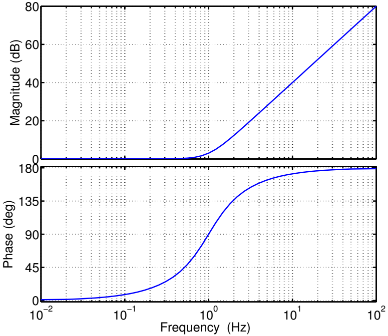

Figure: Bode plot of $\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$

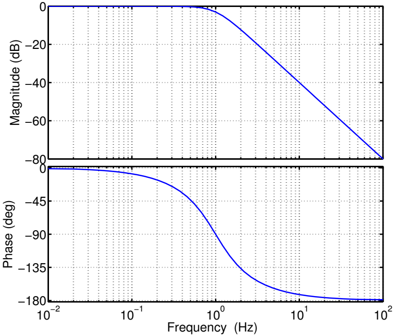

Figure: Bode plot of $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

Detailed 2^{nd}-order Bode plots, [1, p. 550]

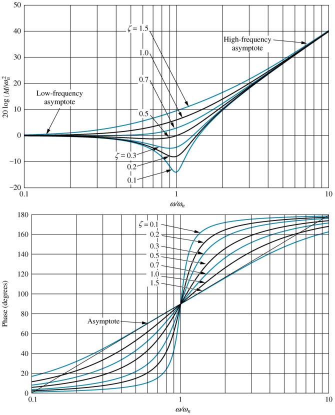

Figure: Bode plot of $\frac{s^2+2\zeta\omega_n s+\omega_n^2}{\omega_n^2}$

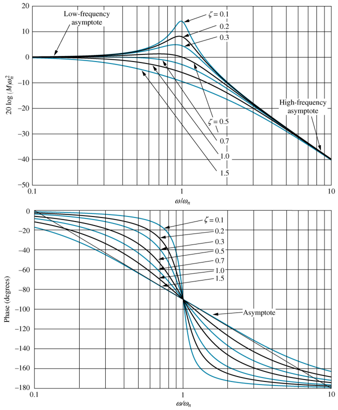

Figure: Bode plot of $\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$

10 Frequency response techniques

History interlude

History (Harry Theodor Nyquist)

- 1889 – 1976
- American engineer
- 1917 – 1934 AT&T
- 1934 – 1954 Bell Telephone Labs
- 1924 – Nyquist-Shannon sampling theorem
- 1926 – Johnson–Nyquist noise
- 1932 – *Nyquist stability criterion*

Figure: Harry Theodor Nyquist

Introduction, [1, p. 559]

- Relates the stability of a CL system to the OL FR and the OL poles and zeros
  - # CL poles in RHP
- Provides information on the transient response and steady-state error

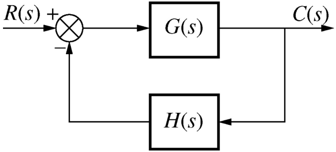

Figure: CL control system

Derivation concepts, [1, p. 560]

$$G(s) = \frac{N_G}{D_G} \quad \text{and} \quad H(s) = \frac{N_H}{D_H}$$

$$G(s)H(s) = \frac{N_G H_G}{D_G D_H}$$

$$F(s) = 1 + G(s)H(s) = \frac{D_G D_H + N_G N_H}{D_G D_H}$$

$$T(s) = \frac{G(s)}{1 + G(s)H(s)} = \frac{N_G N_H}{D_G D_H + N_G N_H}$$

- Poles of $1 + G(s)H(s)$ are the same as the poles of the OL system, $G(s)H(s)$
- Zeros of $1 + G(s)H(s)$ are the same as the poles of the CL system, $T(s)$

Derivation concepts, [1, p. 560]

- *Map* – function
- *Contour* – collection of points
- For our particular scenario, assume

$$F(s) = \frac{(s - z_1)(s - z_2)...}{(s - p_1)(s - p_2)...}$$

and a clockwise direction for mapping the points on the contour A

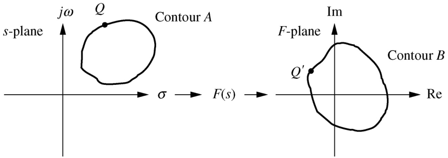

Figure: Mapping contour A through function $F(s)$ to contour B

Derivation concepts, [1, p. 561]

- If $F(s)$ has only zeros or only poles that *are not encircled* by the contour then contour B maps in a *clockwise* direction

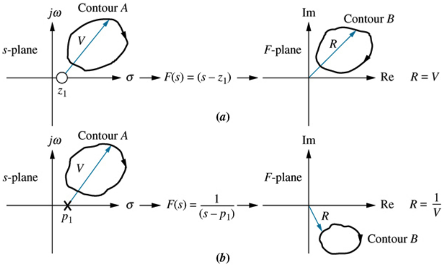

Figure: Contour mapping – without encirclements

Derivation concepts, [1, p. 561]

- If $F(s)$ has *only zeros* that *are encircled* by the contour then contour B maps in a *clockwise* direction
- If $F(s)$ has *only poles* that *are encircled* by the contour then contour B maps in a *counterclockwise* direction
- If $F(s)$ has *only poles or only zeros* that *are encircled* by the contour then contour B map *does encircle the origin*

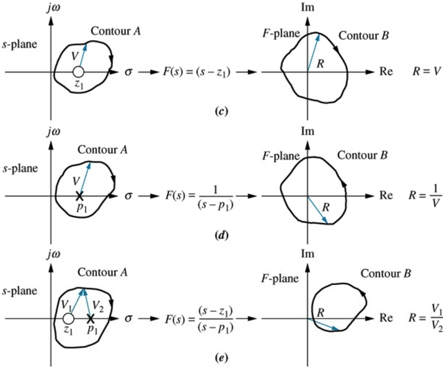

Figure: Contour mapping – with encirclements

Derivation concepts, [1, p. 562]

- If $F(s)$ has $\#_{poles} = \#_{zeros}$ that *are encircled* by the contour then contour B map *does not encircle the origin*

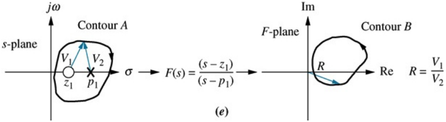

Figure: Contour mapping – with encirclements

Derivation concepts, [1, p. 562]

- Each pole or zero of $1 + G(s)H(s)$ whose vector undergoes a complete rotation of contour A must yield a change of 360° in the resultant, R, or a complete rotation of contour B
- A zero inside a CW contour A yields a CW rotation of contour B
- A pole inside a CW contour A yields a CCW rotation of contour B

- $N = P - Z$
  - N, # CCW rotations of contour B about the origin
  - P, # poles of $1 + G(s)H(s)$ inside contour A
  - Z, # zeros of $1 + G(s)H(s)$ inside contour A

Derivation concepts, [1, p. 562]

Adjustment – extend the contour A to include the entire RHP

- Z, # RHP CL poles
  - CL stability!
- P, # RHP OL poles
  - Easy
- N, # CCW rotations of contour B about origin
  - Difficult

Adjustment – map $G(s)H(s)$ instead of $1 + G(s)H(s)$

- N, # CCW rotations of contour B about -1
  - Less difficult

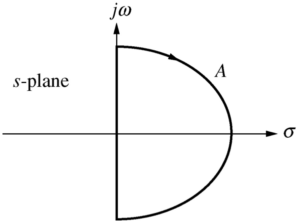

Figure: Contour enclosing RHP to determine stability

Definition, [1, p. 563]

Definition (Nyquist stability criterion)

- If a contour, A, that encircles the entire RHP is mapped through the OL system, $G(s)H(s)$, then the # of RHP CL poles, Z, equals the # of RHP OL poles, P, minus the # of CCW revolutions, N, around -1 of the mapping.

$$Z = P - N$$

- The mapping is called the Nyquist diagram of $G(s)H(s)$.
- FR technique because the mapping of points on the positive $j\omega$-axis through $G(s)H(s)$ is the same as substituting $s = j\omega$ into $G(s)H(s)$ to form the FR function $G(j\omega)H(j\omega)$.

Applying the Nyquist stability criterion, [1, p. 563]

- No RHP CL poles
  - $P = 0$
  - $N = 0$
  - $Z = 0$
  - CL system is stable
- 2 RHP CL poles
  - $P = 0$
  - $N = -2$
  - $Z = 2$
  - CL system is unstable

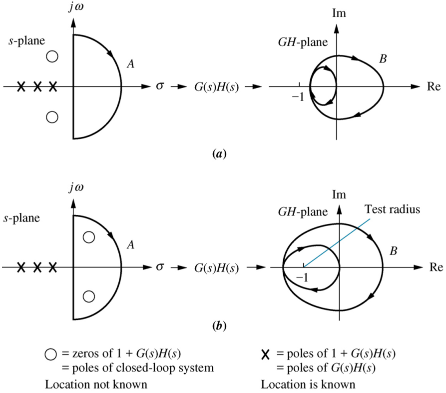

Figure: Mapping examples – with encirclement: a. contour does not enclose CL poles; b. contour does enclose CL poles

Method

Example, [1, p. 564]

Example (general)

- $G(s) = \dfrac{500}{(s+10)(s+3)(s+1)}$

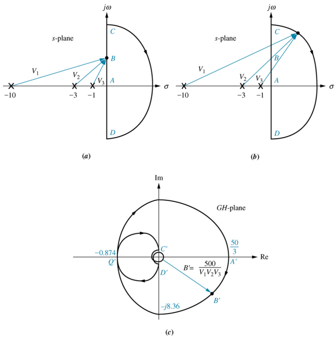

Figure: Vector evaluation of the Nyquist diagram: a. vectors on contour at low frequency, b. vectors on contour around ∞; c. Nyquist diagram

Example, [1, p. 567]

Example (poles on contour)

- $G(s) = \frac{s+2}{s^2}$

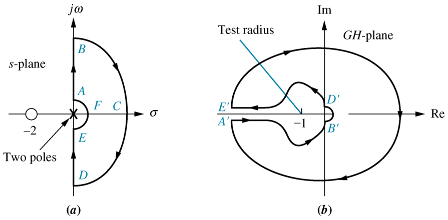

Figure: a. contour, b. Nyquist diagram

10 Frequency response techniques

Example, [1, p. 569]

Example (general)

- $G(s) = \dfrac{K(s+3)(s+5)}{(s-2)(s-4)}$

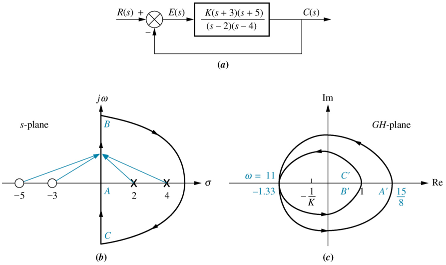

Figure: a. system; b. contour, c. Nyquist diagram

Example, [1, p. 570]

Example (general)

- $G(s) = \frac{K}{s(s+3)(s+5)}$

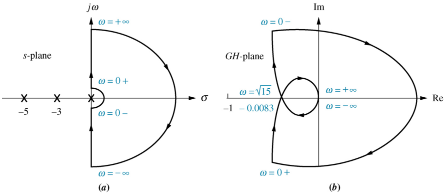

Figure: a. contour; b. Nyquist diagram

Stability via mapping only the positive $j\omega$-axis, [1, p. 571]

10 Frequency response techniques

Definitions, [1, p. 574]

Two quantitative measures of how stable a system is

- *Gain margin, G_M* – the change in OL gain, expressed in dB, required at 180° of phase shift to make the CL system unstable
- *Phase margin, Φ_M* – the change in OL phase shift required at unity gain to make the CL system unstable

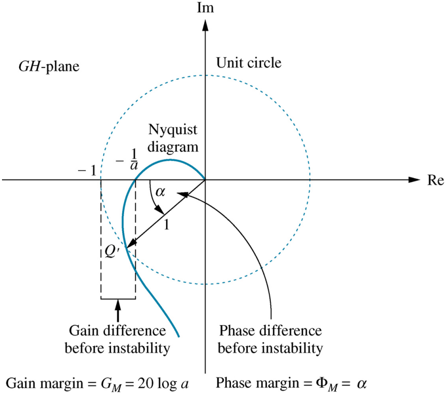

Figure: Nyquist diagram showing gain and phase margins

10 Frequency response techniques

Stability via Bode plots, [1, p. 576]

Method

- Draw a Bode log-magnitude plot
- Determine the range of the gain that ensures that the magnitude is less than 0 dB (unity gain) at that frequency where the phase is $\pm 180^\circ$

Example, [1, p. 577]

Example (general)

- Use Bode plots to determine the range of K within which the unity FB system is stable.

$$G(s) = \frac{k}{(s+2)(s+4)(s+5)}$$

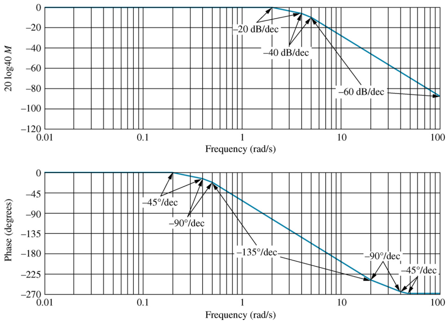

Figure: Bode log-magnitude and phase diagrams

Gain & phase margin via Bode plots, [1, p. 578]

Method

- Gain margin
  - Phase plot $\rightarrow$ $\omega_{G_M} = \omega|_{\Phi=180°}$
  - At ω_{G_M}, magnitude plot $\rightarrow$ gain margin, G_M, which is the gain required to raise the magnitude curve to 0 dB
- Phase margin
  - Magnitude plot $\rightarrow$ $\omega_{\Phi_M} = \omega|_{G=0dB}$
  - At ω_{Φ_M}, phase plot $\rightarrow$ phase margin, Φ_M, which is the difference between the phase value and 180°

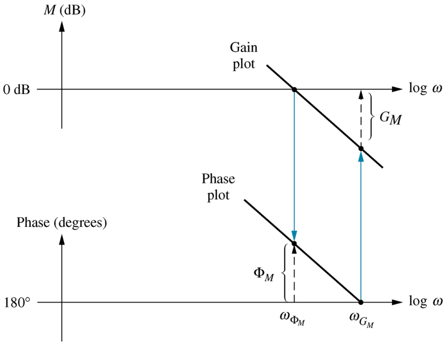

Figure: Gain and phase margins on the Bode diagrams

Example, [1, p. 579]

Example (general)

- If $K = 200$, find the gain and phase margins.

$$G(s) = \frac{k}{(s+2)(s+4)(s+5)}$$

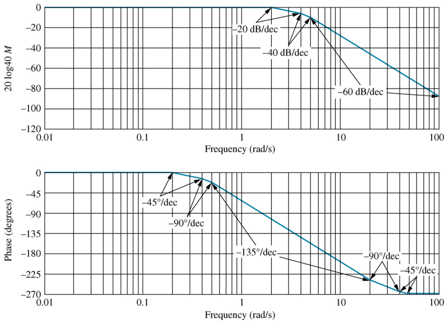

Figure: Bode log-magnitude and phase diagrams

10 Frequency response techniques

Damping ratio & CL FR, [1, p. 580]

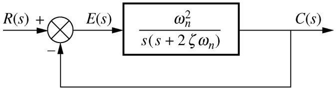

Figure: 2^{nd}-order CL system

Peak magnitude of the CL FR

$$M_p = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

Frequency of the peak magnitude

$$\omega_p = \omega_n\sqrt{1-2\zeta^2}$$

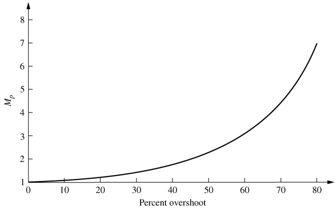

Figure: CL FR peak vs. $\%OS$ for a 2 pole system

Response speed & CL FR, [1, p. 581]

- Bandwidth of a 2-pole system

$$\omega_{BW} = \omega_n \sqrt{(1 - 2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}}$$

- ω_n–T_s relation

$$\omega_n = \frac{4}{T_s \zeta}$$

- ω_n–T_p relation

$$\omega_n = \frac{\pi}{T_p \sqrt{1 - \zeta^2}}$$

- ω_n–T_r relation
  - Found using look-up table

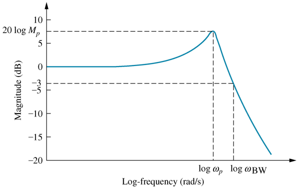

Figure: Representative log-magnitude plot

Response speed & CL FR, [1, p. 582]

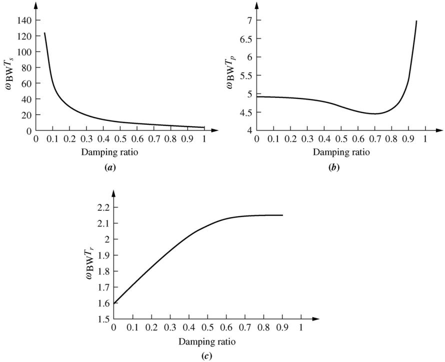

Figure: Normalized bandwidth vs. damping ratio for: a. T_s, b. T_p; c. T_r

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Constant M circles & constant N circles, [1, p. 583]

Skip for now

Damping ratio from M circles, [1, p. 589]

Skip for now

Damping ratio from phase margin, [1, p. 589]

Skip for now

Response speed from OL FR, [1, p. 591]

Skip for now

10 Frequency response techniques

Position constant, [1, p. 593]

Type 0 system

$$G(s) = K\frac{\prod_{i=1}^{n}(s+z_i)}{\prod_{j=1}^{m}(s+p_j)}$$

Initial log-magnitude value

$$20\log M = 20\log K_p$$

Position constant

$$K_p = K\frac{\prod_{i=1}^{n} z_i}{\prod_{j=1}^{m} p_j}$$

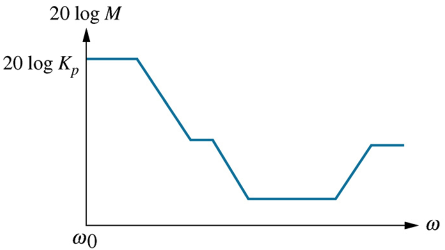

Figure: Typical unnormalized and unscaled Bode log-magnitude plots showing the value of static error constants

Velocity constant, [1, p. 594]

Type 1 system

$$G(s) = K \frac{\prod_{i=1}^{n}(s+z_i)}{s\prod_{j=1}^{m}(s+p_j)}$$

Initial log-magnitude value

$$20\log M = 20\log \frac{K_v}{\omega_0}$$

Velocity constant

$$K_v = K \frac{\prod_{i=1}^{n} z_i}{\prod_{j=1}^{m} p_j}$$

Frequency axis intersect

$$\omega = K_v$$

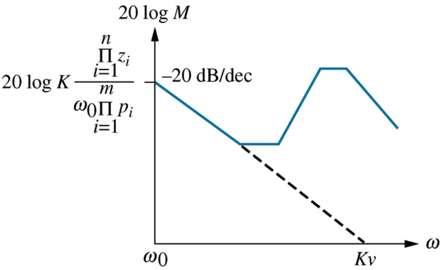

Figure: Typical unnormalized and unscaled Bode log-magnitude plots showing the value of static error constants

Acceleration constant, [1, p. 595]

Type 2 system

$$G(s) = K \frac{\prod_{i=1}^{n}(s+z_i)}{s^2 \prod_{j=1}^{m}(s+p_j)}$$

Acceleration constant

$$K_a = K \frac{\prod_{i=1}^{n} z_i}{\prod_{j=1}^{m} p_j}$$

Initial log-magnitude value

$$20 \log M = 20 \log \frac{K_a}{\omega_0^2}$$

Frequency axis intersect

$$\omega = \sqrt{K_a}$$

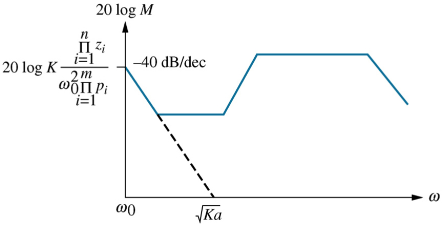

Figure: Typical unnormalized and unscaled Bode log-magnitude plots showing the value of static error constants

10 Frequency response techniques

Modeling time delay, [1, p. 597]

Time delay – delay between the commanded response and the start of the output response

$$G'(s) = e^{-sT}G(s)$$

FR

$$\begin{aligned} G'(j\omega) &= e^{-j\omega T}G(j\omega) \\ &= |G(j\omega)|\angle[-\omega T + \angle G(j\omega)] \end{aligned}$$

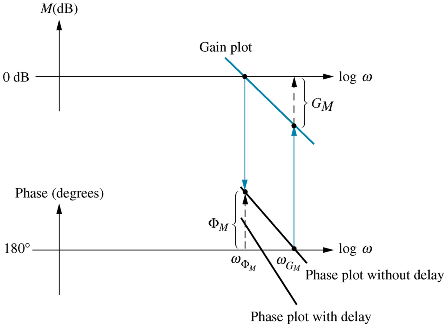

Figure: Effect of delay upon FR

10 Frequency response techniques

Obtaining transfer functions experimentally, [1, p. 602]

Skip for now

Method

1. Estimate the pole-zero configuration from the Bode diagrams
   - Initial slope $\rightarrow$ system type
   - Phase excursions $\rightarrow \#_{\text{poles}}$ & $\#_{\text{zeros}}$
2. Look for obvious 1^{st}- & 2^{nd}-order pole or zero FR characteristics
3. Peaking & depressions $\rightarrow$ underdamped 2^{nd}-order pole & zero, respectively
4. Extract 1^{st}- & 2^{nd}-order characteristics
   - Overlay ± 20 or ± 40 $dB/$decade lines on magnitude curve & $\pm 45^\circ/$decade lines on the phase curve
   - Estimate break frequencies
   - For 2^{nd}-order poles & zeros, estimate ζ & ω_n
5. Form a TF of unity gain using the poles & zeros found
   - Subtract the FR of the model from the measured FR and repeat the process if necessary

Bibliography

Norman S. Nise. *Control Systems Engineering*, 2011.