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DIRICHLET PROBLEMS FOR SOME HAMILTON–JACOBI EQUATIONS WITH INEQUALITY CONSTRAINTS*

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Abstract. We use viability techniques for solving Dirichlet problems with inequality constraints (obstacles) for a class of Hamilton–Jacobi equations. The hypograph of the "solution" is defined as the "capture basin" under an auxiliary control system of a target associated with the initial and boundary conditions, viable in an environment associated with the inequality constraint. From the tangential condition characterizing capture basins, we prove that this solution is the unique "upper semicontinuous" solution to the Hamilton–Jacobi–Bellman partial differential equation in the Barron-Jensen/Frankowska sense. We show how this framework allows us to translate properties of capture basins into corresponding properties of the solutions to this problem. For instance, this approach provides a representation formula of the solution which boils down to the Lax–Hopf formula in the absence of constraints.

Key words. Hamilton–Jacobi equations, viability theory, optimal control, traffic modeling

AMS subject classifications. 49J24, 49L25, 90B20, 58C06

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1. Introduction.

1.1. Motivation. This article is motivated by macroscopic fluid models of highway traffic, following the pioneering work of Lighthill and Whitham [64] and Richards [78]. In their original work, the authors modeled highway traffic flow with a first order hyperbolic *partial differential equation* with concave flux function, called the Lighthill–Whitham–Richards (partial differential) equation. This model is the seminal model for numerous highway traffic flow studies available in the literature today [2, 45, 46, 63, 33, 87, 31]. It models the evolution of the density of vehicles on a highway by a conservation law, in which the mathematical model of the flux function inside the conservation law results from empirical measurements [60].

Solutions to such equations may have shocks (they are set-valued maps), which model abrupt changes in vehicle density on the highway [2], and only model physical phenomena to a certain degree. Hence discontinuous selections of these solutions are investigated, for instance, the *entropy solution* [2] of Oleinik [73], which is acknowledged to be the proper weak solution of this problem. There has been an extensive literature on this problem, of which we single out the work of Bardos, Leroux, and Nedelec [24]; see also Strub and Bayen [83].

Very few results applicable to highway traffic are available for control of first order hyperbolic conservation laws. Differential flatness [50] has been successfully applied to the Burgers equation (and therefore to the Lighthill–Whitham–Richards equation) in [75] order to avoid the formation of such shockwaves. This analysis does not so far

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extend to the presence of shocks. Lyapunov-based techniques have also been applied to the Burgers equation [62]. Adjoint-based methods have been successfully applied to networks of Lighthill–Whitham–Richards equations in [57]; these results seem so far the most promising, but they do not have guarantees to provide an optimal control policy. Questions of interest in controlling first order partial differential equations [66, 74, 82, 86], and in particular, Lighthill–Whitham–Richards equations, are still open and difficult to solve due to the presence of shocks occurring in the solutions of these partial differential equations [3, 20, 21, 32, 36, 42, 47, 48, 49, 58, 59, 61].

In order to alleviate the technical difficulties resulting from shocks present in solutions of the Lighthill–Whitham–Richards equation, an alternate formulation consists in considering the *cumulated number of vehicles*, widely used in the transportation literature as well [70, 71, 72]. The cumulative number of vehicles can be thought of as a primitive of the density over space. Formally, the evolution of the cumulated number $\mathbf{N}(t,x)$ of vehicles is the solution of a *Hamilton–Jacobi (partial differential) equation* of the form

$$\frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \;=\; \psi(v(t)),$$

where the flux function ψ appearing in this Hamilton–Jacobi equation is in fact concave as shown by the empirically measured flux function of the Lighthill–Whitham–Richards equation [64, 78, 24, 83]. The function $v(\cdot)$ will be regarded as a control of the Hamilton–Jacobi equation in forthcoming studies. It could, for example, model the inflow of vehicles at the entrance of a stretch of highway. It is a given datum in this paper.

The solution of this Hamilton–Jacobi equation has no shocks but is not necessarily differentiable. It is only upper semicontinuous. Actually, the nondifferentiability of the cumulated number of vehicles is closely related to the presence of the shocks of the solution to the Lighthill–Whitham–Richards equation (see, for instance, [39, 40, 41]).

Since the Lighthill–Whitham–Richards equation and the Hamilton–Jacobi equation model the same physical phenomenon, and since both formulations are equivalently used in the highway transportation literature, we single out in this paper the study of the evolution of the cumulated number of vehicles in order to leverage the extensive knowledge of Hamilton–Jacobi equations for which control and viability techniques can be applied [65, 67, 68, 69, 76, 77, 81].

1.2. Contributions of the paper. We shall revisit this Hamilton–Jacobi equation by answering new questions as follows:

- introducing a nontrivial right-hand side;
- involving Dirichlet conditions;
- and, above all, imposing *inequality constraints* on the solution, for instance, upper bounds on the cumulated number of vehicles, depending on time and space variables.

For this purpose, we suggest using a novel point of view based on the concept of capture basin of a target viable in an environment extensively studied in the framework of viability theory; i.e., given a closed subset of a finite dimensional vector space regarded as an environment, a closed subset of this environment considered as a target and a control system, the viable capture basin is the subset of initial states of the environment from which starts at least one evolution governed by the control system viable in the environment until the finite time when it reaches the target (see Definition 3.3). It happens that the *hypograph* of the solution to the Hamilton–Jacobi equation satisfying initial and Dirichlet conditions as well as inequality constraints is the capture basin of an auxiliary target (involving initial and boundary conditions) viable in an auxiliary environment (involving inequality constraints) under an auxiliary control system (involving the flux function of the Hamilton–Jacobi equation).

Hence, anticipating this property, we define the *viability hyposolution* of the Dirichlet problem for this Hamilton–Jacobi equation with constraints from this property as being a viable capture basin (see Definition 4.1). Then we proceed by translating properties of viable capture basins (see [7], for instance) in the language of partial differential equations for this particular case. We shall prove that the viability hyposolution

1. is the *unique* generalized solution in the Barron-Jensen/Frankowska sense[1] (a weaker concept of viscosity solution introduced by Crandall, Evans, and Lions in [44, 43] for continuous solutions adapted to the case when solution is only semicontinuous): Theorem 9.1;
2. is equivalently the *unique* upper semicontinuous solution in the contingent Frankowska sense:[2] Theorem 8.1;
3. satisfies the sup-linearity property and depends "hypocontinuously" on the initial and Dirichlet conditions;
4. is represented by the Lax–Hopf formula [1] (see Theorem 5.1) in the absence of inequality constraints, a more involved representation formula (see Theorem 5.5) in the presence of inequality constraints, upper estimates (maximum principle; see Proposition 5.3), and lower estimates (see Proposition 5.4).

The results presented in this article have since been applied to highway traffic data [30, 29], using available algorithms to solve, in particular, viability problems numerically [80, 37, 38].

1.3. Outline of the paper. In order to make the paper more readable, section 3 gathers some definitions, notations, and basic prerequisites of viability theory and convex analysis for the convenience of readers who are not familiar with these topics. We then state the problem and the main assumptions, which will not be repeated. We next define the viability hyposolution to the nonhomogeneous Dirichlet/initial value problem for our class of Hamilton–Jacobi equations under inequality constraints as the capture basin of a target summarizing the Dirichlet/initial data viable in a target associated with inequality constraints. Then, we translate the properties of capture basins into the viability hyposolution, starting with a general representation formula providing Lax–Hopf formulas in the absence of inequality constraints. We next check that the viability hyposolution satisfies the Dirichlet and initial conditions as well as the inequality constraints. The last three sections are devoted to the proof that the viability hyposolution is a solution to the Hamilton–Jacobi partial differential equation

[1]Frankowska proved that the epigraph of the value function of an optimal control problem—assumed to be only lower semicontinuous—is semipermeable (i.e., invariant and backward viable) under a (natural) auxiliary system. Furthermore, when it is continuous, its epigraph is viable and its hypograph invariant [53, 54, 56]. By duality, the latter property is equivalent to the fact that the value function is a viscosity solution of the associated Hamilton–Jacobi equation in the sense of Crandall and Lions. See also [26, 22, 8] for more details. Such concepts have been extended to solutions of systems of first order partial differential equations without boundary conditions by Frankowska and the first author (see [14, 15, 16, 17, 18, 19] and Chapter 8 of [5]). See also [11, 12].

[2]Contingent inequalities were first introduced in [4] for characterizing Lyapunov functions and value functions of a class of control problems and later, used in [84, 85] to investigate infinitesimal properties of Lyapunov and value functions in differential games. The "backward inequality" was introduced for the first time in [55, 56] to prove uniqueness of lower semicontinuous solutions of Hamilton–Jacobi–Bellman equations. See also [23, 25, 27, 28, 34, 35].

in two equivalent dual generalized senses by translating both the viability theorem and the invariance theorem characterizing the capture basin in terms of either tangential conditions or normal conditions, as it was done in a long series of papers by Frankowska. Using tangential conditions, we express the viability hyposolution as a solution to the Hamilton–Jacobi partial differential equation couched in terms of contingent hypoderivatives, whereas using normal conditions, we characterize it in terms of superdifferentials, as it was done independently by Barron-Jensen and Frankowska, in the spirit of nonsmooth analysis and viscosity solutions. The presence of inequality constraints complicates the technical formulation of the concept of solution at points where the solution touches the boundary of the constraint, above all in the superdifferential formulation, justifying the reason why we conclude this paper with this dual characterization.

2. Statement of the problem. This section states the problem of interest for this article. Section 3 provides all prerequisites for the concepts used in the later sections for a reader not familiar with viability theory and convex analysis. No prerequisites from viability theory are required to read this section.

2.1. Notation. For notational convenience, and in order to avoid multiplication of the letters used in the article, we have used the letters σ and τ is several different ways, which depend on context; i.e., for σ, we have the following definitions, based on context:

- Support function for some compact convex subset $A \subset X$, where $\sigma_A(v) := \sigma(A, v) := \sup_{u \in A} \langle u, v \rangle$ is the support function of A. Note that the first argument of σ is a set, while the second is a vector.
- Auxiliary min inf function $\sigma(t, x, u) := \min(t, \tau(x, u))$, defined in Theorem 5.1. Note that this function has three arguments, which are one scalar t and two vectors x and u of X.
- Auxiliary min inf functional $\sigma(t, x, u(\cdot)) = \min(t, \tau(x, u(\cdot)))$, defined in the proof of Theorem 5.1. Note that this function has three arguments, which are one scalar t, one vector $x \in X$, and function $u(\cdot)$ (measurable, integrable).

Similarly for the notation τ is used as a

- Dummy variable τ, for example, in integrals. Note that τ has no argument.
- Pseudotime $\tau(t)$, for example, in (13). Note that $\tau(t)$ has one argument t which corresponds to the running time of the corresponding differential inclusion.
- Auxiliary inf function $\tau(x, u) := \inf_{x+tu \notin K} t$, defined in Theorem 5.1. Note that this function has two inputs, which are vectors x and u of X.
- Auxiliary inf functional $\tau(x, u(\cdot)) := \inf_{x+\int_0^t u(\tau)d\tau \notin K} t$, defined in the proof of Theorem 5.1. Note that this function has two inputs, which is one vector $x \in X$ and function $u(\cdot)$ (measurable, integrable).

We will use this notation in the rest of the article, and in each of the cases of interest, the context, i.e., the number of arguments of τ, provides the proper definition.

2.2. Assumptions. We set $X := \mathbb{R}^n$. Let us consider

1. a concave function $\psi : X \mapsto \mathbb{R}$ satisfying growth conditions
$$\forall\, v \in X, \;\; \beta - \sigma_A(v) \;\le\; \psi(v) \;\le\; \delta - \sigma_A(v)$$
for some compact convex subset $A \subset X$, where $\sigma_A(v) := \sup_{u \in A} \langle u, v \rangle$ is the support function of A and where $\beta \le \delta$.
2. a bounded continuous function $v : \mathbb{R}_+ \mapsto \mathrm{Dom}(\psi)$.

3. an upper semicontinuous initial datum $\mathbf{N}_0 : X \mapsto \mathbb{R}_+$. We set $\mathbf{N}_0(0,x) := \mathbf{N}_0(x)$ and $\mathbf{N}_0(t,x) := -\infty$ if $t > 0$.
4. a closed subset $K \subset X$ with nonempty interior $\mathrm{Int}(K) =: \Omega$ and boundary $\partial K =: \Gamma$.
5. an upper semicontinuous boundary datum $\gamma : \mathbb{R}_+ \times X \mapsto \mathbb{R}$, satisfying[3]

$$\forall\, x \in \partial K, \quad \mathbf{N}_0(x) \ = \ \gamma(0,x) \ \text{ and } \ \forall\, t \geq 0, \ \ \forall\, x \in \mathrm{Int}(K), \ \ \gamma(t,x) \ = \ -\infty.$$

6. a Lipschitz function $\mathbf{b} : \mathbb{R}_+ \times X \mapsto \mathbb{R} \cup \{-\infty\}$ setting the upper constraint.

We shall also assume in this paper that the data satisfy the following consistency conditions:

$$(1) \quad \begin{cases} \text{(i)} & \forall\, x \in \partial K, \quad \mathbf{N}_0(x) \ = \ \gamma(0,x); \\ \text{(ii)} & \forall\, t \geq 0, \ \forall\, x \in K, \ \ \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \ \leq \ \mathbf{b}(t,x); \\ \text{(iii)} & \forall\, 0 \leq r \leq s, \ \forall\, x \in \partial K, \ \forall\, y \in \partial K, \ \ \gamma(r,x) - \gamma(s,y) \ \leq \ \left\langle \dfrac{1}{s-r} \displaystyle\int_r^s v(\tau) d\tau, x - y \right\rangle; \\ \text{(iv)} & \forall\, x \in K, \ \forall\, y \in \partial K, \ \ \mathbf{N}_0(x) \ \leq \ \displaystyle\inf_{s \geq 0} \left(\gamma(s,y) + \left\langle \dfrac{1}{s} \int_0^s v(\tau) d\tau, x - y \right\rangle \right), \end{cases}$$

which are needed only to prove that the Dirichlet/initial conditions are satisfied (see Theorem 6.1). When the function $v(\cdot) \equiv v$ is constant, they boil down to

$$\begin{cases} \text{(i)} & \forall\, x \in \partial K, \quad \mathbf{N}_0(x) \ = \ \gamma(0,x); \\ \text{(ii)} & \forall\, t \geq 0, \ \forall\, x \in K, \ \ \max\left(\mathbf{N}_0(t,x), \gamma(t,x)\right) \ \leq \ \mathbf{b}(t,x); \\ \text{(iii)} & \forall\, 0 \leq r \leq s, \ \forall\, x \in \partial K, \ y \in \partial K, \ \ \gamma(r,x) - \gamma(s,y) \ \leq \ \langle v, x - y \rangle; \\ \text{(iv)} & \forall\, x \in K, \ y \in \partial K, \ \ \mathbf{N}_0(x) \ \leq \ \displaystyle\inf_{s \geq 0} \gamma(s,y) + \langle v, x - y \rangle. \end{cases}$$

Under the above mentioned assumptions that are assumed throughout this paper, we shall solve the existence of a solution to the *nonhomogenous Hamilton–Jacobi equation*

$$(2) \qquad \forall\, t > 0, \ \ x \in \mathrm{Int}(K), \ \ \frac{\partial \mathbf{N}(t,x)}{\partial t} + \psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right) \ = \ \psi(v(t))$$

satisfying the *initial and Dirichlet conditions*

$$(3) \quad \begin{cases} \text{(i)} & \forall\, x \in K, \ \ \mathbf{N}(0,x) = \mathbf{N}_0(x) \text{ (initial condition)}, \\ \text{(ii)} & \forall\, t \geq 0, \ \forall\, x \in \partial K, \ \ \mathbf{N}(t,x) = \gamma(t,x) \text{ (Dirichlet boundary condition)} \end{cases}$$

and the *viability constraints*

$$(4) \qquad \forall\, t \geq 0, \ x \in K, \ \ \mathbf{N}(t,x) \ \leq \ \mathbf{b}(t,x) \text{ (upper inequality constraint)}.$$

[3]This is not mandatory. We can take any function such that $\mathrm{Dom}(\gamma) \subset K$ is strictly contained in K, an instance which may be useful for defining "guards" in impulse or hybrid systems, for instance. Boundary conditions are obtained when $\mathrm{Dom}(\gamma) = \partial K$.

Example. This equation is motivated by a commonly used first order model equation in highway traffic (*Lighthill–Whitham–Richards* equation) when $X := \mathbb{R}$ and $K := [\xi, +\infty[$, ψ a concave flux function vanishing at density 0 and at a jam density $\omega > 0$ and $\mathbf{N}(t,x)$ is the cumulated number of vehicles at time t and at location $x \in K$. Consistency conditions (1) read in this case: $\mathbf{N}_0(\xi) = \gamma(0,\xi)$ and

$$(5) \quad \left\{ \begin{array}{lll} \text{(i)} & \forall\, t \geq 0,\ \forall\, x \in K, & \max\left(\ \mathbf{N}_0(t,x), \gamma(t,x)\right) \ \leq\ \mathbf{b}(t,x); \\ \text{(ii)} & \forall\, 0 \leq r \leq s, & \gamma(r,\xi) - \gamma(s,\xi) \ \leq\ 0 \text{ (monotonocity)}; \\ \text{(iii)} & \forall x \in K, & \mathbf{N}_0(x) \ \leq\ \inf\limits_{s \geq 0} \left(\gamma(s,\xi) + \left\langle \dfrac{1}{s} \displaystyle\int_0^s v(\tau) d\tau, x - \xi \right\rangle \right). \end{array} \right.$$

Then the trapezoidal flux function (such as the one proposed by Daganzo [45, 46]) defined by

$$\psi(v) \ = \left\{ \begin{array}{lll} \nu^\flat v & \text{if} & v \leq \gamma^\flat, \\ \delta & \text{if} & v \in [\gamma^\flat, \gamma^\sharp], \\ \nu^\sharp(\omega - v) & \text{if} & v \geq \gamma^\sharp, \end{array} \right.$$

and the Greenshield flux function

$$\psi(v) \ = \left\{ \begin{array}{lll} \nu v & \text{if} & v \leq 0, \\ \frac{\nu}{\omega} v(\omega - v) & \text{if} & v \in [0, \omega], \\ \nu(\omega - v) & \text{if} & v \geq \omega \end{array} \right.$$

satisfy the assumptions on the function ψ with $A := [-\nu^\flat, +\nu^\sharp]$ and $A := [-\nu, +\nu]$, respectively (see Lemma 7.2 in section 7).

We characterize the solution to this nonhomogenous Dirichlet/initial value problem with inequality constraints through the capture basin of a target defined by the Dirichlet/initial conditions viable in an environment defined by inequality constraints under an adequate control system.

3. Prerequisite from viability theory and convex analysis.

Readers familiar with convex analysis and viability theory can skip this section and proceed directly to section 4.

3.1. Some prerequisites from viability theory.

Here, $X := \mathbb{R}^n$ and $Y := \mathbb{R}^m$ denote finite dimensional vector spaces. Let $f : X \times Y \mapsto X$ be a single-valued map describing the dynamics of a control system and $U : X \rightsquigarrow Y$ the set-valued map describing the state-dependent constraints on the controls.

First, any solution to a control system with state-dependent constraints on the controls

$$\left\{ \begin{array}{lll} \text{(i)} & x'(t) & = \ f(x(t), u(t)), \\ \text{(ii)} & u(t) & \in \ U(x(t)) \end{array} \right.$$

can be regarded as a solution to the differential inclusion $x'(t) \in F(x(t))$, where the right-hand side is defined by $F(x) := f(x, U(x)) := \{f(x,u)\}_{u \in U(x)}$.

We denote by $\mathcal{S}(x) \subset \mathcal{C}(0,\infty;X)$ the set of *absolutely continuous functions* $t \mapsto x(t) \in X$ satisfying

$$\text{for almost all } t \geq 0, \quad x'(t) \ \in \ F(x(t))$$

starting at time 0 at x: $x(0) = x$. The set-valued map $\mathcal{S} : X \rightsquigarrow \mathcal{C}(0,\infty;X)$ is called the *solution map* associated with F.

Therefore, from now on, as long as we do not need to implicate explicitly the controls in our study, we shall replace control problems by differential inclusions.

We shall say that K *is locally viable under* F if from every $x \in K$ starts a solution $x(\cdot)$ to the differential inclusion $x' \in F(x)$ viable in K on the nonempty interval $[0, T_x[$ in the sense that

$$\forall\, t \in [0, T_x[, \quad x(t) \in K$$

and that K is *viable* if we can take $T_x = +\infty$. It is *locally backward invariant under* F if for every $t_0 \in]0,+\infty[$, $x \in K$, for all solutions $x(\cdot)$ to the differential inclusion $x' \in F(x)$ arriving at x at time t_0, there exists $s \in [0, t_0[$ such that $x(\cdot)$ is viable in K on the interval $[s, t_0]$, and *backward invariant* if we can take $s = 0$.

We denote by

$$\text{Graph}(F) := \{(x,y) \in X \times Y \mid y \in F(x)\}$$

the *graph* of a set-valued map $F : X \rightsquigarrow Y$ and by $\text{Dom}(F) := \{x \in X | F(x) \neq \emptyset\}$ its *domain*.

Most of the results of viability theory are true whenever we assume that the dynamics is Marchaud as follows.

DEFINITION 3.1 (Marchaud map). *We shall say that F is a* Marchaud map *if*

$$\begin{cases} \text{(i)} & \textit{the graph of } F \textit{ is closed,} \\ \text{(ii)} & \textit{the values } F(x) \textit{ of } F \textit{ are convex,} \\ \text{(iii)} & \textit{the growth of } F \textit{ is linear:} \\ & \exists\, c > 0 \mid \forall\, x \in X, \|F(x)\| := \sup_{v \in F(x)} \|v\| \ \leq \ c(\|x\| + 1). \end{cases}$$

We shall say that F is λ-Lipschitz if

$$\forall\, x, y \in X, \quad F(x) \ \subset \ F(y) + \lambda \|x - y\| B,$$

where B is the unit ball.

This covers the case of *Marchaud control systems*, where $(x,u) \mapsto f(x,u)$ is continuous, affine with respect to the controls u and with linear growth, and when U is Marchaud.

We recall the following version of the important Theorem 3.5.2 of [5].

THEOREM 3.2 (the stability theorem). *Assume that $F : X \rightsquigarrow X$ is Marchaud. Then the solution map $\mathcal{S}$ is upper semicompact with nonempty values; this means that whenever $x_n \in X$ converge to x in X and $x_n(\cdot) \in \mathcal{S}(x_n)$ is a solution to the differential inclusion $x' \in F(x)$ starting at x_n, there exists a subsequence (again denoted by) $x_n(\cdot)$ converging to a solution $x(\cdot) \in \mathcal{S}(x)$ uniformly on compact intervals.*

We shall also need some other prerequisites from [5].

DEFINITION 3.3 (capture basin of a target). *Let $C \subset K \subset X$ be two subsets, C being regarded as a target, K as a constrained set. The subset* $\text{Capt}(K, C)$ *of initial* *states $x_0 \in K$ such that C is reached in finite time before possibly leaving K by at least one solution $x(\cdot) \in \mathcal{S}(x_0)$ starting at x_0 is called the* viable-capture basin *of C in K. A subset K is a* repeller *under F if all solutions starting from K leave K in finite time. A subset D is* locally backward invariant relative to K *if all backward solutions starting from D viable in K are actually viable in K.*

We recall the following result of [10].

THEOREM 3.4 (fixed-point characterization of capture basins). *The viable-capture basin* $\mathrm{Capt}(K, C)$ *of a target C viable in K is*

1. *the* largest *subset D satisfying $C \subset D \subset K$ and $D \subset \mathrm{Capt}(D, C)$,*
2. *the* smallest *subset D satisfying $C \subset D \subset K$ and $\mathrm{Capt}(K, D) \subset D$, and*
3. *the* unique *subset D satisfying $C \subset D \subset K$ and*

$$D = \mathrm{Capt}(K, D) = \mathrm{Capt}(D, C).$$

The subset $K\backslash C$ denotes the intersection of K and the complement of C; i.e., it is the set of elements of K which do not belong to C. We can derive the following characterization of capture basin (see [7]).

THEOREM 3.5 (viability characterization of capture basins). *Let us assume that F is Marchaud and that the subsets $C \subset K$ and K are closed. If $K\backslash C$ is a repeller (this is the case when K itself is a repeller), then the viable-capture basin* $\mathrm{Capt}(K, C)$ *of the target C under $\mathcal{S}$ is the* unique *closed subset satisfying $C \subset D \subset K$ and*

$$\text{(6)} \qquad \begin{cases} \text{(i)} & D\backslash C \text{ is } \textit{locally viable} \text{ under } \mathcal{S}, \\ \text{(ii)} & D \textit{ is locally backward invariant relative to } K. \end{cases}$$

The *contingent cone* $T_L(x)$ to $L \subset X$ at $x \in L$ is the set of directions $v \in X$ such that there exist sequences $h_n > 0$ converging to 0 and v_n converging to v satisfying $x + h_n v_n \in L$ for every n (see, for instance, [13] or [79] for more details). The *(regular) normal cone* is the polar cone $N_L(x) := (T_L(x))^-$ of the contingent cone.

DEFINITION 3.6 (Frankowska property). *Let us consider a set-valued map $F : X \rightsquigarrow X$ and two subsets $C \subset K$ and K. We shall say that a subset D between C and K satisfies the* Frankowska property *with respect to F if*

$$\text{(7)} \qquad \begin{cases} \text{(i)} & \forall\, x \in D\backslash C, \;\; F(x) \cap T_D(x) \neq \emptyset; \\ \text{(ii)} & \forall\, x \in D \cap \mathrm{Int}(K), \;\; -F(x) \subset T_D(x); \\ \text{(iii)} & \forall\, x \in D \cap \partial K, \;\; -F(x) \cap T_K(x) \subset T_D(x). \end{cases}$$

Actually, conditions (7)(ii), (iii) boil down to the same condition,

$$\forall\, x \in D, \;\; -F(x) \cap T_K(x) \subset T_D(x).$$

When K is further assumed to be backward locally invariant, the above conditions (7) boil down to

$$\text{(8)} \qquad \begin{cases} \text{(i)} & \forall\, x \in D\backslash C, \;\; F(x) \cap T_D(x) \neq \emptyset; \\ \text{(ii)} & \forall\, x \in D, \;\; -F(x) \subset T_D(x). \end{cases}$$

Theorem 3.5 and the viability[4] and invariance theorems imply the following.

[4]See, for instance, Theorems 3.2.4, 3.3.2, and 3.5.2 of [5].

THEOREM 3.7 (tangential characterization of capture basins). *Let us assume that F is Marchaud, that K is closed, and that a closed subset C satisfies* $\mathrm{Viab}_F(K\backslash C) = \emptyset$. *Then the viable-capture basin* $\mathrm{Capt}_F^K(C)$ *is*

1. *the* largest *closed subset D satisfying $C \subset D \subset K$ and*

$$\forall\, x \in D\backslash C, \quad F(x) \cap T_D(x) \;\neq\; \emptyset; \tag{9}$$

2. *the* unique *closed subset D satisfying the Frankowska property* (7) *if F is Lipschitz.*

We provide the dual characterization of the capture basin in terms of normal cones due to Frankowska.

LEMMA 3.8 (normal characterization of capture basins). *Let us assume that*

$$\forall\, x \in K, \quad 0 \in \mathrm{Int}(F(x) + T_K(x)).$$

Then property (7) *is equivalent to the* dual property

$$\left\{\begin{array}{ll} \text{(i)} & \forall\, x \in D\backslash C, \;\; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \geq 0; \\ \text{(ii)} & \forall\, x \in D \cap \mathrm{Int}(K), \;\; \forall\, p \in N_D(x), \;\; \sigma(F(x), -p) \leq 0; \\ \text{(iii)} & \forall\, x \in D \cap \partial K, \;\; \forall\, p \in N_D(x), \;\; \inf_{q \in N_K(x)} \sigma(F(x), q-p) \;\leq\; 0. \end{array}\right. \tag{10}$$

Proof. Whenever $0 \in \mathrm{Int}(F(x)+T_K(x))$, Proposition 3.9 on page 50 of [6] implies that the support function of $-F(x) \cap T_K(x)$ is the inf-convolution of the support functions of $-F(x)$ and $T_K(x)$ as follows:

$$\sigma(-F(x) \cap T_K(x), p) \;=\; \inf_{q \in N_K(x)} \sigma(F(x), q-p).$$

Consequently, inclusion $-F(x) \cap T_K(x) \subset T_D(x)$ is equivalent to

$$\forall\, p, \quad \inf_{q \in N_K(x)} \sigma(F(x), q-p) \;\leq\; \sigma(T_D(x), p),$$

which can be written

$$\forall\, p \in N_D(x), \quad \inf_{q \in N_K(x)} \sigma(F(x), q-p) \;\leq\; 0.$$

This concludes the proof. ∎

3.2. Some prerequisites of convex analysis. We gather in this section notations and some results on convex analysis for the convenience of the reader not familiar with this topic. Since the authors of most books on convex analysis have chosen to study convex functions rather than concave ones, we have chosen to associate with the concave function ψ the Fenchel transform $\varphi^\star$ of $\varphi := -\psi$ rather than the "concave Fenchel" transform $\psi^{\boxtimes}$ defined by the concave function

$$\psi^{\boxtimes}(u) \;:=\; \inf_{p \in \mathrm{Dom}(\psi)} [\langle p, u\rangle - \psi(p)] = -\varphi^\star(-u).$$

The basic theorem of convex analysis states that $\psi = \psi^{\boxtimes\boxtimes}$ if and only if ψ is concave, upper semicontinuous, and nontrivial (i.e., $\mathrm{Dom}(\psi) \;:=\; \{p \mid \varphi(p) > -\infty\} \neq 0$).

The epigraph $\mathcal{E}p(\varphi)$ of an extended function φ is the set of pairs $(x,\lambda) \in X \times \mathbb{R}$ such that $\varphi(x) \leq \lambda$, and the hypograph $\mathcal{H}yp(\psi)$ of a function ψ is the set of pairs $(p,\mu) \in X \times \mathbb{R}$ such that $\mu \leq \psi(p)$. Note that the hypograph of ψ is related to the epigraph of φ by the relation

$$(p,\lambda) \;\in\; \mathcal{H}yp(\psi) \text{ if and only if } (p,-\lambda) \;\in\; \mathcal{E}p(\varphi).$$

An extended function is lower semicontinuous if and only if its epigraph is closed and is upper semicontinuous if and only if its hypograph is closed.

Definition 3.9 (hypoderivatives and superdifferentials). *The hypoderivative $D_{\downarrow}\psi(p)$ and the epiderivative $D_{\uparrow}\varphi(p)$ are related to the tangent cones of the hypograph of ψ and epigraph of φ by the relations*

$$\mathcal{H}yp(D_{\downarrow}\psi(p)) \;:=\; T_{\mathcal{H}yp(\psi)}(p,\psi(p)) \text{ and } \mathcal{E}p(D_{\uparrow}\varphi(p)) \;:=\; T_{\mathcal{E}p(\varphi)}(p,\varphi(p)).$$

The superdifferential $\partial_{+}\psi(p)$ of the concave function ψ at p is defined by

$$u \;\in\; \partial_{+}\psi(p) \text{ if } \;\forall\, v \in X,\;\; \langle u,v\rangle \;\geq\; D_{\downarrow}\psi(p)(v),$$

and the subdifferential $\partial_{-}\varphi(p)$ is defined by

$$u \;\in\; \partial_{-}\varphi(p) \text{ if } \;\forall\, v \in X,\;\; \langle u,v\rangle \;\leq\; D_{\uparrow}\varphi(p)(v).$$

We infer that

$$\forall\, v \in X,\;\; D_{\downarrow}\psi(p)(v) = -D_{\uparrow}\varphi(p)(v)$$

and that

$$u \;\in\; \partial_{+}\psi(p) \text{ if and only if } u \;\in\; -\partial_{-}\varphi(p).$$

The *polar cone* P^{-} of a given set P is defined by

$$P^{-} = \{p \in X^{\star} \mid \forall x \in P,\; \langle p,x\rangle \leq 0\},$$

where $X^{\star}$ is the dual space of X, and the normal cone $N_K(x) := T_K(x)^{-}$ to K at $x \in K$ we use in this paper is the polar cone to the contingent cone to K at $x \in K$. The superdifferential $\partial_{+}\psi(p)$ and the subdifferential $\partial_{-}\varphi(p)$ are related to the normal cones of the hypograph of ψ and epigraph of φ by the relations

$$u \;\in\; \partial_{+}\psi(p) \text{ if and only if } (-u,1) \;\in\; N_{\mathcal{H}yp(\psi)}(p,\psi(p))$$

and

$$u \;\in\; \partial_{-}\varphi(p) \text{ if and only if } (u,-1) \;\in\; N_{\mathcal{E}p(\varphi)}(p,\varphi(p)).$$

Recall the Legendre inversion formula

$$u \;\in\; -\partial_{+}\psi(p) \text{ if and only if } p \in \partial_{-}\varphi^{\star}(u)$$

and the (decreasing) monotonicity property of superdifferential maps $p \rightsquigarrow \partial_{+}\psi(p)$ of a concave function,

$$\forall\, u_i \in \partial_{+}\psi(p_i),\; i = 1,2,\; \langle u_1 - u_2, p_1 - p_2\rangle \;\leq\; 0.$$

The subdifferential $\partial_{-}\sigma(K,p)$ of the support function is defined by the support zone $\{u \in K \text{ such that } \sigma(K,p) = \langle p,u\rangle\}$ of p in K. See [6] or [79] for more details.

4. The viability hyposolution. The assumption that the flux function ψ is concave and upper semicontinuous plays a crucial role for defining the viability hyposolution. Indeed, the Fenchel theorem allows us to characterize it by

$$\psi(p) \; = \inf_{u \in \mathrm{Dom}(\varphi^\star)} [\varphi^\star(u) - \langle p, u\rangle], \tag{11}$$

where $\varphi^\star$ is the Fenchel conjugate function, which is the convex lower semicontinuous function defined by

$$\varphi^\star(u) \; := \sup_{p \in \mathrm{Dom}(\psi)} [\langle p, u\rangle + \psi(p)]. \tag{12}$$

We introduce the auxiliary characteristic control system,

$$\begin{cases} \tau'(t) \; = \; -1, \\ x'(t) \; = \; u(t), \\ y'(t) \; = \; \varphi^\star(u(t)) - \psi(v(\tau(t))), \text{ where } u(t) \; \in \; \mathrm{Dom}(\varphi^\star). \end{cases} \tag{13}$$

The function $\tau(t)$ corresponds to a *countdown*, i.e., a pseudotime decaying at rate -1. This technique of augmentation of a dynamics by $\tau'(t) = -1$ is common in the Hamilton–Jacobi partial differential equation literature; see, for example, [55, 56]. To be rigorous, we have to mention *once and for all* that the controls $u(\cdot)$ are measurable integrable functions with values in $\mathrm{Dom}(\varphi^\star)$, and thus, ranging over $L^1(0,\infty;\mathrm{Dom}(\varphi^\star))$, and that the above system of differential equations is valid for almost all $t \geq 0$.

We set $\mathbf{c}(t,x) := \max(\mathbf{N}_0(t,x), \gamma(t,x))$, defined by

$$\mathbf{c}(t,x) \; := \begin{cases} -\infty & \text{if} \quad t > 0 \text{ and } x \in \Omega := \mathrm{Int}(K), \\ \mathbf{N}_0(x) & \text{if} \quad t = 0 \text{ and } x \in K, \\ \gamma(t,x) & \text{if} \quad t \geq 0 \text{ and } x \in \Gamma := \partial K. \end{cases}$$

We introduce the environment $\mathcal{K} := \mathcal{H}yp(\mathbf{b})$ is the subset of triples $(T,x,y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{b}(T,x)$ (this is the *hypograph* of the function $\mathbf{b}$) and the target $\mathcal{C} := \mathcal{H}yp(\mathbf{c})$ defined as the subset of triples $(T,x,y) \subset \mathbb{R}_+ \times X \times \mathbb{R}$ such that $y \leq \mathbf{c}(T,x)$ (which is the *hypograph* of the function $\mathbf{c}$).

DEFINITION 4.1 (the viability hyposolution). *The capture basin* $\mathrm{Capt}_{(13)}(\mathcal{K},\mathcal{C})$ *of a target* $\mathcal{C}$ *viable in the environment* $\mathcal{K}$ *under control system* (13) *is the subset of initial states* (t,x,y) *such that there exists a measurable control* $u(\cdot)$ *such that the associated solution*

$$s \mapsto \left(t - s, x + \int_0^s u(\tau) d\tau, y + \int_0^s (\varphi^\star(u(\tau)) - \psi(v(t-\tau))) d\tau\right)$$

is viable in $\mathcal{K}$ *until it reaches the target* $\mathcal{C}$.

The viability hyposolution $\mathbf{N}$ *is defined by*

$$\mathbf{N}(t,x) \; := \sup_{(t,x,y) \in \mathrm{Capt}_{(13)}(\mathcal{K},\mathcal{C})} y. \tag{14}$$

Note that $\mathcal{H}yp(\mathbf{M}) \subset \mathcal{H}yp(\mathbf{N})$ if and only if $\mathbf{N}$ is pointwise larger than $\mathbf{M}$. Therefore, using hypographs, the two order relations coincide.

We shall prove the following.

THEOREM 4.2 (nonhomogenous Dirichlet/initial value problem with inequality constraints). *The viability hyposolution* $\mathbf{N}$ *defined by* (14) *is the* largest *upper semicontinuous solution to Hamilton–Jacobi equation* (2) *satisfying initial and Dirichlet conditions* (3) *and inequality constraints* (4) *in both the contingent solution sense* (*see* (28)) *and in the contingent normal sense* (*see* (31)). *If the functions* ψ, $\varphi^\star$, *and* v *are furthermore Lipschitz, then the viability hyposolution* $\mathbf{N}$ *is its* unique *upper semicontinuous solution in both the contingent Frankowska sense* (*see* (29) *and* (30)) *and in the Barron-Jensen/Frankowska sense* (*see* (32), (33) *and Theorems* 8.1 *and* 9.1 *for the precise statement*).

Remark. Note that the concept of "largest solution" coincides with the pointwise one. Inequalities (32) and (33) defining the concept of generalized solutions depend on the type of assumption made on the flux function ψ. The present work uses a standard assumption in transportation engineering, namely that the flux ψ is *concave*, whereas a majority of mathematical studies of Hamilton–Jacobi partial differential equations assume that ψ is *convex*. This change induces an unusual modification of the signs in the inequalities defining the concept of Barron-Jensen/Frankowska solutions. Under the assumption of convex fluxes, this solution would be lower semicontinuous (and sometimes called the lower semicontinuous solution to Hamilton–Jacobi equations). Under the assumption imposed by transportation engineering considerations, the present solution is upper semicontinuous and the signs in inequalities (32) and (33) are changed. The mathematical formulation of the engineering problem thus led to a slightly unusual framework for solving this Hamilton–Jacobi equation. The convex version of this paper will appear in the forthcoming book [9].

We shall derive this theorem and other results from the properties of capture basins gathered in [7, 10]. Since the capture basin of a union of targets is the union of the capture basins of these targets, we infer that whenever $\mathbf{c} := \sup_i \mathbf{c}_i$ is the upper envelope of a family of functions $\mathbf{c}_i$, then the viability hyposolution is the upper envelope

$$\forall\, t \geq 0,\ x \in X,\ \ \mathbf{N}(t,x) \ = \ \sup_i \mathbf{N}_{\mathbf{c}_i}(t,x)$$

of the solutions $\mathbf{N}_{\mathbf{c}_i}$ (sup-linearity property).

In particular, since $\mathbf{c}(t,x) := \max(\ \mathbf{N}_0(t,x), \gamma(t,x))$ (extended to $-\infty$ when $t > 0$ or $x \in \mathrm{Int}(K)$), we obtain the decomposition formula

$$\mathbf{N}(t,x) \ = \ \max\left(\mathbf{N}_{\mathbf{N_0}}(\mathbf{t},\mathbf{x}), \mathbf{N}_\gamma(\mathbf{t},\mathbf{x})\right) \tag{15}$$

in terms of initial condition component $\mathbf{N}_{\mathbf{N}_0}$ and the Dirichlet component $\mathbf{N}_\gamma$ of the viability hyposolution $\mathbf{N}$ defined by

$$\left\{\begin{array}{l} \mathbf{N}_{\mathbf{N_0}}(t,x) \ := \ \sup_{(t,x,y)\in \mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{N}_0))}\, y, \\ \mathbf{N}_\gamma(t,x) \ := \ \sup_{(t,x,y)\in \mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\gamma))}\, y. \end{array}\right.$$

The viability hyposolution depends continuously on the data in the following sense: If the hypographs of a sequence of initial data $\mathbf{c}_j$ converge in the upper Painlevé–Kuratowski sense (see, for instance, [13]) to the hypograph of data $\mathbf{c}$, then the upper

Painlevé–Kuratowski limit of the hypographs of the solutions $\mathbf{N}_j$ associated with data $\mathbf{c}_j$ is contained in the hypograph of the hyposolution $\mathbf{N}$ associated with data $\mathbf{c}$ (upper hypocontinuity property). If the functions ψ, $\varphi^\star$, and v are furthermore Lipschitz, the hypograph of the hyposolution $\mathbf{N}$ associated with data $\mathbf{c}$ is contained in the lower Painlevé–Kuratowski limit of the hypographs of the solutions $\mathbf{N}_j$ associated with data $\mathbf{c}_j$ (lower hypocontinuity property), so that both the upper and lower limits coincide with the hypograph of the hyposolution $\mathbf{N}$ (hypoconvergence of the solutions; see [13] or [79] for a definition). These statements follow from Theorem 6.6 of [7] stating that if the system is both Marchaud and Lipschitz, the capture basin of a Painlevé–Kuratowski limit of targets is the Painlevé–Kuratowski limit of the capture basins of the targets.

5. Lax–Hopf formula and estimates of the solution.

5.1. The Lax–Hopf formula for Dirichlet problems. When there is no inequality constraint, we prove that the viability hyposolution can be represented explicitly as a simple maximization problem involving the Fenchel conjugate $\varphi^\star$ defined by (12).

THEOREM 5.1 (the Lax–Hopf formula). *Let us consider the case without inequality constraints and set*

$$\tau(x,u) \;:=\; \inf_{x+tu\notin K} t \qquad \textit{and} \qquad \sigma(t,x,u) \;:=\; \min(t,\tau(x,u)).$$

Then the viability hyposolution (17) *can be written*

$$(16)\qquad \begin{cases} \mathbf{N}(t,x) \\ = \displaystyle\sup_{\{u\in \mathrm{Dom}(\varphi^\star)\}} \Bigg[\mathbf{c}(t-\sigma(t,x,u), x+\sigma(t,x,u)u) - \sigma(t,x,u)\varphi^\star(u) \\ \qquad\qquad\qquad + \displaystyle\int_{t-\sigma(t,x,u)}^{t} (\psi(v(\tau)))d\tau\Bigg]. \end{cases}$$

Using the decomposition $\mathbf{N}(t,x) = \max\left(\mathbf{N}_{\mathbf{N}_0}(t,x), \mathbf{N}_\gamma(t,x)\right)$, we derive the more explicit formula

$$(17)\qquad \begin{cases} \mathbf{N}_{\mathbf{N}_0}(t,x) \;=\; \displaystyle\sup_{u\in\mathrm{Dom}(\varphi^\star)} (\mathbf{N}_0(x+tu) - t\varphi^\star(u)) + \int_0^t \psi(v\,(\tau))d\tau, \\ \mathbf{N}_\gamma(t,x) \;=\; \displaystyle\sup_{\{u\in\mathrm{Dom}(\varphi^\star)\mid\tau(x,u)\le t\}} \Bigg[\gamma\,(t-\tau(x,u), x+\tau(x,u)u) - \tau(x,u)\varphi^\star(u) \\ \qquad\qquad\qquad + \displaystyle\int_{t-\tau(x,u)}^{t} \psi(v\,(\tau))d\tau\Bigg] \end{cases}$$

involving the initial and Dirichlet conditions.

Proof. Let us associate the following with $u(\cdot)$:

$$\tau(x,u(\cdot)) \;:=\; \inf_{x+\int_0^t u(\tau)d\tau\notin K} t \qquad \text{and} \qquad \sigma(t,x,u(\cdot)) \;=\; \min(t,\tau(x,u(\cdot))).$$

The formula is derived from the general representation formula

$$\begin{cases} \mathbf{N}(t,x) = \sup_{u(\cdot)} \Bigg[\mathbf{c}\left(t-\sigma(t,x,u(\cdot)), x+\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right) - \int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau \\ \qquad\qquad\qquad + \int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau \Bigg] \end{cases}$$

of the viability hyposolution without constraints given by Corollary 5.6.

We proceed in two steps.

1. Taking constant controls $u(\cdot) \equiv u$ and observing that $\tau(x,u) = \tau(x,u(\cdot))$, we infer that

$$\sup_{u \in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u)\right) + \int_{t-\sigma(t,x,u)}^{t} \psi(v(\tau))d\tau \ \leq \ \mathbf{N}(t,x).$$

2. Let us associate with $u(\cdot)$ the function $\widehat{u}$ defined by $\widehat{u}(s) := \frac{1}{s}\int_0^s u(\tau)d\tau$. We first observe that

$$\tau(x,u(\cdot)) \ = \ \tau(x, \widehat{u}(\tau(x,u(\cdot)))).$$

Since $\varphi^\star$ is convex and lower semicontinuous and ψ is concave and upper semicontinuous, the Jensen inequality implies

$$\varphi^\star\left(\frac{1}{s}\int_0^s u(\tau)d\tau\right) \ \leq \ \frac{1}{s}\int_0^s \varphi^\star(u(\tau))d\tau$$

and

$$\frac{1}{s}\int_0^s \psi(v(t-\tau))d\tau \ \leq \ \psi\left(\frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau\right),$$

and thus

$$\int_0^s \psi(v(t-\tau))d\tau - \int_0^s \varphi^\star(u(\tau))d\tau \ \leq \ s\left(\psi\left(\frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau\right) - \varphi^\star\left(\widehat{u}(s)\right)\right). \tag{18}$$

Consequently, setting $t^\sharp := \sigma(t,x,u(\cdot)) = \tau(x, \widehat{u}(\sigma(t,x,u(\cdot))))$ and $u^\sharp := \widehat{u}(t^\sharp)$, we obtain inequalities

$$\begin{cases} \mathbf{c}\left(t-t^\sharp, x+\int_0^{t^\sharp} u(\tau)d\tau\right) - \int_0^{t^\sharp} \varphi^\star(u(\tau))d\tau + \int_{t-t^\sharp}^{t} \psi(v(\tau))d\tau \\ \leq \ \mathbf{c}\left(t-t^\sharp, x+t^\sharp u^\sharp\right) - t^\sharp \varphi^\star(u^\sharp) + \int_{t-t^\sharp}^{t} \psi(v(\tau))d\tau \\ \leq \ \sup_{\{u \in \mathrm{Dom}(\varphi^\star)\}} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u)\right) \\ \qquad\qquad + \int_{t-\sigma(t,x,u)}^{t} \psi(v(\tau))d\tau. \end{cases}$$

Therefore, by taking the supremum, we obtain

$$\mathbf{N}(t,x) \ \leq \ \sup_{u\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u)\right) + \int_{t-\sigma(t,x,u)}^{t} \psi(v(\tau))d\tau.$$

This completes the proof of the Lax–Hopf inequality.

COROLLARY 5.2 (case of the traffic model). *When $X := \mathbb{R}$, $K := [\xi, +\infty[$, ψ is a concave flux function vanishing at density* 0 *and at a jam density $\omega > 0$, and $\mathbf{N}(t,x)$ is the cumulated number of vehicles at time t and at location $x \in K$. Consistency conditions* (5) *imply the existence of a unique upper semicontinuous solution $\mathbf{N}(t,x) = \max\left(\mathbf{N}_{\mathbf{N}_0}(t,x), \mathbf{N}_\gamma(t,x)\right)$ to this problem in the Barron-Jensen/Frankowska sense satisfying the Lax–Hopf formula*

(19)
$$\begin{cases} \mathbf{N}_{\mathbf{N}_0}(t,x) \ = \ \sup\limits_{u\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{N}_0(x+tu) - t\varphi^\star(u) + \displaystyle\int_0^t \psi(v\left(t-\tau\right))d\tau\right), \\ \mathbf{N}_\gamma(t,x) \ = \\ \sup\limits_{\{u\in \mathrm{Dom}(\varphi^\star) | u \leq \frac{\xi-x}{t}\}} \left(\gamma\left(t - \dfrac{\xi-x}{u}, \xi\right) - \dfrac{\xi-x}{u}\varphi^\star(u) + \displaystyle\int_0^{\frac{\xi-x}{u}} \psi(v\left(t-\tau\right))d\tau\right). \end{cases}$$

5.2. A posteriori estimates. The maximum principle, an a priori upper estimate of a solution of a partial differential equation (whether it exists or not) is here obtained as an a posteriori estimate, a property of the *viability hyposolution*.

PROPOSITION 5.3 (upper estimate of the viability hyposolution). *The viability hyposolution satisfies*

$$\mathbf{N}(t,x) \ \leq \ \sup_{u\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle\right).$$

*Consequently, the viability hyposolution satisfies the following (*a posteriori *instead of* a priori*) estimate*

$$\mathbf{N}(t,x) \ \leq \ \sup_{t\geq 0,\ x\in K} \mathbf{c}(t,x) + t \ \mathrm{Diam}(\mathrm{Dom}(\varphi^\star)) \sup_{t\geq 0} \|v(t)\|$$

(maximum principle).

Proof. Fix $u \in \mathrm{Dom}(\varphi^\star)$ and set $\sigma(t,x,u) =: s$. Definition (12) of the conjugate function implies

$$\psi\left(\frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau\right) - \varphi^\star(u) \ \leq \ -\left\langle \frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau, u\right\rangle. \tag{20}$$

Consequently,

$$\begin{cases} \mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \sigma(t,x,u)\varphi^\star(u) + \displaystyle\int_{t-\sigma(t,x,u)}^{t} (\psi(v(\tau)))d\tau \\ \leq \ \mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \displaystyle\int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle \\ \leq \ \sup\limits_{w\in \mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,w), x+\sigma(t,x,w)w\right) - \left\langle w, \displaystyle\int_{t-\sigma(t,x,w)}^{t} v(\tau)d\tau \right\rangle\right). \end{cases}$$

Taking the supremum over $u \in \mathrm{Dom}(\varphi^\star)$, Lax–Hopf formula (16) implies the upper estimate

$$\mathbf{N}(t,x) \ \leq \ \sup_{u\in\mathrm{Dom}(\varphi^\star)} \left(\mathbf{c}\left(t-\sigma(t,x,u), x+\sigma(t,x,u)u\right) - \left\langle u, \int_{t-\sigma(t,x,u)}^{t} v(\tau)d\tau \right\rangle \right).$$

This completes the proof. ∎

In the same way, we provide a lower estimate of the solution.

PROPOSITION 5.4 (lower estimate). *Assume that $v(t) := v$ is constant and, for simplicity, that the function ψ is differentiable. Then*

$$\mathbf{c}(t-\sigma(t,x,-\psi'(v)), x-\sigma(t,x,-\psi'(v))\psi'(v)) + \sigma(t,x,\psi'(v))\,\langle v,\psi'(v)\rangle \ \leq \ \mathbf{N}(t,x).$$

Consequently, the hyposolution is nonnegative on its positivity domain $\mathrm{Dom}_+(\mathbf{N})$, *defined as the subset of pairs $(t,x) \in \mathbb{R}_+ \times K$ such that*

$$\mathbf{c}(t-\sigma(t,x,-\psi'(v)), x-\sigma(t,x,-\psi'(v))\psi'(v)) + \sigma(t,x,-\psi'(v))\,\langle v,\psi'(v)\rangle \ \geq \ 0.$$

Proof. By Definition 3.9 of the superdifferential,

$$\forall\, u \in \partial_+\psi(v), \ \ \psi(v) - \varphi^\star(-u) \ = \ \langle v,u\rangle.$$

Therefore, if ψ is differentiable, taking $u := -\psi'(v)$ as the unique element of $-\partial_+\psi(v) = \partial_-\varphi(v)$, the Legendre equality $\psi(v) - \varphi^\star(-\psi'(v)) \ = \ \langle v,\psi'(v)\rangle$ yields

$$\left\{ \begin{array}{l} \mathbf{c}(t-\sigma(t,x,-\psi'(v)), x-\sigma(t,x,-\psi'(v))\psi'(v)) + \sigma(t,x,-\psi'(v))\,\langle v,\psi'(v)\rangle \\ = \ \mathbf{c}(t-\sigma(t,x,u), x+\sigma(t,x,u)u) - \sigma(t,x,u)\,\langle v,u\rangle \\ = \ \mathbf{c}(t-\sigma(t,x,u), x+\sigma(t,x,u)u) + \sigma(t,x,u)(\psi(v)-\varphi^\star(u)) \leq \mathbf{N}(t,x) \end{array} \right.$$

thanks to the Lax–Hopf formula. ∎

5.3. General representation formula.

We have derived Lax–Hopf formula (16) from a general representation formula (21) valid when there is viability constraints.

THEOREM 5.5 (representation formula of the viability solution (the case with constraints)). *We already set*

$$\tau(x,u(\cdot)) \ := \inf_{x+\int_0^t u(\tau)d\tau \notin K} t \quad \textit{and} \quad \sigma(t,x,u(\cdot)) \ = \ \min(t,\tau(x,u(\cdot))).$$

The viability hyposolution can be represented in the form

$$(21)\quad \left\{ \begin{array}{l} \mathbf{N}(t,x) \ = \ \sup\limits_{u(\cdot)} \Bigg[\min \Bigg(\mathbf{c}\Bigg(t-\sigma(t,x,u(\cdot)), x + \displaystyle\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau \Bigg) \\ - \displaystyle\int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau + \int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau, \\ \inf\limits_{s\in[0,\sigma(t,x,u(\cdot))]} \Bigg(\mathbf{b}\Bigg(t-s, x+\displaystyle\int_0^s u(\tau)d\tau \Bigg) - \int_0^s \varphi^\star(u(\tau))d\tau + \int_{t-s}^{t} \psi(v(\tau))d\tau \Bigg) \Bigg) \Bigg]. \end{array} \right.$$

Using the decomposition $\mathbf{N}(t,x) = \max(\mathbf{N}_{N_0}(t,x), \mathbf{N}_\gamma(t,x))$, this formula boils down to

$$\left\{\begin{array}{l} \mathbf{N}_{N_0}(t,x) \;=\; \sup\limits_{u(\cdot)} \Bigg[\min\Bigg(\mathbf{N}_0\left(x+\int_0^t u(\tau)d\tau\right) - \int_0^t \varphi^\star(u(\tau))d\tau + \int_0^t \psi(v(\tau))d\tau, \\ \qquad\qquad \inf\limits_{s\in[0,t]} \left(\mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right) - \int_0^s \varphi^\star(u(\tau))d\tau + \int_{t-s}^t \psi(v(\tau))d\tau\right)\Bigg]\end{array}\right.$$

and

$$\left\{\begin{array}{l} \mathbf{N}_\gamma(t,x) \;=\; \sup\limits_{\{u(\cdot)|\tau(x,u(\cdot))\le t\}} \Bigg[\min\Bigg(\gamma\left(t-\tau(x,u(\cdot)), x+\int_0^{\tau(x,u(\cdot))} u(\tau)d\tau\right) \\ -\int_0^{\tau(x,u(\cdot))} \varphi^\star(u(\tau))d\tau + \int_{t-\tau(x,u(\cdot))}^t \psi(v(\tau))d\tau, \\ \inf\limits_{s\in[0,\tau(x,u(\cdot))]} \left(\mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right) - \int_0^s \varphi^\star(u(\tau))d\tau + \int_{t-s}^t \psi(v(\tau))d\tau\right)\Bigg)\Bigg].\end{array}\right.$$

Proof. We begin by observing that a solution $(\tau(\cdot), x(\cdot), y(\cdot))$ to control system (13) starting from (t,x,y) is given by $\tau(s) = t-s$, $x(s) = x + \int_0^s u(r)dr$ and

$$y(s) \;=\; y + \int_0^s (\varphi^\star(u(r)) - \psi(v(t-r)))dr$$

for some $u(\cdot)$.

Therefore, to say that (t,x,y) belongs to the capture basin $\mathrm{Capt}_{(13)}(\mathcal{K}, \mathcal{C})$ amounts to saying that there exists a solution $(\tau(\cdot), x(\cdot), y(\cdot))$ to the characteristic control system (13) starting from (t,x,y) and $t^\star \in [0,t]$ such that

1. $(t-t^\star, x(t^\star), y(t^\star))$ belongs to the target $\mathcal{C}$, i.e., such that

$$\begin{aligned} y(t^\star) := y + \int_0^{t^\star} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \;&\le\; \mathbf{c}\,(t-t^\star, x(t^\star)) \\ &=\; \mathbf{c}\left(t-t^\star, x+\int_0^{t^\star} u(\tau)d\tau\right). \end{aligned}$$

2. For all $s \in [0,t^\star]$, $(t-s, x(s), y(s))$ belongs to the environment $\mathcal{K}$, i.e., such that

$$\begin{aligned} y(s) = y + \int_0^s (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \;&\le\; \mathbf{b}(t-s, x(s)) \\ &=\; \mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right). \end{aligned}$$

This implies that

$$y \;\le \min\left(\begin{array}{l} \mathbf{c}\left(t-t^\star, x+\int_0^{t^\star} u(\tau)d\tau\right) - \int_0^{t^\star} (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau, \\ \inf\limits_{s\in[0,t^\star]} \mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right) - \int_0^s (\varphi^\star(u(\tau)) - \psi(v(t-\tau)))d\tau \end{array}\right).$$

Since y is finite, this implies that $\mathbf{c}(t-t^\star, x+\int_0^{t^\star} u(\tau)d\tau)$ must be finite, and thus, that

1. either $t-t^\star=0$, in which case $\mathbf{c}(t-t^\star, x+\int_0^{t^\star} u(\tau)d\tau) = \mathbf{N}_0(x+\int_0^t u(\tau)d\tau)$;
2. or $x(t^\star)\in\partial K$, which means that $t^\star=\tau(x,u(\cdot))=\sigma(t,x,u(\cdot))\le t$, in which case

$$\mathbf{c}\left(t-t^\star, x+\int_0^{t^\star} u(\tau)d\tau\right) = \gamma\left(t-\sigma(t,x,u(\cdot)), x+\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right).$$

This implies that $\mathbf{N}(t,x)\le\mathbf{V}(t,x)$, where

$$\left\{\begin{array}{l} \mathbf{V}(t,x) \;=\; \sup\limits_{u(\cdot)}\left(\min\left(\mathbf{c}\left(t-\sigma(t,x,u(\cdot)), x+\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right)\right.\right. \\ \qquad\qquad\qquad\qquad -\int_0^{\sigma(t,x,u(\cdot))}\varphi^\star(u(\tau))d\tau+\int_{t-\sigma(t,x,u(\cdot))}^t \psi(v(\tau))d\tau, \\ \left.\left.\inf\limits_{s\in[0,\sigma(t,x,u(\cdot))]}\left(\mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right)-\int_0^s\varphi^\star(u(\tau))d\tau+\int_{t-s}^t\psi(v(\tau))d\tau\right)\right)\right). \end{array}\right.$$

For proving the converse inequality, we associate with every $\varepsilon>0$ a control $t\mapsto u_\varepsilon(t)\in\mathrm{Dom}(\varphi^\star)$ such that

$$\left\{\begin{array}{l} \mathbf{V}(t,x)-\varepsilon \;\le\; \min\left(\mathbf{c}\left(t-\sigma(t,x,u_\varepsilon(\cdot)), x+\int_0^{\sigma(t,x,u_\varepsilon(\cdot))} u_\varepsilon(\tau)d\tau\right)\right. \\ \qquad\qquad\qquad\qquad -\int_0^{\sigma(t,x,u_\varepsilon(\cdot))}\varphi^\star(u_\varepsilon(\tau))d\tau+\int_{t-\sigma(t,x,u_\varepsilon(\cdot))}^t \psi(v(\tau))d\tau, \\ \left.\inf\limits_{s\in[0,\sigma(t,x,u_\varepsilon(\cdot))]}\left(\mathbf{b}\left(t-s, x+\int_0^s u(\tau)d\tau\right)-\int_0^s\varphi^\star(u_\varepsilon(\tau))d\tau+\int_{t-s}^t\psi(v(\tau))d\tau\right)\right). \end{array}\right.$$

Therefore, setting $x_\varepsilon(t):=x+\int_0^t u_\varepsilon(s)ds$ and

$$y_\varepsilon(t) \;:=\; \mathbf{V}(t,x)-\varepsilon+\int_0^t\left(\varphi^\star(u_\varepsilon(r))-\psi(v(t-r))\right)dr$$

we observe that the function $s\mapsto(t-s, x_\varepsilon(s), y_\varepsilon(s))$ starts from $(t,x,\mathbf{V}(t,x)-\varepsilon)$, is a solution to characteristic control system (13), viable in $\mathcal{K}$ for $s\le\sigma(t,x,u_\varepsilon(\cdot))$ because

$$y_\varepsilon(s) \;=\; \mathbf{V}(t,x)-\varepsilon+\int_0^s\left(\varphi^\star(u_\varepsilon(r))-\psi(v(t-r))dr\right) \;\le\; \mathbf{b}\left(t-s, x_\varepsilon(s)\right),$$

and reaches the target $\mathcal{C}:=\mathcal{H}yp(\mathbf{c})$ at time $t_\varepsilon:=\sigma(t,x,u_\varepsilon(\cdot))$,

$$y_\varepsilon(t_\varepsilon) \;=\; \mathbf{V}(t,x)-\varepsilon+\int_0^{t_\varepsilon}\left(\varphi^\star(u_\varepsilon(r))-\psi(v(t-r))\right)dr \;\le\; \mathbf{c}\left(t-t_\varepsilon, x_\varepsilon(t_\varepsilon)\right).$$

This implies that $(t,x,\mathbf{V}(t,x)-\varepsilon)$ belongs to the capture basin $\mathrm{Capt}_{(13)}(\mathcal{K},\mathcal{C})$, and thus, that $\mathbf{V}(t,x)-\varepsilon\le\mathbf{N}(t,x)$. Letting ε converge to 0 provides the converse inequality, and thus, the representation formula we were looking for. $\square$

COROLLARY 5.6 (representation formula of the viability solution (the case without constraints)). *Without inequality constraints, the viability hyposolution can be represented in the form*

$$\begin{cases} \boldsymbol{N}(t,x) \;=\; \sup\limits_{u(\cdot)} \left(\displaystyle\int_{t-\sigma(t,x,u(\cdot))}^{t} \psi(v(\tau))d\tau \right. \\ \qquad\qquad \left. + \,\mathbf{c}\left(t-\sigma(t,x,u(\cdot)), x + \displaystyle\int_0^{\sigma(t,x,u(\cdot))} u(\tau)d\tau\right) - \displaystyle\int_0^{\sigma(t,x,u(\cdot))} \varphi^\star(u(\tau))d\tau \right). \end{cases}$$

6. Dirichlet/initial conditions and inequality constraints. We begin by checking that the viability hyposolution satisfies the initial condition, the Dirichlet condition, and the inequality constraints.

THEOREM 6.1 (Dirichlet/initial conditions and inequality constraints). *Consistency conditions* (1) *imply that the viability hyposolution satisfies the initial and Dirichlet conditions* (3) *and inequality constraints* (4).

Proof. Inclusions

$$\mathcal{C} \;:=\; \mathcal{H}yp(\mathbf{c}) \;\subset\; \mathrm{Capt}_{(13)}(\mathcal{K},\mathcal{C}) \;\subset\; \mathcal{K} \;:=\; \mathcal{H}yp(\mathbf{b})$$

imply that

$$\forall\, t \geq 0,\; \forall\, x \in K, \quad \mathbf{c}(t,x) \;\leq\; \mathbf{N}(t,x) \;\leq\; \mathbf{b}(t,x),$$

and thus inequality constraint $\mathbf{N}(t,x) \leq \mathbf{b}(t,x)$ and inequalities $\mathbf{N}_0(x) \leq \mathbf{N}(0,x)$ for all $x \in K$ and $\gamma(t,x) \leq \mathbf{N}(t,x)$ for all $t \geq 0$ and $x \in \partial K$. We now prove by contradiction that consistency conditions (1) imply converse inequalities $\mathbf{N}_0(x) \geq \mathbf{N}(0,x)$ for all $x \in K$ and $\gamma(t,x) \geq \mathbf{N}(t,x)$ for all $t \geq 0$ and $x \in \partial K$ that we summarize in

$$\forall\, (t,x) \in \mathrm{Dom}(\mathbf{c}), \quad \mathbf{N}(t,x) \leq \mathbf{c}(t,x).$$

Assume that there exist $(t,\xi) \in \mathrm{Dom}(\mathbf{c})$ and $\varepsilon > 0$ such that

$$\mathbf{N}(t,\xi) \;=\; \mathbf{c}(t,\xi) + \varepsilon.$$

Since $(t,\xi,\mathbf{N}(t,\xi))$ belongs to the capture basin $\mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$, there exists a solution $(\tau(\cdot),x(\cdot),y(\cdot))$ to the characteristic control system (13) starting from $(t,\xi,\mathbf{N}(t,\xi))$ and $t^\star > 0$ such that $(t-t^\star, x(t^\star), y(t^\star))$ belongs to the hypograph $\mathcal{H}yp(\mathbf{c})$; i.e., setting $x(t^\star) = \xi + \int_0^{t^\star} u(\tau)d\tau = \eta$, we obtain

$$y(t^\star) \;=\; \mathbf{N}(t,\xi) + \int_0^{t^\star} \varphi^\star(u(\tau))d\tau - \int_0^{t^\star} \psi(v(t-\tau))d\tau \;\leq\; \mathbf{c}(t-t^\star,\eta).$$

Inequality (18) and definition (20) imply

$$\int_0^s \psi(v(t-\tau))d\tau - \int_0^s \varphi^\star(u(\tau))d\tau \leq \; -\left\langle \frac{1}{s}\int_{t-s}^{t} v(\tau)d\tau, \widehat{u}(s) \right\rangle. \tag{22}$$

Piecing these inequalities together and taking $s = t^\star$, we infer that

$$\begin{cases} \mathbf{c}(t,\xi) + \varepsilon + \left\langle \dfrac{1}{t^\star} \displaystyle\int_{t-t^\star}^{t} v(\tau)d\tau, \eta - \xi \right\rangle \\ \le \mathbf{N}(t,\xi) + \displaystyle\int_0^{t^\star} \varphi^\star(u(\tau))d\tau - \int_0^{t^\star} \psi(v(t-\tau))d\tau \le \mathbf{c}(t-t^\star,\eta), \end{cases}$$

from which we deduce that

$$\varepsilon \le \mathbf{c}(t-t^\star,\eta) - \mathbf{c}(t,\xi) - \left\langle \frac{1}{t^\star}\int_{t-t^\star}^{t} v(\tau)d\tau, \eta-\xi \right\rangle.$$

Consistency conditions (1) can be written in the form

$$\forall\, 0 \le r \le s,\ \forall\, x \in K,\ y \in \partial K,\ \ \mathbf{c}(r,x) - \mathbf{c}(s,y) \le \left\langle \frac{1}{s-r}\int_r^s v(\tau)d\tau, x-y \right\rangle.$$

Taking $r := t - t^\star$, $s := t$, $x := \eta$, and $y := \xi$, we obtain the contradiction $\varepsilon \le 0$, and thus, we proved that for any $(t,\xi) \in \mathrm{Dom}(\mathbf{c})$, $\mathbf{N}(t,\xi) = \mathbf{c}(t,\xi)$. □

7. Other auxiliary systems. For proving that the viability hyposolution is the solution, in a generalized sense, to the Hamilton–Jacobi partial differential equation derived from the tangential or normal conditions characterizing capture basins, we need assumptions that control system (13) does not satisfy.

The two inequalities characterizing the Barron-Jensen/Frankowska solution follow from the two inclusions characterizing the Frankowska property of the capture basin (Definition 3.6). One is derived from the viability theorem and requires the assumption that F is Marchaud (upper semicontinuous, linear growth, with convex images); the other one is derived from the invariance theorem, valid whenever F is Lipschitz with closed values, without bounds on the size of their images (see Theorems 3.7 and 3.8). This is the reason why we introduce below two new systems, (23) and (24). The first one complies with the "Marchaud assumptions" of the viability theorem, so that the capture basin under it will satisfy the first inclusion of the Frankowska property, the second one to the "Lipschitz assumptions" of the invariance theorem, so that the capture basin under it will satisfy the second inclusion of the Frankowska property. The aim of this section is to derive from the inclusions of the Frankowska property the corresponding inequalities defining the Barron-Jensen/Frankowska property. However, to conclude, we need to prove that the capture basin is the same under the original system and the two new ones. This is achieved by our proof; i.e., the capture basin being the same under the three systems, it captures these two properties, and thus, these two inequalities, each valid under the assumptions made (convexity *with* bounds for the one deriving from the viability theorem and Lipschitz property *without* bounds for the other one deriving from the invariance theorem).

It happens that the capture basin of the hypograph of $\mathbf{c}$ viable in the hypograph of $\mathbf{b}$ under control system (13) is still the capture basin under other auxiliary systems which satisfy these assumptions, so we shall be able to transfer the theorems concerning capture basins.

The function ψ being concave and finite, it is then continuous so that, the function $v(\cdot)$ being bounded, the constant

$$\alpha := \sup_{u \in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u) - \inf_{\tau \ge 0} \psi(v(\tau))$$

is finite by Lemma 7.3. The new characteristic control systems are defined by

$$\text{(23)} \quad \begin{cases} \tau'(t) \ = \ -1, \\ x'(t) \ = \ u(t) \qquad \text{where } u(t) \ \in \ \mathrm{Dom}(\varphi^\star), \\ y'(t) \ = \ -\psi(v(\tau(t))) + \varphi^\star(u(t)) + \pi(t) \\ \qquad\qquad\qquad \text{where } \pi(t) \ \in \ [0, \alpha + \psi(v(\tau(t)) - \varphi^\star(u(t)))] \end{cases}$$

and

$$\text{(24)} \quad \begin{cases} \tau'(t) \ = \ -1, \\ x'(t) \ = \ u(t) \qquad \text{where } u(t) \ \in \ \mathrm{Dom}(\varphi^\star), \\ y'(t) \ = \ -\psi(v(\tau(t))) + \varphi^\star(u(t)) + \pi(t) \qquad \text{where } \pi(t) \ \geq \ 0, \end{cases}$$

where we added a new control π ranging over different intervals.

LEMMA 7.1 (equality between capture basins). *The capture basins of the hypograph of the function* $\mathbf{c}$ *by systems* (13), (23), *and* (24) *coincide as follows:*

$$\mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})).$$

Furthermore,

$$\mathrm{Capt}_{(23)}(\mathcal{H}yp(\boldsymbol{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\boldsymbol{b}), \mathcal{H}yp(\mathbf{c})) - \{0\} \times \{0\} \times \mathbb{R}_+$$

(where in a vector space, $A - B := \{a - b\}_{a \in A,\, b \in B}$*).*

Proof. Inclusions

$$\mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ \subset \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ \subset \ \mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$$

are obvious. For proving that

$$\mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ \subset \ \mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})),$$

let us consider an element $(t, x, y) \in \mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$. This means that there exist $u(\cdot) \in L^1(0, +\infty; \mathrm{Dom}(\varphi^\star))$ and a corresponding solution $(\tau(\cdot), x(\cdot), y(\cdot))$ to the characteristic control system (24) starting from (t, x, y) given by $\tau(s) = t - s$, $x(s) = x + \int_0^s u(r)dr$, and

$$\text{(25)} \qquad y(s) \ \geq \ y - \int_0^s (\psi(v(t - r)) - \varphi^\star(u(r)))\, dr$$

and there exists $t^\star \in [0, t]$ such that $(t - t^\star, x(t^\star), y(t^\star)) \in \mathcal{H}yp(\mathbf{c})$ and, for all $s \in [0, t^\star]$, $(t - s, x(s), y(s)) \in \mathcal{H}yp(\mathbf{b})$. Setting

$$y_0(s) \ := \ y + \int_0^s (\varphi^\star(u(r)) - \psi(v(t - r)))\, dr$$

we infer that $(\tau(\cdot), x(\cdot), y_0(\cdot))$ is a solution to the characteristic control system (13) starting from (t, x, y) viable in the environment $\mathcal{H}yp(\mathbf{b})$ because

$$\forall\, s \in [0, t^\star],\ \ y_0(s) \ \le\ y(s) \ \le\ \mathbf{b}(t-s, x(s))$$

until time $t^\star$, where it reaches the target $\mathcal{H}yp(\mathbf{c})$ because

$$y_0(t^\star) \ \le\ y(t^\star) \ \le\ \ \mathbf{c}(t-t^\star, x(t^\star)).$$

This means that $(t, x, y) \in \mathrm{Capt}_{(13)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$.

We also observe that whenever $(t, x, y) \in \mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$ and $z \le y$, inequality (25) implies that

$$y(s) \ \ge\ y - \int_0^s (\psi(v(t-r)) - \varphi^\star(u(r)))\, dr \ \ge\ z - \int_0^s (\psi(v(t-r)) - \varphi^\star(u(r)))\, dr$$

and thus that (t, x, z) also belongs to the capture basin, so that,

$$\mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ =\ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) - \{0\} \times \{0\} \times \mathbb{R}_+.$$

The proof is completed. ∎

We also need the following.

Lemma 7.2. *Let $\psi : X \mapsto \mathbb{R}$ be an upper semicontinuous concave function. The domain of its Fenchel transform $\varphi^\star$ is contained in a closed convex subset A if and only if the function ψ satisfies inequality*

$$\exists\, \beta \in \mathbb{R} \ \ \textit{such that}\, \forall\, v \in X,\ \ \beta - \sigma_A(v) \ \le\ \psi(v).$$

Its Fenchel transform $\varphi^\star$ is bounded on a convex subset A if and only if the function ψ satisfies

$$\exists\, \delta \in \mathbb{R} \ \ \textit{such that}\, \forall\, v \in X^\star,\ \ \psi(v) \ \le\ \delta - \sigma_A(v).$$

Proof. Since $\psi(0) = \inf_{u\in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u)$, we infer that

$$\forall\, v \in X,\ \ \forall\, u \in \mathrm{Dom}(\varphi^\star),\ \ \psi(0) - \sigma_{\mathrm{Dom}(\varphi^\star)}(v) \ \le\ \varphi^\star(u) - \langle u, v\rangle$$

so that, by taking the infimum over u, we obtain inequality $\psi(0) - \sigma_{\mathrm{Dom}(\varphi^\star)}(v) \le \psi(v)$. It is enough to set $\beta := \psi(0)$ and to take $A := \mathrm{Dom}(\varphi^\star)$. Conversely, assume that for all $v \in X$, $\psi(v) \ \ge\ \beta - \sigma_A(v)$. We shall prove that $\mathrm{Dom}(\varphi^\star) \subset A$. If not, there would exist $u \in \mathrm{Dom}(\varphi^\star) \setminus A$. The separation theorem states there exist $p_0 \in X$ and $\varepsilon > 0$ such that $\varepsilon \le \langle p_0, u\rangle - \sigma_A(p_0)$. Consequently, for every $\lambda > 0$,

$$\lambda\varepsilon \ \le\ \langle \lambda p_0, u\rangle - \sigma_A(\lambda p_0) \le\ \langle \lambda p_0, u\rangle + \psi(\lambda p_0) - \beta \ \le\ \varphi^\star(u) - \beta$$

by assumption and by the definition of $\varphi^\star$. Letting $\lambda \mapsto +\infty$ implies that $\varphi^\star(u) = +\infty$, i.e., that $u \notin \mathrm{Dom}(\varphi^\star)$, a contradiction.

For proving the second statement, we observe that if $\delta := \sup_{u\in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u) < +\infty$ is finite, then

$$\psi(v) \ \le\ \delta + \inf_{u\in \mathrm{Dom}(\varphi^\star)} \langle v, -u\rangle \ =\ \delta - \sigma_{\mathrm{Dom}(\varphi^\star)}(v)$$

so that the inequality holds true with $A := \mathrm{Dom}(\varphi^\star)$. Conversely, inequality $\psi(v) \leq \delta - \sigma_A(v)$ implies that

$$\forall\, u \in A,\ \ \varphi^\star(u) \ \leq \sup_{v\in\mathrm{Dom}(\psi)} [\langle v, u\rangle + \delta - \sigma_A(v)] \ < \ +\infty$$

is bounded on A, and thus, on $\mathrm{Dom}(\varphi^\star)$ whenever this domain is contained in A. □

Control systems (23) and (24) are actually differential inclusions

$$(\tau'(t), x'(t), y'(t)) \ \in \ F(\tau(t), x(t), y(t)),$$

where

$$F(\tau, x, y) \ := \ \{(-1, u, -\psi(v(\tau)) + \varphi^\star(u) + \pi)\}_{u\in\mathrm{Dom}(\varphi^\star),\ \pi\in[0,\alpha+\psi(v(\tau))-\varphi^\star(u)]} \tag{26}$$

and

$$(\tau'(t), x'(t), y'(t)) \ \in \ F_\infty(\tau(t), x(t), y(t)),$$

where

$$F_\infty(\tau, x, y) \ := \ \{(-1, u, -\psi(v(\tau)) + \varphi^\star(u) + \pi)\}_{u\in\mathrm{Dom}(\varphi^\star),\ \pi\geq 0}\,, \tag{27}$$

respectively.

LEMMA 7.3. *The set-valued map F is Marchaud and, if the functions ψ, $\varphi^\star$, and v are Lipschitz, the set-valued map F_∞ is Lipschitz with closed images.*

Proof. For proving that the set-valued map F is Marchaud, we shall check successively that

1. *the values $F(\tau, x, y)$ of the set-valued map F are convex.* Indeed, for convex weight $\lambda_i \geq 0$ such that $\sum \lambda_i = 1$, we can write

$$\sum \lambda_i(-1, u_i, -\psi(v(\tau)) + \varphi^\star(u_i) + \pi_i) \ = \ (-1, \overline{u}, \varphi^\star(\overline{u}) - \psi(v(\tau)) + \overline{\pi}),$$

where $\overline{u} := \sum \lambda_i u_i$ and

$$\overline{\pi} \ := \ \sum \lambda_i \varphi^\star(u_i) - \varphi^\star\left(\sum \lambda_i u_i\right) + \sum \lambda_i \pi_i.$$

Since the domain of $\varphi^\star$ is convex, $\overline{u} \in \mathrm{Dom}(\varphi^\star)$. We observe that $\overline{\pi}$ is nonnegative and smaller than or equal to $\alpha + \psi(v(\tau)) - \varphi^\star(\overline{u})$ because

$$\begin{cases} \overline{\pi} \ \leq \ \sum \lambda_i \varphi^\star(u_i) - \varphi^\star\left(\sum \lambda_i u_i\right) + \sum \lambda_i \left(\alpha + \psi(v(\tau)) - \varphi^\star(u_i)\right) \\ = \ \alpha + \psi(v(\tau)) - \varphi^\star\left(\sum \lambda_i u_i\right). \end{cases}$$

2. *the graph of the set-valued map F is closed.* Indeed, let us consider a sequence of elements $((\tau_n, x_n, y_n), (-1, u_n, \lambda_n))$ of the graph of F converging to $((\tau, x, y), (-1, u, \lambda))$, where $\lambda_n := -\psi(v(\tau_n)) + \varphi^\star(u_n) + \pi_n$ and where $\pi_n \in [0, \alpha + \psi(v(\tau_n)) - \varphi^\star(u_n)]$.

Since the function $(\tau, x, y, u) \mapsto \varphi^\star(u) - \psi(v(\tau))$ is lower semicontinuous and since

$$(\tau_n, x_n, y_n, u_n, \lambda_n) \ = \ (\tau_n, x_n, y_n, u_n, -\psi(v(\tau_n)) + \varphi^\star(u_n) + \pi_n)$$

belongs to the epigraph of this function (because π_n is positive by construction), which is closed, we deduce that the limit (τ, x, y, u, λ) also belongs to this epigraph, i.e., that $\lambda \geq \varphi^\star(u) - \psi(v(\tau))$. It is enough to set $\pi := \lambda - \varphi^\star(u) - \psi(v(\tau)) \geq 0$, which from now on defines π. Recall that $\pi_n = \lambda_n + \psi(v(\tau_n)) - \varphi^\star(u_n) \leq \alpha + \mathbf{l}(\tau_n, x_n) - \varphi^\star(u_n)$ by construction of π_n. Therefore, $\lambda_n \leq \alpha$. Therefore, taking the limit, $\lambda = \pi + \varphi^\star(u) - \psi(v(\tau)) \leq \alpha$. In summary, the limit $((\tau, x, y), (-1, u, \lambda))$ of elements $((\tau_n, x_n, y_n), (-1, u_n, \lambda_n))$ belongs to the graph of F since $\lambda = -\psi(v(\tau)) + \varphi^\star(u) + \pi$, where $\pi \in [0, \alpha + \psi(v(\tau)) - \varphi^\star(u)]$.

3. *the images $F(\tau, x, y)$ of F are bounded.* This follows from Lemma 7.2 because $\mathrm{Dom}(\varphi^\star)$ is bounded and

$$\varphi^\star(u) - \psi(v(\tau)) + \pi \ \leq \ \alpha \ := \sup_{u \in \mathrm{Dom}(\varphi^\star)} \varphi^\star(u) - \inf_{\tau \geq 0} \psi(v(\tau))$$

is finite since $\varphi^\star$ is bounded above. Therefore

$$\|(-1, u, \varphi^\star(u) - \psi(v(\tau)) + \pi)\| \leq \max(1, \|\mathrm{Dom}(\varphi^\star)\|, \alpha).$$

Hence, we have proved that the set-valued map F is Marchaud. The fact that F_∞ is Lipschitz is obvious since the functions ψ, $\varphi^\star$, and v, are assumed to be Lipschitz and since the controls u range over $\mathbb{R}_+$ which, being constant, is Lipschitz. □

We thus deduce the following.

PROPOSITION 7.4 (upper semicontinuity of the solution). *The viability hyposolution is upper semicontinuous and its hypograph satisfies*

$$\mathcal{H}yp(\mathbf{N}) \ = \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})).$$

The viability hyposolution is concave whenever the functions $\mathbf{b}$ *and* $\mathbf{c}$ *are concave.*

Proof. The first statement follows from Proposition 4.3 of [5] stating that under a Marchaud control system, the capture basin of a target is closed whenever the target $\mathcal{H}yp(\mathbf{c})$ and the environment $\mathcal{H}yp(\mathbf{b})$ are closed and the complement of the target in the environment is a repeller; this is the case because the first component of the system is $\tau'(t) = -1$ which implies that all solutions $(t - s, x(s), y(s))$ starting from any (t, x, y) leave $\mathbb{R}_+ \times X \times \mathbb{R}$, and thus, $\mathcal{H}yp(\mathbf{b}) \subset \mathbb{R}_+ \times X \times \mathbb{R}$. Since we have proved that

$$\mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c})) - \{0\} \times \{0\} \times \mathbb{R}_+,$$

we infer that $\mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}), \mathcal{H}yp(\mathbf{c}))$ is a hypograph, and thus, the hypograph of the viability hyposolution. □

8. Contingent solution to the Hamilton–Jacobi equation. We shall prove that the viability hyposolution to Hamilton–Jacobi equation (2) (see Definition 4.1) is the *contingent solution* by characterizing them in terms of tangent cones and translating them into terms of contingent hyposolutions.

THEOREM 8.1 (contingent Frankowska solution). *The viability hyposolution* $\mathbf{N}$ *is the* largest *upper semicontinuous solution satisfying*

$$\psi(v(t)) \ \geq \inf_{u \in \mathrm{Dom}(\varphi^\star)} (\varphi^\star(u) - D_\downarrow \mathbf{N}(t, x)(-1, u)) \tag{28}$$

and the initial/Dirichlet conditions and the inequality constraints. If the functions ψ, $\varphi^\star$, and v are furthermore Lipschitz, then $\mathbf{N}$ *is the* smallest *upper semicontinuous solution satisfying the following:*

1. *If* $\mathbf{N}(t,x) < \mathbf{b}(t,x)$, *then*

$$\psi(v(t)) \;\leq\; \inf_{u\in \mathrm{Dom}(\varphi^\star)} \left(D_\downarrow \mathbf{N}(t,x)(1,-u)+\varphi^\star(u)\right). \tag{29}$$

2. *If* $\mathbf{N}(t,x) = \mathbf{b}(t,x)$, *then*

$$\psi(v(t)) \;\leq\; \inf_{\{u|\psi(v(t))\;\leq\; D_\downarrow \mathbf{b}(t,x)(1,-u)+\varphi^\star(u)\}} (D_\downarrow \mathbf{N}(t,x)(1,-u)+\varphi^\star(u)). \tag{30}$$

We need the following technical lemma on tangent cones to hypographs for proving Theorem 8.1.

LEMMA 8.2 (tangent cones to hypographs). *If* $\psi : X \mapsto \mathbf{R}_+ \cup \{-\infty\}$ *is an extended function and if* $D_\downarrow\psi(p)(dp)$ *is finite, then, for every* $w < \psi(p)$ *and every* $\mu \in \mathbf{R}$, *the pair* (dp,μ) *belongs to the contingent cone* $T_{\mathcal{H}yp(\psi)}(p,w)$ *to the hypograph of* ψ *at* (p,w).

Proof. Let (dp,λ) belong to $T_{\mathcal{H}yp(\psi)}(p,\psi(p))$. Then we know that there exist sequences $h_n > 0$ converging to 0, dp_n converging to dp, and λ_n converging to λ such that $(p+h_n dp_n, \psi(p)+h_n\lambda_n)$ belongs to $\mathcal{H}yp(\psi)$. Therefore, for $w < \psi(p)$ and $\mu \in \mathbf{R}$ and h_n small enough,

$$(p+h_n dp_n, w+h_n\mu) \;=\; (p+h_n dp_n, \psi(p)+h_n\lambda_n)+(0, w-\psi(p)+h_n(\mu-\lambda_n)) \;\in\; \mathcal{H}yp(\psi)$$

belongs to the hypograph of ψ because $w-\psi(p)+h_n(\mu-\lambda_n) \leq 0$ for h_n small enough. Therefore, since $dp_n \to p$ and $\mu_n := \mu \to \mu$, we infer that $(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w)$. ☐

Proof of Theorem 8.1. Observe first that

$$(t,x,y) \;\in\; \mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c}) \text{ if and only if } t>0,\ x\in \mathrm{Int}(K), \text{ and } y \;\leq\; \mathbf{N}(t,x).$$

Indeed, $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is the set of (t,x,y) such that $\mathbf{c}(t,x) < y \leq\ \mathbf{N}(t,x)$. This is automatically satisfied when $t > 0$ and $x \in \mathrm{Int}(K)$ whenever $y \leq \mathbf{N}(t,x)$ since in this case, $\mathbf{c}(t,x) = -\infty$. It is impossible otherwise since, by Theorem 6.1 $\mathbf{N}(t,x) = \mathbf{c}(t,x)$.

Theorem 4.6 of [7] states that since F is Marchaud by Lemma 7.3, the capture basin is the largest closed subset between the hypograph of $\mathbf{c}$ and $\mathbb{R}_+ \times X \times \mathbb{R}$ such that $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F.

Theorems 3.2.4 and 3.3.4 of [7] state that $\mathcal{H}yp(\mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F if and only if for all $t > 0$, for all $x \in X$, for all $y \leq \mathbf{N}(t,x)$, $\exists u \in \mathrm{Dom}(\varphi^\star)$, $\exists \pi \in [0, \alpha+\psi(v(\tau))-\varphi^\star(u)]$, such that

$$(-1, u, -\psi(v(t))+\varphi^\star(u)+\pi) \;\in\; T_{\mathcal{H}yp(\mathbf{N})}(t,x,y).$$

If $y = \mathbf{N}(t,x)$, then

$$T_{\mathcal{H}yp(\mathbf{N})}(t,x,\mathbf{N}(t,x)) \;=:\; \mathcal{H}yp(D_\downarrow \mathbf{N}(t,x))$$

so that we infer that there exists $u \in \mathrm{Dom}(\varphi^\star)$

$$-\psi(v(t))+\varphi^\star(u)+\pi \;\leq\; D_\downarrow \mathbf{N}(t,x)(-1,u)$$

from which inequality (28) ensues.

Conversely, since $D_\downarrow \mathbf{N}(t,x)(-1,\cdot)$ is upper semicontinuous and the domain of $\varphi^\star$ is compact, inequality (28) implies the existence of $u \in \mathrm{Dom}(\varphi^\star)$ such that

$$(-1, u, -\psi(v(t))+\varphi^\star(u)+\pi) \;\in\; T_{\mathcal{H}yp(\mathbf{N})}(t,x,\mathbf{N}(t,x)).$$

When $y < \mathbf{N}(t,x)$, then Lemma 8.2 implies that

$$(-1, u, -\psi(v(t)) + \varphi^\star(u) + \pi) \in T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

because $(-1, u)$ belongs to the domain of $D_\downarrow \mathbf{N}(t,x)$.

By Theorems 4.7 and 4.10 of [7], the capture basin is the smallest closed subset between the hypographs of $\mathbf{c}$ and $\mathbf{b}$ such that $\mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$. Since F_∞ is Lipschitz by Lemma 7.3 whenever the functions ψ, $\varphi^\star$, and v are Lipschitz, the invariance theorem (Theorem 5.3.4 in [5]) states that $\mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$ under F_∞ if and only if

$$\forall\, (t,x,y) \in \mathcal{H}yp(\mathbf{N}), \quad F_\infty(t,x,y) \cap T_{\mathcal{H}yp(\mathbf{b})}(t,x,y) \subset T_{\mathcal{H}yp(\mathbf{N})}(t,x,y).$$

Since the function $\mathbf{b}$ is assumed to be continuous,

$$\mathrm{Int}(\mathcal{H}yp(\mathbf{b})) = \{(t,x,y) \text{ such that } y < \mathbf{b}(t,x)\}.$$

Therefore, we have to investigate the following two cases:

1. For all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \mathrm{Int}(\mathcal{H}yp(\mathbf{b}))$. Then

$$\begin{gathered}\forall t \geq 0, \ \ \forall x \in X, \ \ \forall\, y \leq \mathbf{N}(t,x), \ \ \forall\, u \in \mathrm{Dom}(\varphi^\star), \ \ \forall\, \pi \geq 0,\\ (1, -u, \psi(v(t)) - \varphi^\star(u) - \pi) \in T_{\mathcal{H}yp(\mathbf{N})}(t,x,y).\end{gathered}$$

If $y = \mathbf{N}(t,x)$, then we infer that for all $u \in \mathrm{Dom}(\varphi^\star)$,

$$\psi(v(t)) - \varphi^\star(u) \leq D_\downarrow \mathbf{N}(t,x)(1,-u)$$

from which we derive inequality (29). Conversely, since for all $u \in \mathrm{Dom}(\varphi^\star)$, $(1,-u)$ belongs to the domain of $D_\downarrow \mathbf{N}(t,x)$, we derive that

$$(1, -u, \psi(v(t)) - \varphi^\star(u) - \pi) \in T_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$$

holds true.

2. For all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \partial(\mathcal{H}yp(\mathbf{b}))$, and in this case, $y = \mathbf{N}(t,x) = \mathbf{b}(t,x)$. Then, $\forall t \geq 0$, $\forall x \in X$, $\forall\, u \in \mathrm{Dom}(\varphi^\star)$, $\forall\, \pi \geq 0$ such that

$$(1, -u, \psi(v(t)) - \varphi^\star(u) - \pi) \in T_{\mathcal{H}yp(\mathbf{b})}(t,x,y)$$

we have

$$(1, -u, \psi(v(t)) - \varphi^\star(u) - \pi) \in T_{\mathcal{H}yp(\mathbf{N})}(t,x,y).$$

This means that whenever

$$\psi(v(t)) \leq D_\downarrow \mathbf{b}(t,x)(1,-u) + \varphi^\star(u),$$

then

$$\psi(v(t)) \leq D_\downarrow \mathbf{N}(t,x)(1,-u) + \varphi^\star(u),$$

which is (30).

Theorem 5.5 states that the viability hyposolution is the valuation function (21) of the underlying optimal control problem (13).

The *associated regulation map* R for regulating the optimal evolutions is thus defined by

$$\forall\, t > 0,\ x \in X,\ \ R(t,x) \ := \ \{u \mid 0 \ \leq \ D_{\downarrow}\mathbf{N}(t,x)(-1,u) - \varphi^{\star}(u) + \psi(v(t))\}.$$

One can prove that the optimal solutions of the control problem are governed by the control system

$$\begin{cases} \tau'(s) \ = \ -1, \\ x'(s) \ = \ u(s) \ \in \ R(\tau(s), x(s)), \\ y'(s) \ = \ \varphi^{\star}(u(s)) - \psi(v(\tau(s))). \end{cases}$$

This motivates a further study of the regulation map. If the solution $\mathbf{N}$ is differentiable, the regulation map can be written in the form

$$R(t,x) \ := \ \left\{u \mid \ 0 \ \leq \ -\frac{\partial \mathbf{N}(t,x)}{\partial t} + \frac{\partial \mathbf{N}(t,x)}{\partial x} u - \varphi^{\star}(u) + \psi(v(t))\right\}.$$

The elements u maximizing the right-hand side are the elements belonging to

$$-\partial_{+}\psi\left(\frac{\partial \mathbf{N}(t,x)}{\partial x}\right).$$

Consequently,

$$-\partial_{+}\psi\left(\frac{\partial\ \mathbf{N}(t,x)}{\partial x}\right) \ \subset \ R(t,x).$$

Actually, approximations of the regulation map and thus, optimal evolutions, as well as the solution to the Hamilton–Jacobi–Bellman equation are provided by the capture basin algorithm.

9. Barron-Jensen/Frankowska solution to the Hamilton–Jacobi equation.

Instead of characterizing capture basins in terms of tangent cones and translating them into terms of contingent Frankowska hyposolutions, we translate them into the equivalent formulation of Barron-Jensen/Frankowska solutions, a weaker concept of viscosity solutions requiring only the upper semicontinuity of the solution instead of its continuity.

THEOREM 9.1 (Barron-Jensen/Frankowska solution). *The viability hyposolution* $\mathbf{N}$ *is the* largest *upper semicontinuous solution between* $\mathbf{c}$ *and* $\mathbf{b}$ *satisfying*

$$(31)\quad \begin{cases} \text{(i)} \quad \forall t > 0,\ \forall x \in \mathrm{Int}(K),\ \forall (p_t, p_x) \in \partial_{+}\mathbf{N}(t,x),\ \ p_t + \psi(p_x) \ \leq \ \psi(v(t)), \\ \text{(ii)} \quad \forall t > 0,\ \forall x \in \mathrm{Int}(K),\ \forall (p_t, p_x) \in (\mathrm{Dom}(D_{\downarrow}\mathbf{N}(t,x)))^{-}, \\ \qquad\qquad p_t - \sigma(\mathrm{Dom}(\varphi^{\star}), p_x) \ \leq \ 0. \end{cases}$$

If the functions ψ, $\varphi^{\star}$, and v are furthermore Lipschitz, then $\mathbf{N}$ *is the* smallest *upper semicontinuous solution between* $\mathbf{c}$ *and* $\mathbf{b}$ *satisfying the following:*

1. *If* $\mathbf{N}(t,x) < \mathbf{b}(t,x)$, *then*

$$(32)\quad \begin{cases} \text{(i)} \quad \forall t \geq 0,\ \forall x \in K \text{ such that} \\ \qquad\qquad \mathbf{N}(t,x) < \mathbf{b}(t,x),\ \forall (p_t, p_x) \in \partial_{+}\mathbf{N}(t,x), \\ \qquad\qquad p_t + \psi(p_x) \ \geq \ \psi(v(t)); \\ \text{(ii)} \quad \forall t \geq 0,\ \forall x \in K \text{ such that} \\ \qquad \mathbf{N}(t,x) < \mathbf{b}(t,x),\ \forall (p_t, p_x) \in (\mathrm{Dom}(D_{\downarrow}\mathbf{N}(t,x)))^{-}, \\ \qquad\qquad p_t - \sigma(\mathrm{Dom}(\varphi^{\star}), p_x) \ \geq \ 0. \end{cases}$$

2. *If* $\mathbf{N}(t,x) = \mathbf{b}(t,x)$, *then*

$$(33)\quad \begin{cases} \forall\, (p_t,p_x) \in \partial_+\mathbf{N}(t,x),\ \ \exists\, (q_t,q_x) \in \partial_+\mathbf{b}(t,x) \text{ and } 0<\mu<1 \text{ such that} \\ \text{either}\quad p_t - q_t - \sigma(\mathrm{Dom}(\varphi^\star), p_x - q_x) \ \ge\ 0 \\ \text{or}\quad \dfrac{p_t - \mu q_t}{1-\mu} + \psi\left(\dfrac{p_x - \mu q_x}{1-\mu}\right) \ \ge\ \psi(v(t)). \end{cases}$$

Thus, the unique *upper semicontinuous solution satisfies all these properties.*

Observe that under the Lipschitz assumptions, the viability hyposolution satisfies

$$(34)\qquad \begin{cases} \text{(i)}\quad \forall t>0,\ \forall x \in \mathrm{Int}(K)\ \text{ such that } \mathbf{N}(t,x) < \mathbf{b}(t,x), \\ \qquad \forall (p_t,p_x) \in \partial_+\mathbf{N}(t,x),\ \ p_t + \psi(p_x) \ =\ \psi(v(t)), \\ \text{(ii)}\quad \forall (p_t,p_x) \in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-,\ \ p_t - \sigma(\mathrm{Dom}(\varphi^\star),p_x) \ =\ 0. \end{cases}$$

We need the following technical lemma on normal cones to hypographs for proving Theorem 9.1.

LEMMA 9.2 (normal cones to hypographs). *A pair* (u,λ) *belongs to the normal cone* $N_{\mathcal{H}yp(\psi)}(p,w)$ *to the hypograph of* ψ *at* (p,w) *if and only*

1. *if* $w = \psi(p)$*; then either*
   - $\lambda = 0$ *and* $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$ *or*
   - $\lambda > 0$ *and* $u \in -\lambda\partial_+\psi(p)$.
2. *if* $w < \psi(p)$*; then* $\lambda = 0$ *and* $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$.

In particular, if the domain of $D_\downarrow\psi(p)$ *is dense in* X*, then* (u,λ) *belongs to the normal cone* $N_{\mathcal{H}yp(\psi)}(p,w)$ *to the hypograph of* ψ *at* (p,w) *if and only if* $\lambda = 0$ *and* $u = 0$. *This is the case whenever* ψ *is Lipschitz around* p.

Proof. Let us consider now a pair (u,λ) belonging to the normal cone $N_{\mathcal{H}yp(\psi)}(p,w) := (T_{\mathcal{H}yp(\psi)}(p,w))^-$ to the hypograph of ψ at (p,w). Therefore,

$$\forall\, (dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w),\ \ \langle (dp,\mu),(u,\lambda)\rangle \ = \langle u, dp\rangle + \lambda\mu \ \le\ 0.$$

Examine first the case when $w = \psi(p)$, for which $(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,\psi(p))$ if and only if $dp \in \mathrm{Dom}(D_\downarrow\psi(p))$ and $\mu \le D_\downarrow\psi(p)(dp)$. If $\lambda < 0$, we obtain a contradiction because, when $\mu \to -\infty$, $\langle u, dp\rangle + \lambda\mu \to +\infty$. Hence

- either $\lambda > 0$, and thus, dividing by λ and taking $\mu := D_\downarrow\psi(p)(dp)$, we obtain

  $$\forall\, dp \in \mathrm{Dom}(D_\downarrow\psi(p)),\ \ \left\langle \frac{u}{\lambda}, dp\right\rangle + D_\downarrow\psi(p)(dp) \ \le\ 0$$

  which means that $-\frac{u}{\lambda} \in \partial_+\psi(p)$;
- or $\lambda = 0$ and we obtain

  $$\forall\, dp \in \mathrm{Dom}(D_\downarrow\psi(p)),\ \ \langle u, dp\rangle \ \le\ 0,$$

  which means that $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$ by definition of the polar cone.

When $w < \psi(p)$, inequalities

$$\forall\, (dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w),\ \ \langle (dp,\mu),(u,\lambda)\rangle \ = \langle u, dp\rangle + \lambda\mu \ \le\ 0$$

imply that $\lambda = 0$ thanks to Lemma 8.2; otherwise, $\lambda\mu$ converges to $+\infty$ when $\mu \to +\infty$ when $\lambda > 0$, and when $\mu \to -\infty$ when $\lambda < 0$ since μ is allowed to range over $\mathbb{R}$. Therefore $u \in (\mathrm{Dom}(D_\downarrow\psi(p)))^-$ because whenever $dp \in \mathrm{Dom}(D_\downarrow\psi(p))$ and $\mu \in \mathbb{R}$, then $(dp,\mu) \in T_{\mathcal{H}yp(\psi)}(p,w)$. If the domain $D_\downarrow\psi(p)$ is dense in X, then the polar cone $(\mathrm{Dom}(D_\downarrow\psi(p)))^-$ is $\{0\}$, and thus $u = 0$. $\square$

Proof of Theorem 9.1. Proposition 7.4 states that the hypograph of the viability hyposolution satisfies

$$\mathcal{H}yp(\mathbf{N}) \ = \ \mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})) \ = \ \mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c})).$$

Theorem 3.4 states that $\mathrm{Capt}_{(23)}(\mathcal{H}yp(\mathbf{b}),\mathcal{H}yp(\mathbf{c}))$ is the *largest* subset $\mathcal{D}$ between $\mathcal{C}$ and $\mathcal{K}$ such that $\mathcal{D}\backslash\mathcal{C}$ is locally viable.

Taking $\mathcal{D} := \mathcal{H}yp(\mathbf{N})$, Theorems 3.2.4 and 3.3.4 of [5] state that $\mathcal{H}yp(\ \mathbf{N})\backslash\mathcal{H}yp(\mathbf{c})$ is locally viable under F if and only if for all $t > 0$, for all $x \in \mathrm{Int}(K)$, for all $y \le \mathbf{N}(t,x)$, $\exists u \in \mathrm{Dom}(\varphi^\star)$, $\exists \pi \in [0,\alpha+\psi(v(t))-\varphi^\star(u)]$, such that for all $(-p_t,-p_x,\lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$,

$$(35) \qquad \begin{cases} \langle(-p_t,-p_x,\lambda),(-1,u,-\psi(v(t))+\varphi^\star(u)+\pi)\rangle \\ = \ p_t - \langle p_x,u\rangle + \lambda(-\psi(v(t))+\varphi^\star(u)+\pi) \ \le \ \ 0. \end{cases}$$

By Lemma 9.2, if $y = \mathbf{N}(t,x)$, $(-p_t,-p_x,\lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$ means that either $\lambda > 0$, and that, taking $\lambda = 1$, $(p_t,p_x) \in \partial_+\mathbf{N}(t,x)$, or that $\lambda = 0$, and that $(p_t,p_x) \in (\mathrm{Dom}(D_\downarrow\mathbf{N}(t,x)))^-$. If $y < \mathbf{N}(t,x)$, $(-p_t,-p_x,\lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y)$ means also that $\lambda = 0$, and that $(p_t,p_x) \in (\mathrm{Dom}(D_\downarrow\mathbf{N}(t,x)))^-$.

Consequently, condition (35) can be written in the following form:

- The case when $y = \mathbf{N}(t,x)$ and $\lambda = 1$:

$$\begin{cases} \forall t > 0,\ \forall x \in \mathrm{Int}(K),\ \forall\ (p_t,p_x) \in \partial_+\mathbf{N}(t,x),\ \text{then} \\ p_t - \psi(v(t)) + \inf_{u\in\mathrm{Dom}(\varphi^\star)}[\varphi^\star(u) - \langle p_x,u\rangle] \\ = \ p_t - \psi(v(t)) + \psi(p_x) \le 0. \end{cases}$$

- The case when $y \le \mathbf{N}(t,x)$ and $\lambda = 0$:

$$\begin{cases} \forall t > 0,\ \forall x \in \mathrm{Int}(K),\ \forall\ (p_t,p_x) \in (\mathrm{Dom}(D_\downarrow\ \mathbf{N}(t,x)))^-,\ \text{then} \\ p_t - \sup_{u\in\mathrm{Dom}(\varphi^\star)}\langle p_x,u\rangle \ = \ p_t - \sigma(\mathrm{Dom}(\varphi^\star),p_x) \le 0. \end{cases}$$

(Recall that this condition disappears whenever the viability hyposolution $\mathbf{N}$ is hypodifferentiable, and, in particular, when the hyposolution is Lipschitz.)

Proof of inequalities (32) *and* (33). Theorem 3.4 states that $\mathrm{Capt}_{(24)}(\mathcal{H}yp(\mathbf{b}),$ $\mathcal{H}yp(\mathbf{c}))$ is the *smallest* subset $\mathcal{D}$ between $\mathcal{C}$ and $\mathcal{K}$ such that $\mathcal{D}$ is backward invariant with respect to $\mathcal{K}$. Theorem 3.7 and Lemma 3.8 state that $\mathcal{D} := \mathcal{H}yp(\mathbf{N})$ is backward invariant with respect to $\mathcal{H}yp(\mathbf{b})$ under (24) if and only if one of the following holds:

1. For all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \mathrm{Int}(\mathcal{H}yp(\mathbf{b}))$,

$$\forall\ (-p_t - p_x,\lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y),\ \ \sigma(F_\infty(x),(p_t,p_x,-\lambda)) \ \le \ 0.$$

Since the function $\mathbf{b}$ is assumed to be continuous,

$$\mathrm{Int}(\mathcal{H}yp(\mathbf{b})) \ = \ \{(t,x,y) \ \text{ such that } y \ < \ \mathbf{b}(t,x)\},$$

the first case means that $y \le \mathbf{N}(t,x) < \mathbf{b}(t,x)$ and the above condition implies that

$$(36) \qquad \begin{cases} \forall\ (-p_t - p_x,\lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y), \\ \langle(p_t,p_x,-\lambda),(-1,u,-\psi(v(t))+\varphi^\star(u)+\pi)\rangle \\ = -p_t + \langle p_x,u\rangle + \lambda(\psi(v(t)) - \varphi^\star(u) - \pi) \ \le \ \ 0. \end{cases}$$

This implies that $\lambda \ge 0$.

Consequently, condition (36) can be written in the following form:

- The case when $y = \mathbf{N}(t,x) < \mathbf{b}(t,x)$ and $\lambda = 1$:

$$\begin{cases} \forall t > 0,\ \forall x \in X,\ \forall\ (p_t, p_x) \in \partial_+ \mathbf{N}(t,x),\ \text{then} \\ -p_t + \psi(v(t)) + \sup_{u \in \mathrm{Dom}(\varphi^\star)}[\langle p_x, u\rangle - \varphi^\star(u)] \\ =\ -p_t + \psi(v(t)) - \psi(p_x) \leq 0. \end{cases}$$

- The case when $y \leq \mathbf{N}(t,x)$ and $\lambda = 0$:

$$\begin{cases} \forall t > 0, \forall x \in X,\ \forall\ (p_t, p_x) \in (\mathrm{Dom}(D_\downarrow \mathbf{N}(t,x)))^-,\ \text{then} \\ -p_t + \sup_{u \in \mathrm{Dom}(\varphi^\star)} \langle p_x, u\rangle\ =\ -p_t + \sigma(\mathrm{Dom}(\varphi^\star), p_x) \leq 0. \end{cases}$$

2. For all $(t,x,y) \in \mathcal{H}yp(\mathbf{N}) \cap \partial(\mathcal{H}yp(\mathbf{b}))$, and in this case, $y = \mathbf{N}(t,x) = \mathbf{b}(t,x)$ and

$$\begin{cases} \forall\ (-p_t - p_x, \lambda) \in N_{\mathcal{H}yp(\mathbf{N})}(t,x,y), \quad \exists\ (-q_t - q_x, \mu) \in N_{\mathcal{H}yp(\mathbf{b})}(t,x,y) \\ \qquad\qquad \text{such that} \quad \sigma(F_\infty(x), (p_t - q_t, p_x - q_x, \mu - \lambda))\ \leq\ 0, \end{cases}$$

where $\lambda \geq 0$ and $\mu > 0$ since we have assumed that $\mathbf{b}$ is Lipschitz, and thus hypodifferentiable. This can be translated into the following form:

$$-p_t + q_t + \sup_u(\langle p_x - q_x, u\rangle + (\mu - \lambda)(\varphi^\star(u)) + \sup_{\pi \geq 0}(\mu - \lambda)[\pi - \psi(v(t))])\ \leq\ 0.$$

This implies that $\lambda \geq \mu > 0$.

- The case when $\lambda - \mu = 0$. It happens when both $(p_t, p_x) \in \partial_+ \mathbf{N}(t,x)$ and $(q_t, q_x) \in \partial_+ \mathbf{b}(t,x)$. In this case, the above inequality boils down to

$$-p_t + q_t + \sigma(\mathrm{Dom}(\varphi^\star), p_x - q_x)\ \leq\ 0.$$

- The case when $\lambda - \mu > 0$. The condition states that for every $\lambda > 0$ and $(p_t, p_x) \in \partial_+ \mathbf{N}(t,x)$, there exist $0 < \mu < \lambda$ and $(q_t, q_x) \in \partial_+ \mathbf{b}(t,x)$ such that

$$-\frac{\lambda p_t - \mu q_t}{\lambda - \mu} + \sup_u \left(\left\langle \frac{\lambda p_x - \mu q_x}{\lambda - \mu}, u \right\rangle - \varphi^\star(u) \right) + \psi(v(t))\ \leq\ 0,$$

which can be written

$$-\frac{\lambda p_t - \mu q_t}{\lambda - \mu} - \psi\left(\frac{\lambda p_x - \mu q_x}{\lambda - \mu} \right) + \psi(v(t))\ \leq\ 0.$$

This completes the proof. ∎

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