Proceedings of the 45th IEEE Conference on Decision & Control
Manchester Grand Hyatt Hotel
San Diego, CA, USA, December 13-15, 2006

Continuous adjoint method for Air Traffic Flow Management

Issam S. STRUB and Alexandre M. BAYEN
University of California, Berkeley

***Abstract*— This article develops a model of Air Traffic Flow using an Eulerian description with hyperbolic partial differential equations. Existence and uniqueness (well-posedness) of a solution to the system of partial differential equations on a network is established. Subsequently, an optimal control problem is studied with the junction coefficients as control variables. We use a continuous adjoint approach and we implement it on a network with 16 links and 5 junctions, demonstrating the computational efficiency of this method.**

I. INTRODUCTION

The projected continued growth of air traffic in the next decades, and in particular the expected development of air transportation in the Middle East and Asia has increased the congestion of airspaces, as well as the complexity of the task assigned to Air Traffic Controllers across the globe. For example, the 20 million yearly passengers currently using air transportation in India are expected to become 90 million by the year 2010; the two main Indian airports of Delhi and Mumbai are already operating at twice their capacity causing considerable delays and strain on local Air Traffic Controllers. This forecast has led to the development of decision support tools to model, simulate and optimize air traffic with the objective of helping Air Traffic Controllers in their daily tasks.

Air Traffic Control (ATC) is operated at the sector level, where a sector is a small portion of the airspace controlled by a single human Air Traffic Controller. *Traffic Flow Management* (TFM) typically deals with traffic at the *Center* level, i.e. 10 to 20 sectors. TFM problems include maintaining the aircraft count in each sector below a legal threshold in order to ease the human ATC workload, as well as to ensure the safety of the flights [1]. This task is quite cumbersome; furthermore, extensive traffic forecast simulations (including all airborne aircraft) are computationally too expensive to include systematic investigations of traffic patterns that lead to sector overload. As a result, a new class of traffic flow models has emerged from recent studies: *Eulerian* models, which are control volume based. This is in contrast to *Lagrangian* models, which are trajectory-based and take into account all aircraft trajectories. Unlike Lagrangian models which focus on the history of a given material element therefore using the position vector of the material element and time as variables, the Eulerian models provides a picture of the spatial distribution of the flow in function of position in space and time.

I. S. STRUB is a Ph.D. student in the Department of Civil and Environmental Engineering, University of California, Berkeley CA 94720-1710. strub@ce.berkeley.edu.

A. M. BAYEN is Assistant Professor in the Department of Civil and Environmental Engineering, University of California, Berkeley CA 94720-1710. bayen@ce.berkeley.edu.

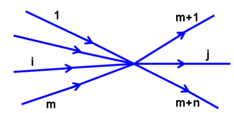

Fig. 1. *A junction with m incoming links* ($1 \leqslant i \leqslant m$) *and n outgoing links* ($m+1 \leqslant j \leqslant m+n$)

Eulerian models have two main advantages over Lagrangian models. (i) They are computationally tractable, and their computational complexity does not depend on the number of aircraft, but only on the size of the physical problem of interest. (ii) Their control theoretic structure enables the use of standard methodologies to analyse them.

The field of Eulerian models for the NAS has been pioneered by the article of Menon et al. [2]. Adjoint based techniques were subsequently developed for a fully continuous [3] NAS model (i.e. using partial differential equations), which has then been further used for modeling behavior of single agents (airlines) in the NAS [4], [5]. In order to alleviate the problems due to network splits (this problem will be explained in detail later in this article and affects all previous models), a delay system model based on network flow techniques (inspired by the work [6]) was finally proposed [7] and successfully implemented.

The Eulerian viewpoint will be used throughout this article. We will start by establishing the existence and uniqueness of a solution to the partial differential equations modelling air traffic on a network. Then we will solve an optimal control problem using a continuous adjoint method on a network that includes the arrival airspace at Oakland International Airport (OAK). We wish to make it clear that this article is fundamentally different from the earlier work [3] as the existence and uniqueness on a general network are proved in this article, the adjoint method that we develop uses new mathematical techniques and the control variables are different.

II. A MODEL OF AIR TRAFFIC NETWORKS

We will now present an Eulerian model of air traffic flow. We describe the flow streams as a continuum and attempt to study their evolution.

We divide the airspace into line elements on which we model the density of aircraft. These line elements are called

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paths and in practice often coincide with jetways. We represent a link on a path as a segment $[0,L]$ and we denote by $u(x,t)$ the number of aircraft between distances 0 and x at time t. In particular, $u(0,t)=0$ and $u(L,t)$ is the total number of aircraft in the path modeled by $[0,L]$ at time t. We make the additional assumption of a steady velocity profile $v(x)>0$ which depicts the mean velocity of aircraft flow at position x and time t. Applying the conservation of mass to a control volume comprised between positions x and $x+h$, and letting h tend to 0, one easily finds the following relation between the spatial and temporal derivatives of $u(x,t)$:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t}+v(x)\frac{\partial u(x,t)}{\partial x}=q(t) & (x,t)\in(0,L)\times(0,T] \\ u(x,0)=u_0(x) & x\in[0,L] \\ u(0,t)=0 & t\in[0,T] \end{cases}$$

where $q(t)$ represents the inflow at the entrance of the link $(x=0)$ or in terms of the density $q(t)=\rho(0,t)v(0)$.

We can define the density of aircraft as the weak derivative of $u(x,t)$ with respect to x: $\rho(x,t)=\frac{\partial u(x,t)}{\partial x}$. The aircraft density is a solution of the partial differential equation:

$$\begin{cases} \frac{\partial \rho(x,t)}{\partial t}+v(x)\frac{\partial \rho(x,t))}{\partial x} & \\ +v'(x)\rho(x,t)=0 & (x,t)\in(0,L)\times(0,T] \\ \rho(x,0)=\rho_0(x) & x\in[0,L] \\ \rho(0,t)=\frac{q(t)}{v(0)} & t\in[0,T] \end{cases}$$

This is a linear advection equation with positive velocity $v(x)$ and a source term: $v'(x)\rho(x,t)$. Clearly, these two partial differential equations are equivalent and model the same physical phenomenon. We will use the latter in this article as it enables us to impose constraints in terms of aircraft density.

We now consider a junction with m incoming links numbered from 1 to m and n outgoing links numbered from $m+1$ to $m+n$; each link k is represented by an interval $[0,L_k]$ (Figure 1). One can see that any network is composed of a number of such junctions. We define an allocation matrix $M=(m_{ij}(t))$ for $1\leqslant i\leqslant m,\ m+1\leqslant j\leqslant m+n$ where $0\leqslant m_{ij}(t)\leqslant 1$ denotes the proportion of aircraft from incoming link i going to the outgoing link j; we should also have $\sum_{j=m+1}^{m+n} m_{ij}(t)=1$ for $1\leqslant i\leqslant m$. The system of partial differential equations on the network can be written as:

$$\begin{cases} \frac{\partial \rho_k(x,t)}{\partial t}+v_k(x)\frac{\partial \rho_k(x,t))}{\partial x} & \\ +v_k'(x)\rho_k(x,t)=0 & 1\leqslant k\leqslant m+n \\ & (x,t)\in(0,L_k)\times(0,T] \\ \rho_k(x,0)=\rho_{0,k}(x) & x\in[0,L_k] \\ \rho_i(0,t)=\frac{q_i(t)}{v_i(0)} & 1\leqslant i\leqslant m \\ & t\in[0,T] \\ \rho_j(0,t)=\frac{\sum_{i=1}^m m_{ij}(t)\rho_i(L_i,t)v_i(L_i)}{v_j(0,t)} & m+1\leqslant j\leqslant m+n \\ & t\in[0,T] \end{cases}$$

We will now show that on such a network, the preceding system of partial differential equations admits a unique solution hence that the problem is well-posed.

First we consider the case of a single link $[0,L]$. Since the velocity is always positive, a boundary condition shall be set on the left $(x=0)$ but not on the right $(x=L)$. Using classical partial differential equations techniques, more precisely the theory of characteristics to compute the solution and prove the existence and energy methods for the uniqueness, it can be shown that the advection equation will have a unique solution on this interval (see for example [8] or [3] for a proof). On a network, this ensures the existence and uniqueness of a solution on the incoming links. For the outgoing links, we need to impose a boundary condition on the left, that is immediately after the junction. This is done using the coefficients of the allocation matrix. Indeed for the j-th outgoing link, the density at the origin will be related to the densities at the right extremity of the incoming links by:

$$\rho_j(0,t)=\frac{\sum_{i=1}^m m_{ij}(t)\rho_i(L_i,t)v_i(L_i)}{v_j(0,t)}$$

Then the advection equation on each outgoing link has a unique solution, thus defining uniquely a density on both the incoming and outgoing links. Therefore, the problem for any network, which is made of several such junctions, is well-posed.

III. Continuous adjoint approach for the optimal control of Air Traffic Networks

In this section, we study an optimal control problem for a network. We try to mitigate congestion on the network by acting on the coefficients of the allocation matrix. To evaluate the gradient of the objective function, we will use a continuous adjoint system. We consider the following problem:

Minimise the functional $H(m_{ij})=\sum_{k=1}^{m+n}\int_0^T\int_0^{L_k}\rho_k(x,t)dxdt$

with the additional constraints

$$0\leqslant m_{ij}(t)\leqslant 1 \text{ for } 1\leqslant i\leqslant m,\ m+1\leqslant j\leqslant m+n$$

$$\sum_{j=m+1}^{m+n} m_{ij}(t)=1 \text{ for } 1\leqslant i\leqslant m$$

$$\rho_k(x,t)\leqslant \rho_k^{max} \text{ for } 1\leqslant k\leqslant m+n$$

Minimising this functional is equivalent to maximising the outflow of the network; indeed the value of H represents the total amount of time aircraft spend in the network. The control variables are the coefficients of the allocation matrix $(m_{ij}(t))$. This is in fact a case of boundary control since as explained earlier, the density at the left of an outgoing link is directly related to the value of $(m_{ij}(t))$ by:

$$\rho_j(0,t)=\frac{\sum_{i=1}^m m_{ij}(t)\rho_i(L_i,t)v_i(L_i)}{v_j(0,t)},\ m+1\leqslant j\leqslant m+n, t\in[0,T]$$

The first two constraints are used to make sure that the model is realistic; all the aircraft have to leave an incoming link and enter an outgoing link. The third constraint implements a maximum density not to be exceeded for each link.

Adjoint methods were first introduced in the late 1980s as a tool for shape optimisation, in particular aircraft design (see [9], [10]). The direct approach which consists in calculating the gradient of the cost functional using finite differences is only possible when the number of control variables is small. In most real life problems, this number is too large making this approach unfeasible. A more efficient way of calculating gradients is to use the adjoint equations and boundary conditions, which can be solved using numerical schemes to yield the gradient of the cost functions.

We illustrate this technique on a simple optimisation problem:

$$\min_x F(x,u(x)) \text{ subject to the constraint } C(x,u(x)) = 0$$

where $u = (u_1,\dots,u_n)$, $C = (C_1,\dots,C_n)$, $x = (x_1,\dots,x_m)$.

For a small variation of the control variable $x \to x + \varepsilon\bar{x}$, the variation in u will be:

$$u(x) \to u(x+\varepsilon\bar{x}) = u(x) + \varepsilon\bar{u} + O(\varepsilon^2)$$

A linearisation of the constraint equation gives a relation between $\bar{x}$ and $\bar{u}$:

$$\frac{\partial C}{\partial x}\bar{x} + \frac{\partial C}{\partial u}\bar{u} = 0$$

The variation of F is:

$$F(x+\varepsilon\bar{x}, u(x+\varepsilon\bar{x})) - F(x,u(x)) = \delta F$$
$$= \varepsilon\left(\bar{x}^T\frac{\partial F}{\partial x} + \bar{u}^T\frac{\partial F}{\partial u}\right) + O(\varepsilon^2)$$

which would suggest using $\bar{x}^T\frac{\partial F}{\partial x} + \bar{u}^T\frac{\partial F}{\partial u}$ as a descent direction. However this quantity depends on $\bar{u}$, the first variation of u which can only be computed after setting a direction of change $\bar{x}$ for the control variable x. To avoid these complications, we try to eliminate $\bar{u}$ from the variation of the functional F.

Noticing that $\bar{u}^T\frac{\partial C}{\partial u}^T + \bar{x}^T\frac{\partial C}{\partial x}^T = 0$, we multiply this quantity by an arbitrary vector $\lambda = (\lambda_1,\dots,\lambda_n)$ and we add this term (of value zero) to the variation of the functional. Then choosing a specific value for λ, we will eliminate the dependence in $\bar{u}$:

$$\delta F = \varepsilon\left(\bar{x}^T\frac{\partial F}{\partial x} + \bar{u}^T\frac{\partial F}{\partial u}\right) + \varepsilon\left(\bar{u}^T\frac{\partial C}{\partial u}^T + \bar{x}^T\frac{\partial C}{\partial x}^T\right)\lambda + O(\varepsilon^2)$$

which can be written as:

$$\delta F = \varepsilon\bar{x}^T\left(\frac{\partial F}{\partial x} + \frac{\partial C}{\partial x}^T\lambda\right) + \varepsilon\bar{u}^T\left(\frac{\partial F}{\partial u}^T + \frac{\partial C}{\partial u}^T\lambda\right) + O(\varepsilon^2)$$

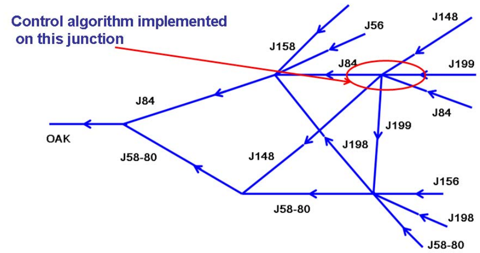

Fig. 2. *Network used for the optimisation containing 16 links and 5 junctions. This is an idealised figure representing the portion of the airspace considered, and links are not drawn to scale. The links are numbered according to the jetways they represent. We will focus on the junction circled in red.*

If we choose λ such that $\frac{\partial F}{\partial u}^T + \frac{\partial C}{\partial u}^T\lambda = 0$, the variation of the functional becomes:

$$\delta F = \varepsilon\bar{x}^T\left(\frac{\partial F}{\partial x} + \frac{\partial C}{\partial x}^T\lambda\right) + O(\varepsilon^2)$$

with the additional constraint:

$$\frac{\partial F}{\partial u}^T + \frac{\partial C}{\partial u}^T\lambda = 0$$

which is called the adjoint equation. The variation of the functional no longer depends explicitly on $\bar{u}$, and a descent direction is:

$$\bar{x} = -\left(\frac{\partial F}{\partial x} + \frac{\partial C}{\partial x}^T\lambda\right)$$

Using these results, the adjoint method can be written as the following algorithm:

1. For a fixed x, solve $C(x,u(x)) = 0$ for $u(x)$.
2. For a fixed x and $u(x)$, solve $\frac{\partial C}{\partial u}^T\lambda + \frac{\partial F}{\partial u} = 0$ for λ.
3. Update x as $x \leftarrow x - \delta\left(\frac{\partial F}{\partial x} + \frac{\partial C}{\partial x}^T\right)\lambda$.

We will use this technique to determine the gradient of the functional H. We consider links of length L_k, which in our example will be equal to the actual length of the corresponding links of the Air Traffic network considered, and we assume the network is empty at time $t = 0$. We bring the reader's attention to the fact that the following results can be applied to any functional

$$H(m_{ij}) = \sum_{k=1}^{m+n}\int_0^T\int_0^{L_k} h_k(\rho_k(x,t))dxdt$$

for any functions $h_k(x)$.

In order to compute the gradient of H we form the adjoint system corresponding to the partial differential equations developed at the end of Section II:

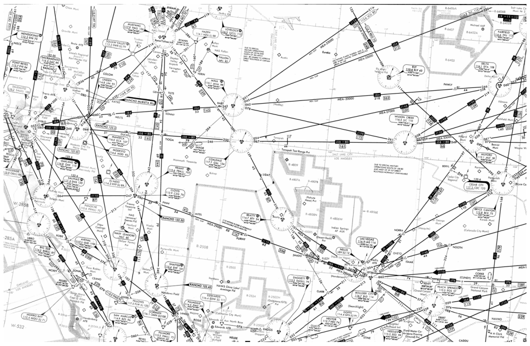

Fig. 3. *Map of the enroute high altitude jetways used for this study.*

$$\frac{\partial u_k(x,t)}{\partial t}+v'_k(x)u_k(x,t)+v_k(x)\frac{\partial u_k(x,t)}{\partial x}=0,\ 1\leqslant k\leqslant m+n$$

with terminal and boundary conditions:

$$u_k(x,T)=0,\ 1\leqslant k\leqslant m+n$$

$$u_i(0,t)=\frac{q_i(t)}{v_i(0)},\ 1\leqslant i\leqslant m$$

$$u_j(0,t)=\frac{\sum_{i=1}^{m} m_{ij}(t)\frac{du_i(L,t)}{dt}v_i(L)}{v_j(0)}+\frac{\sum_{i=1}^{m}\delta m_{ij}(t)u_i(L,t)v_i(L)}{v_j(0)}$$

$$m+1\leqslant j\leqslant m+n$$

We construct the Lagrangian:

$$L(m_{ij})=\sum_{k=1}^{m+n}\int_0^T\int_0^L\rho_k(x,t)$$

$$+\sum_{k=1}^{m+n}\int_0^T\int_0^L u_k(x,t)\left(\frac{\partial\rho_k(x,t)}{\partial t}+\frac{\partial(v_k(x)\rho_k(x,t))}{\partial x}\right)dxdt$$

We then linearise the Lagrangian to find the gradient of L with respect to the control variables m_{ij}:

$$\frac{\partial L(m_{ij})}{\partial m_{ij}}=\sum_{k=1}^{m+n}\int_0^T\int_0^L(-\frac{\partial u_k(x,t)}{\partial t}$$

$$-(v'_k(x)\rho_k(x,t)-v_k(x)\frac{\partial\rho_k(x,t)}{\partial x})\frac{\partial u_k(x,t)}{\partial x})\frac{\partial\rho_k(x,t)}{\partial m_{ij}}dxdt$$

$$+\sum_{k=1}^{m+n}\int_0^L\frac{\partial\rho_k(x,T)}{\partial m_{ij}}(u_k(x,T)\frac{\partial\rho_k(x,T)}{\partial m_{ij}}$$

$$-u_k(x,0)\frac{\partial\rho_k(x,0)}{\partial m_{ij}})dx$$

$$+\sum_{k=1}^{m+n}\int_0^T(u_k(L,t)(v'_k(x)\rho_k(L,t)$$

$$-v_k(x)\frac{\partial\rho_k(L,t)}{\partial x})\frac{\partial\rho_k(L,t)}{\partial m_{ij}}-u_k(0,t)\frac{\partial\rho_k(0,t)}{\partial m_{ij}})dt$$

After using the boundary conditions to simplify the linearised expression, we can now compute the gradient of H:

$$\frac{\partial H(m_{ij})}{\partial m_{ij}}=\int_0^T(u_i(L,t)v_i(L)(u_j(0,t)-\sum_{k=m+1,k\neq j}^{m+n}u_k(0,t))$$

$$-\sum_{j=m+1}^{m+n}\frac{du_j(0,t)}{dt}\delta m_{ij}(t))dt$$

At each iteration, we solve the original and adjoint equations using an upwind finite difference scheme and modify the descent direction accordingly.

Several methods are available to solve optimisation problems. The Newton method when applied to a function f with a starting point x_0 approximates f near x_0 by its second order Taylor expansion:

$$f(x_0+\varepsilon\bar{x}_0)=f(x_0)+\varepsilon\bar{x}_0^T\nabla f+\frac{1}{2}\varepsilon^2\bar{x}_0^T\nabla^2 f\bar{x}_0$$

where ∇f and $\nabla^2 f$ denote the gradient and Hessian of f. Assuming the Hessian is definite positive, the corresponding descent direction would be $-(\nabla^2(x))^{-1}\nabla f(x)$; therefore this method requires computing the Hessian at each step which involves evaluating $O(n^2)$ partial derivatives. While one might be tempted to use a finite difference approximation of the Hessian, this would require at least n evaluations of the gradient ∇f. Additionally, the Hessian may not be positive definite.

These weaknesses of the Newton method led to the development of quasi-Newton methods in which the Hessian is approximated by a symmetric positive definite matrix. Indeed, the Taylor expansion for the gradient of f:

$$\nabla f(x_0+\varepsilon\bar{x}_0)=\nabla f(x_0)+\varepsilon\nabla^2 f(\bar{x}_0)\bar{x}_0+O(\varepsilon^2)$$

gives a possible approximation of the Hessian of f in the direction $\bar{x}_0$ without computing approximations of the individual elements of the Hessian:

$$\bar{x}_0^T\nabla^2 f\bar{x}_0\approx\bar{x}_0^T(\nabla f(x_0+\varepsilon\bar{x}_0)-\nabla f(x_0))$$

Therefore, we can replace the Hessian in the Newton method algorithm by any symmetric matrices verifying:

$$M_{k+1}(x_{k+1}-x_k)=\nabla f(x_{k+1})-\nabla f(x_k)$$

In several dimensions, this relation does not determine uniquely the matrices M_k and several choices are possible. One possibility is given by the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method:

$$M_{k+1}=M_k-\frac{M_k(x_{k+1}-x_k)(M_kx_{k+1}-x_k)^T}{(x_{k+1}-x_k)^TM_k(x_{k+1}-x_k)}$$

$$+\frac{(\nabla f(x_{k+1})-\nabla f(x_k))(\nabla f(x_{k+1})-\nabla f(x_k))^T}{(\nabla f(x_{k+1})-\nabla f(x_k))^T(x_{k+1}-x_k)}$$

The BFGS method can be implemented according to the algorithm:

1 For a starting point x_0 and tolerance ε
  compute the inverse of the Hessian approximation M
2 $k \leftarrow 0$
3 while $\|\nabla f(x_k)\| > \varepsilon$
  compute search direction: $-M_k \nabla f(x_k)$
4 $x_{k+1} = x_k - M_k \nabla f(x_k)$
5 Compute M_{k+1} using the above formula
6 $k \leftarrow k+1$
7 end(while)

While the BFGS method is efficient for small scale problems, it becomes quite impractical when dealing with large scale systems for which the cost of manipulating the Hessian approximations is too elevated. In this case, limited memory methods are preferred, which store a few vectors rather than a fully dense matrix thereby reducing storage requirements. One such method is the limited memory BFGS known as L-BFGS-B based on the BFGS formula. Rather than storing the main Hessian approximation M_k, one can use a small number of vector pairs $\{x_{k+1} - x_k, \nabla f(x_{k+1}) - \nabla f(x_k)\}$ that are used in computing M_k. The product $M_k \nabla f(x_k)$ can be obtained by computing a number of inner products and vector summations involving $\nabla f(x_k)$ and the preceding vector pairs. After each iteration, the oldest vector pair is replaced by the new one.

We chose to use this method on the optimisation problem at hand through the L-BFGS-B routines developed by Byrd, Lu, Nocedal and Zhu in [11], [12], [13]. Several other software for nonlinear optimisation are available, such as MINOS or NPSOL.

The following algorithm was implemented and converged to a minimum of the optimisation program:

1 Solve the partial differential equations for the
  density on each link.
2 Solve the adjoint equations.
3 Evaluate the gradient of the cost functional.
4 Use this result in an optimisation method
  (L-BFGS-B in the present case).
5 Return to step 1.

We now implement this optimisation method on a network represented in Figure 2; the links are taken from the high altitude enroute jetways between Salt Lake City and Oakland International Airport. We use jetways J56, J58-80, J84, J148, J156, J158, J198, and J199. The input is constructed using ETMS data to keep a sense of realism.

We use the methods developed in the article for congestion mitigation, more precisely, we try to keep the aircraft flow on the three outgoing links of the junction circled in red in Figure 2 under a threshold value of 20 aircraft per hour. We represent the flow on the outgoing and incoming links with and without a control strategy being applied in Figure 4. We note that without control the flows are often above

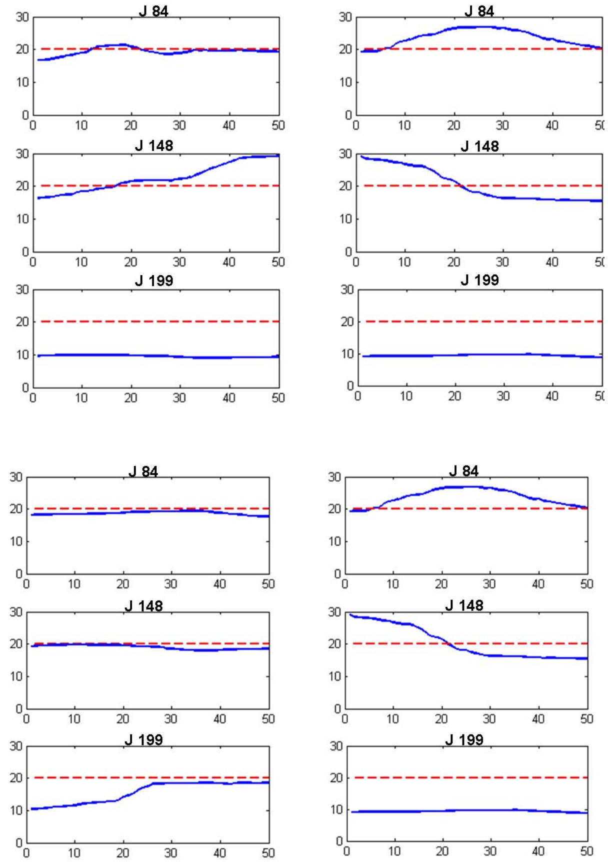

Fig. 4. *Evolution of the aircraft flow on the 6 links of the junction considered. The outgoing links are represented on the left and the incoming links on the right. The horizontal axis is the location and the vertical axis is the flow in aircraft per hour. Air Traffic flow goes from right to left. The top subfigure shows the situation without aircraft being reallocated while the bottom one depicts the results when a control strategy is implemented. As can be seen the threshold (represented by the red dotted line) is exceeded in the first case while the flow is always below the limit with reallocation of aircraft.*

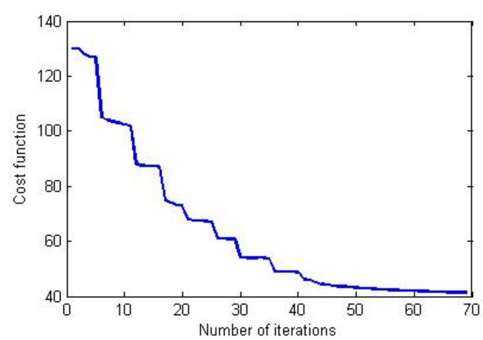

Fig. 5. *Evolution of the cost function with respect to the number of iterations; a large number of iterations are required to approach the minimum value.*

the desirable threshold while with the application of the optimal control strategy, we manage to maintain the flows under the limit at all times. The method used here consists in finding an optimal routing through the coefficients of the allocation matrix that will prevent sudden jumps in aircraft density. These coefficients are automatically adjusted in order to allow the best repartition of aircraft on the outgoing links; if a given link is becoming congested, the allocation coefficient that regulates the inflow on this link will decrease and correspondingly the other coefficients at this junction will increase, thus redirecting the aircraft on less congested links. Thus, we are able to maintain a regular spacing between the aircraft even if sudden increases in aircraft density are registered on the incoming links. In the absence of control, these jumps in aircraft density are not mitigated and eventually raise the aircraft flows above the limit. Another implementation was presented in the article [14] where the objective was to control the aircraft flow on the final approach link to Oakland International Airport using a given input into the network. In that case, a discrete adjoint method was used and similarly, the flow without control exceeded the threshold while when the optimal allocation strategy was in place, the aircraft flow was maintained under the desired limit at all times.

The variations of the cost function after each iteration are represented in Figure 5. The functional H is not convex and as a consequence a large number of iterations are needed before approaching the minimum. The steep drops in the cost function are due to the use of logarithmic barriers to enforce the constraints.

IV. Conclusion

A continuous flow Eulerian model for Air Traffic Networks was analysed, and the existence and uniqueness of a solution to the system of hyperbolic partial differential equations established. The second part of the article dealt with an optimal control problem for which the control variables were the junction coefficients. A continuous adjoint approach was applied to a network with 16 links and 5 junctions and a numerical simulation implemented. A comparison was made between the aircraft flows on the three outgoing links of a given junction with and without a control strategy. In the absence of control, the aircraft flows frequently increased over the required limit while they were maintained below this limit at all times when the routing strategy was applied. The results obtained show the efficiency of this method in alleviating airspace congestion through the optimal routing of flights. Future endeavours would include a comparison with the discrete adjoint method established in [14] for this problem, and last but not least, developing a new model of Air Traffic flows that would allow optimal control problems similar to the one studied in this article to be numerically solvable at the scale of an entire center (a center containing on average 200 to 300 links) in a short amount of time.

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