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Minimal error certificates for detection of faulty sensors using convex optimization

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Abstract— **This article proposes a new method for sensor fault detection, applicable to systems modeled by conservation laws. The state of the system is modeled by a *Hamilton-Jacobi* equation, in which the Hamiltonian is uncertain. Using a Lax-Hopf formula, we show that any local measurement of the state of the system restricts the allowed set of possible values of other local measurements. We derive these constraints explicitly for arbitrary Hamilton-Jacobi equations. We apply this framework to sensor fault detection, and pose the problem finding the minimal possible sensor error (minimal error certificate) as a set of convex programs. We illustrate the performance of the resulting algorithms for a highway traffic flow monitoring sensor network in the San-Francisco Bay Area.**

I. INTRODUCTION

In control theory and information theory, the problems of *system identification* [14] and *estimation* [3] are closely linked. System identification uses data to infer the value of parameters in the model of a system. Estimation takes a given mathematical model for which the model parameters have been obtained (for instance using system identification) and computes an estimate of the state of the system, from the model. In artificial intelligence, these two steps are respectively referred to as *learning* and *inference* [12]. In control theory, traditional estimates are optimal in the least square sense, for example the Kalman filter recursively provides a least square minimizer of the error functional between measurements and estimates. Extensions of the Kalman filter have led to various generalizations of these minimizers.

When numerical values for model parameters are collected (as in system identification), they usually form a distribution. A possible approach to deal with distributions, is to maximize the log likelihood of the model, and to select the result as a constitutive model, subsequently used to do inference or estimation.

An alternative is to account for the uncertainty in the model by creating a family of models (within a class), represented by the distribution. The present article follows a similar methodology for a class of nonlinear distributed parameter systems. More specifically, the article follows the framework and assumptions summarized below.

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Learning set: We assume the availability of a learning set. This set encompasses prior physical realizations of the system, known to be physically acceptable.

Family of model parameters: The model describing the physical system of interest is described by a distributed parameter (in the present case, a function). There exists a family of distributed model parameters which characterize the model (for example a parametric family of functions, or the subset of a functional space).

Certificate: We assume the availability of a certificate, *i.e.* a procedure to label a model parameter as acceptable or not, based on the learning set.

Measurements: Measurements of the state of the system are given, which we seek to evaluate.

An example of this framework is given in section II for the specific case of highway traffic sensors, for illustrative purposes.

Since a single model parameter does not fully capture the underlying physics of the system, it is in general impossible to exactly reconstruct the state of the system using the model and partial measurements of the state. Equivalently, a partial measurement of the state will not completely constraint the value of another partial measurement. In other words, the value of a given measurement only partially restricts the possible values which other measurements could have to be compatible. This is due to the fact that the system of interest is not described by a single model, but by a family of models. Since we want to detect faults in sensors measuring the state of the system, we want to check if the measurements are physically compatible with the family of models and the learning set certificate or not. The natural question to ask is thus:

Given some exact measurements, given the family of models and the learning set certificate, can we find a set of necessary conditions that these measurements have to satisfy to be compatible? As a corollary, can we compute the distance (minimal error) between noisy measurements and the set of measurements which satisfy the family of models and the learning set certificate?

The contributions of the article are as follows. It solves the problem outlined above for a specific class of nonlinear *partial differential equations* (PDEs) constitutive models: the *Hamilton-Jacobi* (HJ) PDE. It makes the assumptions that the Hamiltonians of the HJ PDEs are not known but belong to a family. It derives necessary conditions that the state of the system must obey, assuming that the model parameters describing its evolution belong to the learning set. These necessary conditions yield consistency conditions for the sensor measurements, which can be used directly to detect any inconsistency in a set of exact sensor measurements.

The proposed method leads to a practical application. It poses the problem of finding the minimal possible error of the sensors in the L_1, L_2 and L_∞ sense as a set of convex programs, and demonstrates the applicability of the method to sensor network fault detection. The applicability of this framework is demonstrated with the example of a network of traffic sensors, by computing a certificate assessing if the minimal possible error of the sensors for a given set of measurements is within the sensors certified range or not.

The rest of the article is organized as follows. Section II presents the properties of the state considered in this article, as well as the HJ PDE which models it. Section III derives compatibility conditions between local state measurements, provided that these measurements are exact. We apply these results in section IV to the minimal error certificate problem, in which we find the minimal **possible** error associated with the measurements of a pair of sensors.

II. SOLUTIONS TO SCALAR HAMILTON-JACOBI EQUATIONS

A. First order scalar hyperbolic conservation laws with concave Hamiltonians

First order scalar hyperbolic conservation laws [11] are partial differential equations derived from the conservation principle of a physical quantity P (for instance mass, electric charge or energy). If $\rho(\cdot,\cdot)$ represents the density of P, and $q(\cdot,\cdot)$ represents the flow of P, the conservation of P at location $x \in X$ and for a time $t \in [0, t_{\max}]$ is expressed by:

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial q(t,x)}{\partial x} \;=\; 0 \qquad (1)$$

In the remainder of the article, we assume that the spatial domain X is defined by $X := [\xi, \chi]$, where ξ and χ represent the upstream and downstream boundaries of the domain. In numerous application fields such as thermodynamics, fluid mechanics or traffic flow theory, there exists a constitutive equation between the density $\rho(\cdot,\cdot)$ and the flow $q(\cdot,\cdot)$ of the form $q(\cdot,\cdot) = \psi(\rho(\cdot,\cdot))$. This relation leads to the following form for equation (1):

$$\frac{\partial \rho(t,x)}{\partial t} + \frac{\partial \psi(\rho(t,x))}{\partial x} \;=\; 0 \qquad (2)$$

In conservation laws, the function $\psi(\cdot)$ is known as *flux function*. In the context of Hamilton-Jacobi equations investigated later, it is called *Hamiltonian*. In the remainder of the article, we assume that the Hamiltonian $\psi(\cdot)$ is an upper semicontinuous concave function. Numerous fields make the assumption of a single valued $\psi(\cdot)$ function to capture the dynamics of the system, whereas experimental data strongly suggests that a family of such $\psi(\cdot)$ would be required to fully capture the state of the system.

B. Hamilton-Jacobi equations with concave Hamiltonians

Instead of solving equation (2) directly, we consider an equivalent formulation obtained using the following standard variable change:

Definition 2.1: **[Integral formulation of the conservation law]**. Given a density function $\rho(\cdot,\cdot)$ and a flow function $q(\cdot,\cdot)$, we define the integral form $\mathbf{M}(\cdot,\cdot)$ as:

$$\begin{cases} \mathbf{M}(0,0) := 0 \\ \mathbf{M}(t_2,x_2) - \mathbf{M}(t_1,x_1) := \int_{x_1}^{x_2} -\rho(t_1,x)dx \\ +\int_{t_1}^{t_2} q(t,x_2)dt, \quad \forall (t_1,x_1,t_2,x_2) \in ([0,t_{\max}] \times X)^2 \end{cases} \qquad (3)$$

In the remainder of this article, $\mathbf{M}(\cdot,\cdot)$ is denoted as *state* of our problem. The state evolution equation is the following *Hamilton-Jacobi* (HJ) PDE , obtained by integration of equation (2):

$$\frac{\partial \mathbf{M}(t,x)}{\partial t} - \psi\left(-\frac{\partial \mathbf{M}(t,x)}{\partial x}\right) \;=\; 0 \qquad (4)$$

In the illustrative applications presented later, the state represents the *cumulative number of vehicles function* [16], [9], [10], [5], which is a possible way of describing the flow of vehicles on a highway section.

C. State estimation

For clarity, we define two different functions which represent the true state of the system, and its estimate, obtained using a model as well as partial state information (estimated state).

Definition 2.2: **[True state and estimated state]** The true state $\overline{\mathbf{M}}(\cdot,\cdot)$ represents the state of the system, which could be obtained if measured by errorless sensors covering the entire space-time domain $[0,t_{\max}] \times X$.
The *estimated state* of the system represents the value of the state inferred from partial knowledge of the true state (denoted as value condition) on a subset $\mathcal{D}$ of $[0,t_{\max}] \times X$, and the state evolution equation (4):

$$\begin{cases} \mathbf{M}(\cdot,\cdot) \text{ solves equation (4) on } [0,t_{\max}] \times X \backslash \mathcal{D} \\ \mathbf{M}(t,x) = \overline{\mathbf{M}}(t,x), \;\; \forall (t,x) \in \mathcal{D} \end{cases} \qquad (5)$$

Note that $\mathcal{D}$ can be any subset of the space time domain $[0,t_{\max}] \times X$. The set $\mathcal{D}$ is not necessarily included in the boundary [6] of $[0,t_{\max}] \times X$, nor of empty interior [8]. The traditional *Cauchy problem* is defined by $\mathcal{D} := \{\{0\} \times X\} \bigcup \{[0,t_{\max}] \times \{\xi\}\} \bigcup \{[0,t_{\max}] \times \{\chi\}\}$.
The modeled state of the system depends upon the choice of the model parameters, *i.e.* the choice of the Hamiltonian $\psi(\cdot)$ in the present case. Because of modeling errors, the true state is in general not identical to the estimated state, as can be seen in Figure 1, in the context of traffic. As can be seen in the figure, the solution $\mathbf{M}(\cdot,\cdot)$ of (4) with the proper value of $\overline{\mathbf{M}}(\cdot,\cdot)$ as initial and boundary conditions does not lead to $\overline{\mathbf{M}}(\cdot,\cdot)$ everywhere in the domain. This is the motivation of the present article.

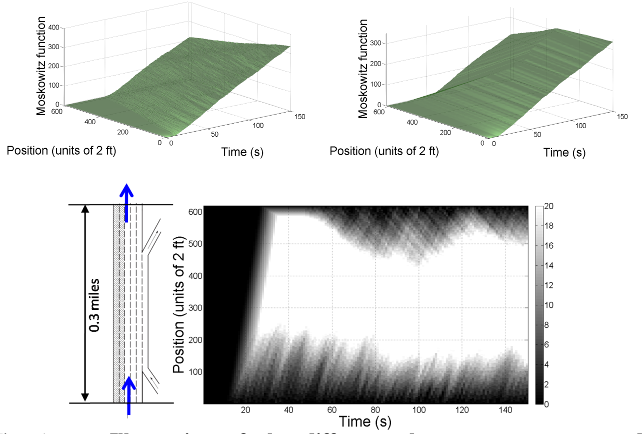

Fig. 1. **Illustration of the difference between true state and modeled state**. We consider here the traffic flow on a highway section near Emeryville, CA. The state of traffic is represented by a scalar function of time and space known as *cumulative number of vehicles function* [16], [9], [10]. The upper left figure represents the true state of the system (obtained in the present case using video cameras monitoring the entire domain of interest [5], [7]). The upper right figure represents the modeled state, reconstructed from the knowledge of the initial, upstream and downstream conditions (illustrated in Figure 3). The difference (in absolute value) between the true state and the modeled state is shown in the lower figure as a colormap. The dark areas represent areas where the true state is close to the modeled state, while the true state is far from the modeled state in pale areas.

The fact that the state $\overline{\mathbf{M}}(\cdot,\cdot)$ does not solve (4) exactly can also be inferred from Figure 2. Indeed, as can be seen from the figure, there is no single valued relation between $-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}$ and $\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t}$, due to the set valued nature of the past realizations.

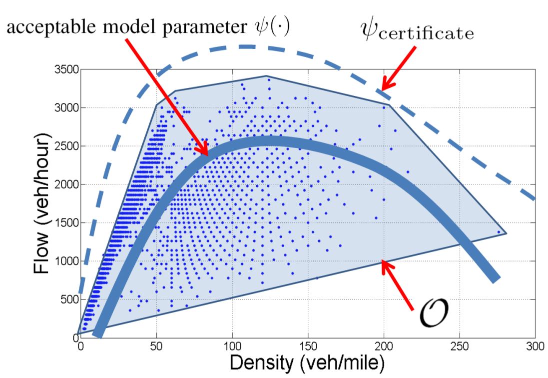

Fig. 2. **Illustration of an experimental flow-density diagram $\psi(\cdot)$.** This plot represents the set of possible values of flow and density obtained from an experimental data set [4]. The horizontal axis represents the density, whereas the vertical axis represents the flow. Each point in this plot has coordinates $(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}, \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t})$ for some $(t,x) \in [0, t_{\max}] \times X$. Note that $\overline{\mathbf{M}}(t,x)$ is a solution (in the weak sense) of (4) if and only if *almost all* points of coordinates $(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}, \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t})$ (for all $(t,x) \in [0, t_{\max}] \times X$) belong to the graph of some concave function ψ, which obviously cannot be the case in the present situation.

Example 2.3: **[Learning set, model parameters, certificate and measurements in the context of traffic flow sensing]** In the context of traffic flow sensing, the learning set is the set of points $\mathcal{P} := \left\{(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}, \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t}),\ \forall (t,x) \in [0, t_{\max}] \times X\right\}$. The convex hull of $\mathcal{P}$ is denoted $\mathcal{O}$. The certificate is a concave and lower semicontinuous function $\psi_{\text{certificate}}(\cdot)$ such that $\mathcal{O} \subset \mathcal{H}yp(\psi_{\text{certificate}})$. Any concave and lower semicontinuous function $\psi(\cdot)$ such that $\psi(\cdot) \le \psi_{\text{certificate}}(\cdot)$ pointwise is an acceptable model parameter. These definitions are illustrated in Figure 2.

In the present application of interest, the measurements come from flow sensors, which measure $\frac{\partial \overline{\mathbf{M}}(t,\xi)}{\partial t}$ and $\frac{\partial \overline{\mathbf{M}}(t,\chi)}{\partial t}$ for all $t \in [0, t_{\max}]$.

Fact 2.4: **[Physical properties of the state]** The true state $\overline{\mathbf{M}}(\cdot,\cdot)$ is assumed to satisfy the following properties:

1) $\overline{\mathbf{M}}(\cdot,\cdot)$ is Lipschitz-continuous, and thus differentiable almost everywhere.
2) The space and time derivatives of $\overline{\mathbf{M}}(t,x)$ are bounded and continuous almost everywhere.

The two above properties hold whenever the flow $\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t}$ and density $-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x}$ associated with the conservation equation (1) are continuous almost everywhere and bounded.

D. Value conditions

Solving equation (4) requires the knowledge of *value conditions* (for instance initial, boundary and/or internal conditions [5], [6]) to characterize the existence and uniqueness of the solution.

Definition 2.5: **[Value condition]** Let the true state $\overline{\mathbf{M}}(\cdot,\cdot)$ be given. A *value condition* $\mathbf{c}(\cdot,\cdot)$ is a function of $[0, t_{\max}] \times X$ satisfying the following condition:

$$\mathbf{c}(t,x) = \overline{\mathbf{M}}(t,x), \quad \forall (t,x) \in \mathrm{Dom}(\mathbf{c}) \tag{6}$$

where $\mathrm{Dom}(\mathbf{c})$ is a subset of $[0, t_{\max}] \times X$.

A value condition $\mathbf{c}(\cdot,\cdot)$ represents partial knowledge of the state $\overline{\mathbf{M}}(\cdot,\cdot)$ on some domain $\mathrm{Dom}(\mathbf{c})$. Figure 3 illustrates the domains of definition of some value conditions.

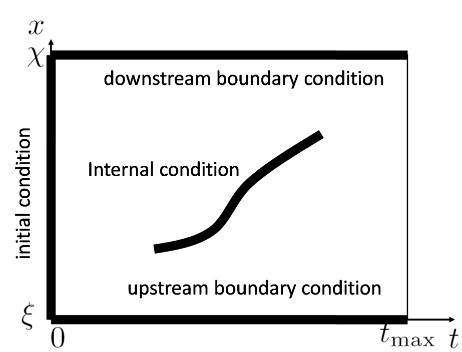

Fig. 3. **Illustration of the domains of the possible value conditions used to construct the solution to the HJ PDE (4)**. The initial condition $\mathcal{M}_0$ is defined on $\{0\} \times X$. The upstream and downstream boundary conditions are defined on $[0, t_{\max}] \times \{\xi\}$ and $[0, t_{\max}] \times \{\chi\}$ respectively. Internal conditions are defined on an empty interior domain included in $]0, +\infty[\times]\xi, \chi[$.

Other types of value conditions exist, such as the hybrid conditions [8]. In this article, we focus on upstream and downstream boundary conditions, though the proposed algorithm can be extended to the case of internal conditions.

Definition 2.6: **[Boundary condition functions]** The upstream and downstream boundary conditions, denoted by $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are value conditions (6) defined on $[0, t_{\max}] \times \{\xi\}$ and $[0, t_{\max}] \times \{\chi\}$ respectively.

E. Properties of the solutions to the HJ PDE problem

In order to define the properties of the solution, we first need to define a convex transform of the Hamiltonian $\psi(\cdot)$ as follows.

Definition 2.7: **[Convex transform]** For a concave function $\psi(\cdot)$ defined as previously, the convex transform φ^* is given by:

$$\varphi^*(u) := \sup_{p \in \text{Dom}(\psi)} [p \cdot u + \psi(p)] \tag{7}$$

In the remainder of this article, we use the results of [2], [7], [5], [6] to characterize the solution to equation (4) by a Lax-Hopf formula. The Lax-Hopf formula is derived in [2], [5], using results from viability and control theory.

Proposition 2.8: **[Lax-Hopf formula]** The solution $\mathbf{M_c}(\cdot,\cdot)$ associated with a lower semicontinuous value condition $\mathbf{c}(\cdot,\cdot)$ can be computed using the following Lax-Hopf formula:

$$\mathbf{M_c}(t,x) = \inf_{(u,T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} (\mathbf{c}(t-T, x+Tu) + T\varphi^*(u)) \tag{8}$$

The Lax-Hopf formula provides an explicit solution of the problem defined in [2]. Equation (8) also implies a very important inf-morphism property [2], [5], [6], [7], which is a key property used to build the algorithms used in this article. This property was initially derived using capture basins [2].

Proposition 2.9: **[Inf-morphism property]** Let us assume that the value condition $\mathbf{c}$ is the minimum of a finite number of functions $\mathbf{c}_i$, namely:

$$\forall (t,x) \in [0, t_{\max}] \times X, \;\; \mathbf{c}(t,x) := \min_{i \in I} \mathbf{c}_i(t,x) \tag{9}$$

With the above assumption, the solution $\mathbf{M_c}$ defined by (8) can be written as:

$$\forall (t,x) \in [0, t_{\max}] \times X, \;\; \mathbf{M_c}(t,x) = \min_{i \in I} \mathbf{M}_{\mathbf{c}_i}(t,x) \tag{10}$$

Proof — The definition (9) of $\mathbf{c}(\cdot,\cdot)$ and the Lax-Hopf formula (8) imply:

$$\mathbf{M_c}(t,x) = \inf_{(u,T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} \left(\min_{i \in I} (\mathbf{c}_i(t-T, x+Tu)) + T\varphi^*(u) \right) \tag{11}$$

Equation (10) is obtained by reversing the order of the infimums in equation (11). ■

The inf-morphism property is a practical tool to integrate new value conditions for computing the modeled state $\mathbf{M}(\cdot,\cdot)$. In addition, we can use it to break a complex problem involving multiple value conditions into a set of more tractable subproblems [5], [6], [8].

In the next section, we use (8) to derive a set of compatibility conditions that have to be satisfied by the value conditions (obtained through measurements). These conditions will enable us to construct convex optimization programs for sensor error analysis in section IV.

III. Constraints of the model on the value conditions

A. Upper estimate of the Hamiltonian

We first need to define a particular class of Hamiltonians satisfying the following property.

Proposition 3.1: **[Learning set certificate]**. For any given state $\overline{\mathbf{M}}$ satisfying the set of properties outlined in section 2.4, there exists a concave and upper semicontinuous function $\psi_{\text{certificate}}(\cdot)$ such that:

$$\forall (t,x) \in [0, t_{\max}] \times X, \quad \frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t} \leq \psi_{\text{certificate}} \left(-\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial x} \right) \tag{12}$$

Proof — The proof of this proposition is immediate. Indeed, we can always choose for $\psi_{\text{certificate}}(\cdot)$ any constant function greater than the upper bound of $\frac{\partial \overline{\mathbf{M}}(t,x)}{\partial t}$. ■

Note that the choice of a function $\psi_{\text{certificate}}(\cdot)$ compatible with (12) is not unique. An example of choice of $\psi_{\text{certificate}}(\cdot)$ satisfying (12) is illustrated in Figure 2.

B. Compatibility conditions

We now show that, for any finite set of value conditions, there exist compatibility conditions between solutions and value conditions.

Proposition 3.2: **[Compatibility conditions]** Let us define a set of value condition functions $\mathbf{c}_i(\cdot,\cdot)$ as in (6), an upper semicontinuous and concave Hamiltonian $\psi_{\text{certificate}}(\cdot)$ satisfying (12), and its associated convex transform φ^* as in (7). We also define a set of solutions $\mathbf{M}_{\mathbf{c}_i}(\cdot,\cdot)$ associated with $\mathbf{c}_i(\cdot,\cdot)$ as in (8). Given these assumptions, the following set of conditions must be satisfied:

$$\begin{array}{l} \mathbf{M}_{\mathbf{c}_j}(t,x) \geq \mathbf{c}_i(t,x), \;\; \forall (t,x) \in \text{Dom}(\mathbf{c}_i), \\ \forall (t,x) \in \text{Dom}(\mathbf{c}_i), \forall i \in [1,n], \;\; \forall j \in [1,n] \end{array} \tag{13}$$

Proof — Let us consider $i \in [1,n]$, $j \in [1,n]$ and $(t,x) \in \text{Dom}(\mathbf{c}_i)$.

We first express $\mathbf{M}_{\mathbf{c}_i}(t,x)$ in terms of $\mathbf{c}_i(\cdot,\cdot)$ using the Lax-Hopf formula (8):

$$\mathbf{M}_{\mathbf{c}_i}(t,x) = \inf_{(u,T) \in \text{Dom}(\varphi^*) \times \mathbb{R}_+} (\mathbf{c}_i(t-T, x+Tu) + T\varphi^*(u)) \tag{14}$$

In the above formula, φ^* is the convex transform of the Hamiltonian $\psi_{\text{certificate}}(\cdot)$, which satisfies equation (7).

Since the value condition $\mathbf{c}_i(\cdot,\cdot)$ satisfies (6) and $(t,x) \in \text{Dom}(\mathbf{c}_i)$, we have that $\overline{\mathbf{M}}(t,x) = \mathbf{c}_i(t,x)$. Hence, we can write the inequality $\mathbf{M}_{\mathbf{c}_j}(t,x) \geq \mathbf{c}_i(t,x)$ as:

$$\inf_{(T,u) \in [0, t_{\max}] \times \text{Dom}(\varphi^*)} \left(\overline{\mathbf{M}}(t-T, x+Tu) + T\varphi^*(u) \right) \geq \overline{\mathbf{M}}(t,x) \tag{15}$$

Since the derivatives of $\overline{\mathbf{M}}(\cdot,\cdot)$ are continuous almost everywhere and bounded, we can write:

$$\begin{array}{l} \overline{\mathbf{M}}(t-T, x+Tu) + T\varphi^*(u) - \overline{\mathbf{M}}(t,x) = \\ \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial t} + u \frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial x} + \varphi^*(u) \right) d\tau \end{array} \tag{16}$$

Since $\psi_{\text{certificate}}(\cdot)$ is concave and upper semicontinuous, it is equal to its Legendre-Fenchel biconjugate. Hence, we have [5], [8] that $\psi_{\text{certificate}}(\rho) = \inf_{u \in \text{Dom}(\varphi^*)} (-\rho u + \varphi^*(u))$, and thus that $\psi_{\text{certificate}}(\rho) \leq -\rho u + \varphi^*(u)$ for all $u \in \text{Dom}(\varphi^*)$. This result enables us to derive the following inequality from equation (16):

$$\overline{\mathbf{M}}(t-T, x+Tu) + T\varphi^*(u) - \overline{\mathbf{M}}(t,x) \geq \int_0^T \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial t} + \psi_{\text{certificate}} \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial x} \right) \right) d\tau \tag{17}$$

Since $\psi_{\text{certificate}}(\cdot)$ satisfies (12), we have that $-\frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial t} + \psi_{\text{certificate}} \left(-\frac{\partial \overline{\mathbf{M}}(t-\tau, x+\tau u)}{\partial x} \right) \geq 0$ for all $(\tau, u) \in [0, T] \times \text{Dom}(\varphi^*)$. Since $T > 0$, the right hand side of equation (17) is nonnegative, which implies the following inequality:

$$\begin{array}{l} \forall (T,u) \in \mathbb{R}_+ \times \text{Dom}(\varphi^*), \\ \overline{\mathbf{M}}(t-T, x+Tu) + T\varphi^*(u) - \overline{\mathbf{M}}(t,x) \geq 0 \end{array} \tag{18}$$

Equation (15) is obtained from equation (18) by taking the infimum over $(T,u) \in \mathbb{R}_+ \times \text{Dom}(\varphi^*)$, which completes the proof. ■

We now present a practical application of the above compatibility conditions: the computation of the minimal error certificate for a pair of sensors.

IV. Application: Linear/Quadratic programming formulation of the minimal error certificate problem

A. Problem definition

Motivated by applications of sensor networks for highway traffic monitoring systems, we consider a stretch of highway located between the upstream and downstream boundaries called ξ and χ respectively, as illustrated in Figure 4. The state of the system is described here by the cumulative number of vehicles function, which is formally defined in [9], [10], [15], [16]. Two fixed sensors located at ξ and χ measure the flow, *i.e.* the time derivative of the state, as illustrated in Figure 4

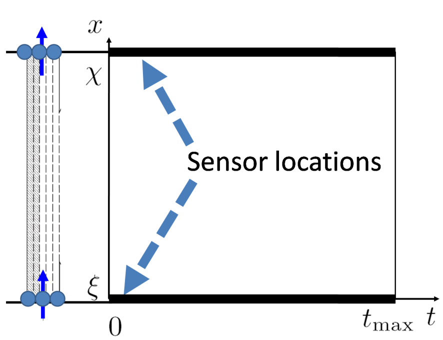

Fig. 4. **Illustration of the sensor layout**. The traffic flow data is obtained from two fixed sensors (arrows) located at the upstream and downstream boundaries of the domain, and measuring the time derivative of the state for all times $t \in [0, t_{\max}]$.

Our objective is to compute the minimal error certificate on the observed data, which can be thought of as follows. The vector of observed data s^{meas} can be viewed as a point in a multidimensional space $\mathcal{S}$. The set $\mathcal{C}$ of *consistent* data, that is the set of data points which obey (13) can also be viewed as a subset of $\mathcal{S}$. The exact data $\overline{s}$, which is unknown to us, is an element of $\mathcal{C}$ by Proposition 3.2. The distance d between s^{meas} and the set $\mathcal{C}$ is defined by:

$$d := \inf_{s \in \mathcal{C}} |s - s^{\text{meas}}|$$

Since $\overline{s} \in \mathcal{C}$, the error $e := |\overline{s} - s^{\text{meas}}|$ satisfies $e \geq d$. Note that the error is unknown since the exact data $\overline{s}$ is unknown, but the smallest possible value of e is d. This property is illustrated in Figure 5.

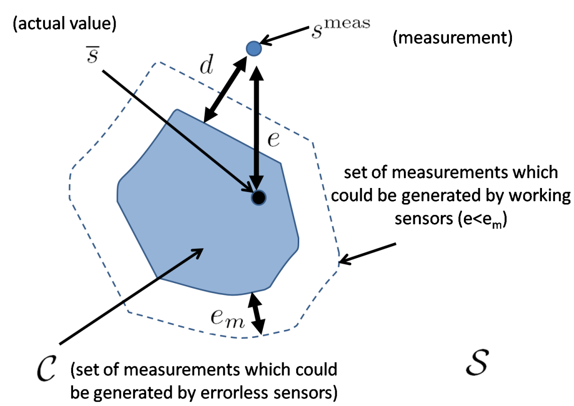

Fig. 5. **Illustration of the minimal error problem**. This figure illustrates the problem of interest in the space of data points $\mathcal{S}$. The set of physically possible elements of $\mathcal{S}$ (*i.e.* the set of points which satisfy 3.2) is denoted as $\mathcal{C}$ (arrow). The real value of the data is a point $\overline{s} \in \mathcal{C}$, and is usually unknown. The minimal error of the sensors generating a data point s is measured by the distance d between s and $\mathcal{C}$. If this distance is greater than the maximal admissible error for a pair of working sensors (dashed line), then the point d is necessarily created by a faulty pair of sensors. Note however that the converse does not imply that the sensors are not faulty.

The maximal level of error e_m of a working sensor is usually established by the manufacturer as part of the sensor design. A distance d greater than e_m implies an error e greater than e_m, which indicates the failure of at least one of the sensors (wrong data, bad hypothesis on the sensor placement,...). However, note that a distance d lower than this threshold does not indicate that the sensors are working correctly. Indeed, the error e can be greater than e_m even if d is lower than e_m.

The following section details the measurements and error norms.

B. Sensor measurements

In the remainder of this article, we assume that the upstream and downstream sensors generate the following measurements:

Definition 4.1: **[Sensor measurements]** Let N be the set of integers $[0, n_{\max}]$ such that $(n_{\max}+1)T = t_{\max}$, where

$t_{\max}$ is a finite time horizon. The discrete time measurements of the upstream and downstream sensors are defined by:

$$\left\{ \begin{array}{lll} q_{\text{in}}^{\text{meas}}(n), & \forall n \in N & \text{upstream sensor flow measurements} \\ q_{\text{out}}^{\text{meas}}(n), & \forall n \in N & \text{downstream sensor flow measurements} \end{array} \right. \tag{19}$$

The exact values of the parameters are denoted by:

$$\left\{ \begin{array}{lll} q_{\text{in}}(n), & \forall n \in N & \text{true upstream flow value} \\ q_{\text{out}}(n), & \forall n \in N & \text{true downstream flow value} \end{array} \right. \tag{20}$$

They are not known in practice.

The difference between sensor measurements and exact values is determined by the sensor error, which is usually expressed using some norm, for instance in the L_1, L_2, or L_∞ sense. These norms are defined for q_{in} (the case q_{out} follows similarly) as:

$$\begin{aligned} e_{L_1} &:= \sum_{i=0}^{n} \left| \frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)} \right| \\ e_{L_2} &:= \sum_{i=0}^{n} \left(\frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)} \right)^2 \\ e_{L_\infty} &:= \max_{i \in N} \left| \frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)} \right| \end{aligned} \tag{21}$$

Using the slack variables $e(\cdot)$ and f, we can also express equation (21) as:

$$\begin{aligned} e_{L_1} = \ & \min \left(\sum_{i=0}^{n} e(i) \right) \\ & \text{s.t.} \left\{ \begin{array}{l} e(i) \geq \frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)}, \ \forall i \in N \\ \text{and } e(i) \geq -\frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)}, \ \forall i \in N \end{array} \right. \end{aligned} \tag{22}$$

$$\begin{aligned} e_{L_2} = \ & \min \left(\sum_{i=0}^{n} e(i) \right) \\ & \text{s.t. } e(i) \geq \left(\frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)}\right)^2, \ \forall i \in N \end{aligned} \tag{23}$$

$$\begin{aligned} e_{L_\infty} = \ & \min f \\ & \text{s.t.} \left\{ \begin{array}{l} f \geq \frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)}, \ \forall i \in N \\ \text{and } f \geq -\frac{q_{\text{in}}(i) - q_{\text{in}}^{\text{meas}}(i)}{q_{\text{in}}^{\text{meas}}(i)}, \ \forall i \in N \end{array} \right. \end{aligned} \tag{24}$$

C. Decision variables

The vector of decision variables used later in the convex programs is defined as follows.

Definition 4.2: **[Decision variables]** Let Δ represent the value of $\overline{\mathbf{M}}(0,\xi) - \overline{\mathbf{M}}(0,\chi)$. Let $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$ be defined as in (20). Let $e_{\text{in}}(\cdot)$, $e_{\text{out}}(\cdot)$, f_{in} and f_{out} be slack variables defined as in (22), (23) and (24), and associated respectively with the upstream and downstream sensors. The vector of decision variables of interest is defined as:

$$\begin{aligned} X := \Big(& \Delta, q_{\text{in}}(0), \ldots, q_{\text{in}}(n), q_{\text{out}}(0), \ldots, q_{\text{out}}(n), \\ & e_{\text{in}}(0), \ldots, e_{\text{in}}(n), e_{\text{out}}(0), \ldots, e_{\text{out}}(n), f_{\text{in}}, f_{\text{out}} \Big)^T \end{aligned} \tag{25}$$

We now express the physical constraints (13) for the specific problem of interest (two fixed sensors) in terms of the decision variables. For this, we need to express the value condition functions $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ of Definition 2.6, and the solutions $\mathbf{M}_\gamma(\cdot,\cdot)$ and $\mathbf{M}_\beta(\cdot,\cdot)$ to (4) as a function of the decision variables $q_{\text{in}}(\cdot)$, $q_{\text{out}}(\cdot)$. This is the object of the two following sections.

D. Piecewise affine boundary conditions

In this section, we define the piecewise affine boundary conditions obtained from sensor measurements. The sensor data is supposed to be sampled temporally, so that it takes piecewise affine values on each sampling interval. Thus, $q_{\text{in}}(\cdot)$ and $q_{\text{out}}(\cdot)$ are piecewise constant, and the boundary conditions functions (integrated over time) are piecewise affine. This property is very important, since the solution to (4) associated with piecewise affine boundary conditions can be computed semi-explicitly [6].

Definition 4.3: **[Piecewise affine boundary conditions]** Let the vector of decision variables X be defined as in (25). The upstream and downstream boundary condition functions $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are defined as:

$$\begin{array}{ll} \gamma(t,x) = \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t-nT) & \text{if } x=\xi \text{ and } \\ & \exists n \in N \text{ s. t. } \\ & t \in [nT,(n+1)T] \\ \beta(t,x) = -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T + q_{\text{out}}(n)(t-nT) & \text{if } x=\chi \text{ and } \\ & \exists n \in N \text{ s. t. } \\ & t \in [nT,(n+1)T] \end{array} \tag{26}$$

The functions $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ are piecewise affine. For all $n \in N$, the affine blocks $\gamma_n(\cdot,\cdot)$ and $\beta_n(\cdot,\cdot)$ are defined by:

$$\begin{aligned} \gamma_n(t,x) &:= \begin{cases} \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t-nT) & \text{if } x=\xi \text{ and } t \in [nT,(n+1)T] \\ +\infty & \text{otherwise} \end{cases} \\ \beta_n(t,x) &:= \begin{cases} -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T + q_{\text{out}}(n)(t-nT) & \text{if } x=\chi \text{ and } t \in [nT,(n+1)T] \\ +\infty & \text{otherwise} \end{cases} \end{aligned} \tag{27}$$

Using equation (27), we can express (26) as:

$$\left\{ \begin{array}{l} \gamma(\cdot,\cdot) = \inf_{n \in N} (\gamma_n(\cdot,\cdot)) \\ \beta(\cdot,\cdot) = \inf_{n \in N} (\beta_n(\cdot,\cdot)) \end{array} \right. \tag{28}$$

We now compute the solutions $\mathbf{M}_\gamma(\cdot,\cdot)$ and $\mathbf{M}_\beta(\cdot,\cdot)$ respectively associated with the value conditions $\gamma(\cdot,\cdot)$ and $\beta(\cdot,\cdot)$ analytically using the method developed in [6].

E. Solutions associated with affine boundary conditions

For this specific application, we assume that the Hamiltonian $\psi_{\text{certificate}}(\cdot)$ is given by the following formula:

$$\psi_{\text{certificate}}(\rho) := \left\{ \begin{array}{ll} v\rho & \text{if } \rho \leq k_c \\ w(k_m - \rho) & \text{if } k_c \leq \rho \end{array} \right. \tag{29}$$

The function $\psi_{\text{certificate}}(\cdot)$ defined by (29) is both concave and upper semicontinuous (indeed, continuous). It is commonly used in traffic flow modeling [9], [16]. Its parameters v, w, k_c and k_m are chosen such that the condition (12) is satisfied on all available experimental data. This last condition enables us to use the constraints (13) for our present application. The convex transform $\varphi^*(\cdot)$ of $\psi_{\text{certificate}}(\cdot)$ is given [2], [5], [6] by:

$$\varphi^*(u) = \begin{cases} k_c(u+v) & \text{if } u \in [-v, w] \\ +\infty & \text{otherwise} \end{cases} \tag{30}$$

Since $\gamma(\cdot,\cdot) = \inf_{n\in N}(\gamma_n(\cdot,\cdot))$ and $\beta(\cdot,\cdot) = \inf_{n\in N}(\beta_n(\cdot,\cdot))$, the inf-morphism property (10) implies:

$$\begin{cases} \mathbf{M}_\gamma(\cdot,\cdot) = \inf_{n\in N}(\mathbf{M}_{\gamma_n}(\cdot,\cdot)) \\ \mathbf{M}_\beta(\cdot,\cdot) = \inf_{n\in N}(\mathbf{M}_{\beta_n}(\cdot,\cdot)) \end{cases} \tag{31}$$

where $\mathbf{M}_{\gamma_n}(\cdot,\chi)$ and $\mathbf{M}_{\beta_n}(\cdot,\xi)$ are the solutions to (4) associated with $\gamma_n(\cdot,\cdot)$ and $\beta_n(\cdot,\cdot)$ defined by (28). These solutions can be computed analytically using the framework developed in [6].

Proposition 4.4: **[Explicit computation of the solutions]** The solutions $\mathbf{M}_{\gamma_n}(\cdot,\cdot)$ and $\mathbf{M}_{\beta_n}(\cdot,\cdot)$ associated with $\gamma_n(\cdot,\cdot)$ and $\beta_n(\cdot,\cdot)$ are given by the following explicit formulas:

$$\mathbf{M}_{\gamma_n}(t,x)=\begin{cases} +\infty \quad \text{if } t \le nT + \frac{x-\xi}{v} \\ \displaystyle\sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - \frac{x-\xi}{v} - nT) \\ \qquad \text{if } nT + \frac{x-\xi}{v} \le t \le (n+1)T + \frac{x-\xi}{v} \\ \displaystyle\sum_{i=0}^{n} q_{\text{in}}(i)T + k_c v(t-(n+1)T - \frac{x-\xi}{v}) \\ \qquad \text{otherwise} \end{cases}$$

$$\mathbf{M}_{\beta_n}(t,x)=\begin{cases} +\infty \quad \text{if } t \le nT + \frac{\chi-x}{w} \\ -\Delta + \displaystyle\sum_{i=0}^{n-1} q_{\text{out}}(i)T + q_{\text{out}}(n)(t - \frac{\chi-x}{w} \\ \qquad -nT) + \frac{\chi-x}{w}k_c(v+w) \\ \qquad \text{if } nT + \frac{\chi-x}{w} \le t \le (n+1)T + \frac{\chi-x}{w} \\ -\Delta + \displaystyle\sum_{i=0}^{n-1} q_{\text{out}}(i)T + k_c v(t-(n+1)T - \frac{\chi-x}{w}) \\ \qquad \text{otherwise} \end{cases} \tag{32}$$

Proof — This proposition is an instantiation of the result of [6] for the specific Hamiltonian (29). ■

F. Expression of the compatibility conditions

The conditions (13) can be expressed in the present case as:

$$\begin{cases} (i) & \mathbf{M}_\gamma(t,\xi) \ge \gamma(t,\xi) & \forall t \in [0, t_{\max}] \\ (ii) & \mathbf{M}_\beta(t,\chi) \ge \beta(t,\chi) & \forall t \in [0, t_{\max}] \\ (iii) & \mathbf{M}_\gamma(t,\chi) \ge \beta(t,\chi) & \forall t \in [0, t_{\max}] \\ (iv) & \mathbf{M}_\beta(t,\xi) \ge \gamma(t,\xi) & \forall t \in [0, t_{\max}] \end{cases} \tag{33}$$

In the present case, checking that the constraints (33) are satisfied amounts to checking that a set of linear inequalities is feasible. The two following propositions enable us to find the expression of (33) as linear inequalities.

Proposition 4.5: **[Consistency check of a single boundary condition]** The conditions (33) (i) and (ii) are satisfied if and only if the following inequalities are satisfied:

$$\begin{cases} (i) & q_{\text{in}}(n) \le vk_c & \forall n \in N \\ (ii) & q_{\text{out}}(n) \le vk_c & \forall n \in N \end{cases} \tag{34}$$

Proof — Let us choose $t \in [nT, (n+1)T]$. The function $\mathbf{M}_\gamma(t,\xi)$ can be computed by combining (31) and (32):

$$\mathbf{M}_\gamma(t,\xi) = \min\left[\min_{k\in[0,n-1]}\left(\sum_{i=0}^{k} q_{\text{in}}(i)T + (t-(k+1)T)vk_c\right), \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t-nT)\right] \tag{35}$$

Since $\gamma(t,\xi) = \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t-nT)$, we have that $\mathbf{M}_\gamma(t,\xi) \ge \gamma(t,\xi)$ for all $t \in [nT, (n+1)T]$ if and only if:

$$\sum_{i=k+1}^{n-1} q_{\text{in}}(i)T + (t-nT)q_{\text{in}}(n) - (t-(k+1)T)vk_c \le 0 \quad \forall k \in [0, n-1] \tag{36}$$

By choosing $t = nT$, equation (36) yields the necessary condition

$$\forall k \in [0, n-1], \quad \sum_{i=k+1}^{n-1} q_{\text{in}}(i)T - (n-k-1)Tvk_c \le 0 \tag{37}$$

Choosing $k = n-2$ in (37) yields the necessary condition $q_{\text{in}}(n-1) \le vk_c$. Similarly, choosing $k = n-3$ yields $q_{\text{in}}(n-2) \le vk_c$. This can be extended trivially by induction for all $n \in N$, and yields inequality (i) of (34). Conversely, the condition (i) of (34) is sufficient. Indeed, condition (36) can be rewritten as:

$$\sum_{i=k+1}^{n-1} (q_{\text{in}}(i) - vk_c)T + (t-nT)(q_{\text{in}}(n) - vk_c) \le 0 \quad \forall k \in [0, n-1] \tag{38}$$

Since $T > 0$ and $t - nT \ge 0$, the inequality (38) is trivially satisfied when condition (i) of (34) is satisfied, which completes the proof of (i). The proof of (ii) is identical to the proof of (i). ■

Proposition 4.6: **[Consistency check of multiple boundary conditions]** We assume that the conditions (34) hold. The conditions (33) (iii) are satisfied if and only if the following inequalities are satisfied:

$$\begin{cases} (i) & \sum_{i=0}^{\lfloor n-\frac{\chi-\xi}{vT}\rfloor -1} q_{\text{in}}(i)T + q_{\text{in}}(\lfloor n-\frac{\chi-\xi}{vT}\rfloor)(nT \\ & -\frac{\chi-\xi}{v} - \lfloor n-\frac{\chi-\xi}{vT}\rfloor T) \geq -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T \\ & \forall n \in N \\ (ii) & \sum_{i=0}^{n-1} q_{\text{in}}(i)T \geq -\Delta + \sum_{i=0}^{\lfloor n+\frac{\chi-\xi}{vT}\rfloor -1} q_{\text{out}}(i)T \\ & +q_{\text{out}}(\lfloor n+\frac{\chi-\xi}{vT}\rfloor)(nT + \frac{\chi-\xi}{v} - \lfloor n+\frac{\chi-\xi}{vT}\rfloor T) \\ & \forall n \in N \\ (iii) & -\Delta + \sum_{i=0}^{\lfloor n-\frac{\chi-\xi}{wT}\rfloor -1} q_{\text{out}}(i)T + q_{\text{out}}(\lfloor n-\frac{\chi-\xi}{wT}\rfloor)(nT \\ & -\frac{\chi-\xi}{w} - \lfloor n-\frac{\chi-\xi}{wT}\rfloor T) \geq \sum_{i=0}^{n-1} q_{\text{in}}(i)T \\ & \forall n \in N \\ (iv) & -\Delta + \sum_{i=0}^{n-1} q_{\text{out}}(i)T \geq \sum_{i=0}^{\lfloor n+\frac{\chi-\xi}{wT}\rfloor -1} q_{\text{in}}(i)T \\ & +q_{\text{in}}(\lfloor n+\frac{\chi-\xi}{wT}\rfloor)(nT + \frac{\chi-\xi}{w} - \lfloor n+\frac{\chi-\xi}{wT}\rfloor T) \\ & \forall n \in N \end{cases} \tag{39}$$

Proof — The conditions (33) can be expressed in the present case as:

$$\begin{cases} (i) & \mathbf{M}_\gamma(t,\chi) \geq \beta(t,\chi) \quad \forall t \in [0, t_{\max}] \\ (ii) & \mathbf{M}_\beta(t,\xi) \geq \gamma(t,\xi) \quad \forall t \in [0, t_{\max}] \end{cases} \tag{40}$$

We only prove that the constraint (i) of (40) yields inequalities (i) and (ii) of (39).

Let us fix $t \in [nT + \frac{\chi-\xi}{v}, (n+1)T + \frac{\chi-\xi}{v}]$. We first express the solution $\mathbf{M}_\gamma(t,\chi)$ using (32) and the inf-morphism property (31):

$$\mathbf{M}_\gamma(t,\chi) = \begin{cases} \min\left[\sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - \frac{x-\xi}{v} - nT),\right. \\ \left.\min_{k\in[0,n-1]}\left(\sum_{i=0}^{k} q_{\text{in}}(i)T + k_c v(t-(k+1)T - \frac{x-\xi}{v})\right)\right] \end{cases} \tag{41}$$

Since $q_{\text{in}}(k) \leq k_c v$ for all $k \in N$, we have that $\sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - \frac{x-\xi}{v} - nT) \leq \sum_{i=0}^{k} q_{\text{in}}(i)T + k_c v(t - (k+1)T - \frac{x-\xi}{v})$ for all $k \in [0, n-1]$. Hence, we have that

$$\mathbf{M}_\gamma(t,\chi) = \sum_{i=0}^{n-1} q_{\text{in}}(i)T + q_{\text{in}}(n)(t - \frac{\chi-\xi}{v} - nT) \tag{42}$$

Thus, $\mathbf{M}_\gamma(t,\chi)$ is affine for all $t \in [nT + \frac{\chi-\xi}{v}, (n+1)T + \frac{\chi-\xi}{v}]$, for all $n \in N$. By construction, $\beta(t,\chi)$ is also affine for all $t \in [nT, (n+1)T]$, for all $n \in N$. Hence, checking that condition (i) of (40) is satisfied amounts to checking that

$$\begin{cases} \mathbf{M}_\gamma(nT,\chi) \geq \beta(nT,\chi) & \forall n \in N \\ \mathbf{M}_\gamma(nT + \frac{\chi-\xi}{v},\chi) \geq \beta(nT + \frac{\chi-\xi}{v},\chi) & \forall n \in N \end{cases} \tag{43}$$

Indeed, checking that the continuous and piecewise affine function $\mathbf{M}_\gamma(t,\chi) - \beta(t,\chi)$ is positive for all $t \in [0, t_{\max}]$ amounts to checking that it is positive on all its kinks.

In addition, remarking that $t \in [nT, (n+1)T]$ if and only if $n = \lfloor \frac{t}{T} \rfloor$ and $t \in [nT + \frac{\chi-\xi}{v}, (n+1)T + \frac{\chi-\xi}{v}]$ if and only if $n = \lfloor \frac{t - \frac{\chi-\xi}{v}}{T} \rfloor$, we can express the functions $\mathbf{M}_\gamma(\cdot,\chi)$ and $\beta(t,\chi)$ explicitly as:

$$\begin{aligned} \mathbf{M}_\gamma(t,\chi) =& \sum_{i=0}^{\lfloor \frac{t-\frac{\chi-\xi}{v}}{T}\rfloor -1} q_{\text{in}}(i)T + q_{\text{in}}(\lfloor \frac{t-\frac{\chi-\xi}{v}}{T}\rfloor)(t - \frac{\chi-\xi}{v} \\ & -\lfloor \frac{t-\frac{\chi-\xi}{v}}{T}\rfloor T), \quad \forall t \in [0, t_{\max}] \\ \beta(t,\chi) =& -\Delta + \sum_{i=0}^{\lfloor \frac{t}{T}\rfloor -1} q_{\text{out}}(i)T + q_{\text{out}}(\lfloor \frac{t}{T}\rfloor)(t - \lfloor \frac{t}{T}\rfloor T) \\ & \forall t \in [0, t_{\max}] \end{aligned} \tag{44}$$

The conditions (i) and (ii) of equation (39) are obtained from (43), using formula (44) for expressing the boundary conditions. One can similarly prove that inequality (ii) of (40) yields inequalities (iii) and (iv) of (39). ■

G. Convex programming formulation of the minimal error certificate problem

Proposition 4.7: **[Minimal error of a sensor]** The minimal error ζ_{in} (respectively ζ_{out}) of the fixed upstream (respectively downstream) sensor in the L_1 sense is the solution to the following *linear program* (LP):

$$\begin{aligned} \zeta_{\text{in/out}} := \quad & \min\left(\sum_{i=0^n} e_{\text{in/out}}(i)\right) \\ \text{s.t.} & \begin{cases} e_{\text{in}}(i) \geq \frac{q_{\text{in}}(i) - q_{\text{in}}(i)^{\text{meas}}}{q_{\text{in}}(i)^{\text{meas}}}, \ \forall i \in N \\ e_{\text{in}}(i) \geq -\frac{q_{\text{in}}(i) - q_{\text{in}}(i)^{\text{meas}}}{q_{\text{in}}(i)^{\text{meas}}}, \ \forall i \in N \\ e_{\text{out}}(i) \geq \frac{q_{\text{out}}(i) - q_{\text{out}}(i)^{\text{meas}}}{q_{\text{out}}(i)^{\text{meas}}}, \ \forall i \in N \\ e_{\text{out}}(i) \geq -\frac{q_{\text{out}}(i) - q_{\text{out}}(i)^{\text{meas}}}{q_{\text{out}}(i)^{\text{meas}}}, \ \forall i \in N \\ \text{(34) holds} \\ \text{(39) holds} \end{cases} \end{aligned} \tag{45}$$

The minimal error η_{in} (respectively η_{out}) of the fixed upstream (respectively downstream) sensor in the L_2 sense is the solution to the following convex *quadratically constrained quadratic program* (QCQP):

$$\begin{aligned} \eta_{\text{in/out}} := \quad & \min\left(\sum_{i=0^n} e_{\text{in/out}}(i)\right) \\ \text{s.t.} & \begin{cases} e_{\text{in}}(i) \geq (\frac{q_{\text{in}}(i) - q_{\text{in}}(i)^{\text{meas}}}{q_{\text{in}}(i)^{\text{meas}}})^2, \ \forall i \in N \\ e_{\text{out}}(i) \geq (\frac{q_{\text{out}}(i) - q_{\text{out}}(i)^{\text{meas}}}{q_{\text{out}}(i)^{\text{meas}}})^2, \ \forall i \in N \\ \text{(34) holds} \\ \text{(39) holds} \end{cases} \end{aligned} \tag{46}$$

The minimal error θ_{in} (respectively θ_{out}) of the fixed upstream (respectively downstream) sensor in the L_∞ sense is the solution to the following LP:

$$\theta_{\text{in/out}} := \quad \min f \quad \text{s.t.} \begin{cases} f \geq \frac{q_{\text{in/out}}(i) - q_{\text{in/out}}(i)^{\text{meas}}}{q_{\text{in/out}}(i)^{\text{meas}}}, \ \forall i \in N \\ f \geq -\frac{q_{\text{in/out}}(i) - q_{\text{in/out}}(i)^{\text{meas}}}{q_{\text{in/out}}(i)^{\text{meas}}}, \ \forall i \in N \\ (34) \text{ holds} \\ (39) \text{ holds} \end{cases} \tag{47}$$

Proof — The proof follows from the definition of the error terms (22), (23) and (24), and from the inequality constraints (13) (that is, from (34) and (39) with our assumptions), since the Hamiltonian (29) satisfies (12). Note that the inequalities (34) and (39) are linear in terms of the decision variable (25). ■

H. Fault detection in traffic sensor networks

We apply the above results to sensors of the PeMS system [1], which is a network of loop detectors measuring traffic on California highways. The PeMS system is one of the data feeds currently integrated in the *Mobile Millennium* traffic monitoring system [17], [18], operated jointly by Nokia and UC Berkeley. One of the main challenges arising when using data from the PeMS system is the automated identification of the mislocated or faulty sensors. Previous approaches such as [13] have successfully implemented sensor fault detection algorithms based on statistical correlation with adjacent sensors. In the present case, the use of a flow model enables us to provide a fault certificate, by proving that the level of error in a pair of sensors exceeds its design value.

The main difficulty solved by this approach is the identification of "realistic looking" faulty data. The data in question, when looked at for a single sensor is not abnormal. However, when checked with the data from other sensors, the approach developed earlier enables the identification of incompatibility, thus resulting in the detection of faults, which results from the proper use of the model and the method.

We solve the fault detection problem by applying the LP (47) on all pairs of consecutive sensors present on the highway network. We first create a learning set based on the data of a subset of traffic sensors which are known to be valid. This learning set is shown in Figure 2. We then choose a function $\psi_{\text{certificate}}(\cdot)$ of the form (29) which satisfies the condition (12). This function $\psi_{\text{certificate}}(\cdot)$ will be our learning set certificate.

We assume that the maximal allowable error of a PeMS sensor is 15%. There are multiple sources of uncertainty arising when dealing with loop detectors, such as pavement depth, loop layout, which typically creates maximal errors of this magnitude. The maximal allowable error on a pair of PeMS sensors will thus be set to 30% in the considered scenario.

As an application, we consider the results of (47) for five consecutive sensors, labeled 401339, 401714, 401376, 400609 and 400835 respectively, as illustrated in Figure 6. For each one of the four adjacent pairs of sensors, we compute the minimal value of the error $\theta_{\text{in}} + \theta_{\text{out}}$ in equation (47), during a one month period at the frequency of one day. The distribution of these results is shown in Figure 6.

Figure 6 shows that there is no indication of malfunction for the first and the last pairs of sensors. Note that the success to the minimal error test does not guarantee that a pair of sensors is working, since the actual error of the pair of sensors can be above the maximal allowable error.

The second and third pairs exhibit L_∞ errors that are higher than 30%, which indicates a malfunction of the corresponding pairs. Further analysis has shown that the pair $401714 - 400609$ is passing the minimal error test, and thus that sensor 401376 is likely either failing or incorrectly mapped.

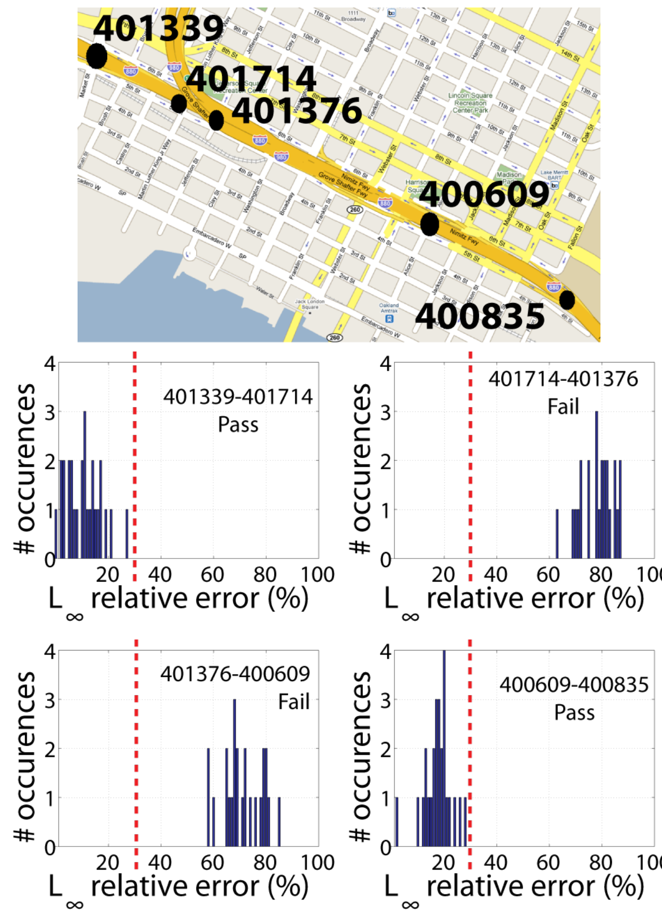

Fig. 6. **Faulty sensor detection**. We consider here the traffic flow on highway I880-S near Oakland, CA. The linear program (47) is run each day on a one-month period. The sensors of interests are highlighted in the top figure, and their corresponding minimal error distribution over the one-month period is represented in the four bottom figures. **Bottom:** The top left and bottom right subfigures represents the minimal errors of the pair $401339 - 401714$ and $40609 - 400835$. These minimal errors fall in the allowable range. In contrast, the minimal errors of the pair $401714 - 401376$ and $401376 - 400609$ are above the allowable range. This means that there must exist a fault in one of the sensors 401714, 401376, or 400609.

The present method is currently implemented in the *Mobile Millennium* system, as a filtering method for the PeMS feed to the system. The results have shown to greatly improve the performance of the system [18].

V. Conclusion

This article presents a new method for fault detection in networks of sensors monitoring systems modeled by Hamilton-Jacobi partial differential equations. The state of the system of interest does not usually follow a partial differential equation characterized by a well known parameter, but by a distribution of parameters. Based on the solution methods of Hamilton-Jacobi equations, we show that any local knowledge of the state provides constraints on the possible values of the state in the space-time domain. We subsequently use this fact to pose the problem of minimal error certificate of a pair of sensors as a set of convex programs. The minimal error certificate is the minimal possible error of the sensors given a set of measurements. It is a very effective tool for detecting inconsistencies. When the error is above a preset threshold depending on the sensor type, the corresponding set of sensors is identified as faulty. This work has been successfully implemented in the *Mobile Millennium* [18] system, and is part of the consistency check methods used in this system. It checks the consistency of the data generated by more than 1400 sensors every 30 seconds in real time with streaming data for all PeMS sensors in northern California.

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