Assessment of Uncertainty Propagation in the Dynamic Response of Single-Degree-of-Freedom Structures Using Reachability Analysis

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Abstract: A novel method to compute the bounds of the response of structures to dynamic loads, including earthquakes, is presented. This method, based on reachability analysis, deterministically predicts the sets of states an elastic structural system can reach under uncertain dynamic excitation starting from uncertain initial conditions, where deterministic uncertainty ranges describe uncertainties. Ellipsoidal approximations of these reachable sets for three canonical dynamic problems are presented to demonstrate the applicability of this method to single-degree-of-freedom (SDOF) systems. The principle of superposition is formulated as a concatenation of ellipsoidal reachable sets using their semigroup properties. Using this extension, computation of the external (worst-case) ellipsoidal approximation of reachable sets for a SDOF system under earthquake excitation is presented. Possible applications of this method for software validation and hybrid simulation are discussed. **DOI: 10.1061/(ASCE)EM.1943-7889.0000676.**

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Introduction

A structure dynamically responding to a time-varying excitation is considered a real-time continuous dynamic system. The design parameters of this system (mass, strength damping, and stiffness) are assumed to be known with high certainty, whereas the initial conditions and excitation of the structure are not known with certainty due to measurement errors. The aim of this paper is to evaluate worst-case scenario bounds to assess, for example, the maximum possible displacements a structure might experience when affected by a plausible but uncertain excitation. To do this, the process of dynamic system verification that consists of computing the sets of states (displacements and velocities of the structure's masses at a point in time) of a dynamic system excited by an admissible but uncertain excitation that can be reached starting from an uncertain initial state will be used. Such sets of dynamic system states are called *reachable sets*.

Although the verification of discrete state systems is a relatively well-explored field for which efficient tools have been successfully developed (Bryant 1986; Hu et al. 1993; Chutinan and Krogh 2003; Asarin et al. 2003; Henzinger et al. 1998), algorithms for verification of continuous-state systems have only been developed relatively recently (Mitchell et al. 2005; Lygeros 2004; Tomlin et al. 2000). Verifying an uncountable (infinite) set of states represented by continuous variables requires a numerical treatment that is both theoretically more challenging than that for discrete systems and harder to implement in practice. A possible approach is to use the Hamilton-Jacobi (HJ) partial differential equation (PDE). The benefit of this approach, sometimes called *reachability analysis*, is that it provides a proof for the utilized mathematical models of the system that the system state trajectory will remain inside an envelope and reach the target. This is in contrast with Monte-Carlo methods, which do not provide any guarantee for trajectories that are not part of the simulation, and the random vibration analysis method, which provides probabilistic characterizations of the likelihood that systems will follow a given state trajectory. Both the Monte Carlo and random vibration analysis methods have historically been used to explore possible trajectories a system might follow from uncertain initial conditions under uncertain dynamic excitation, but neither is capable of producing deterministic trajectory bounds.

The validity of proof at the basis of reachability analysis goes back to the discovery of the viscosity solution (Crandall and Lions 1983; Crandall et al. 1984) of HJ PDE. Prior to this, methods based on differential games (Isaacs 1965) (or optimal control for only one player) provided, at best, certificates that specific trajectories of the system stayed inside the envelope but did not provide guarantees on sets. The advent of level set methods (Osher and Sethian 1988; Sethian 1999; Osher and Fedkiw 2002) enabled numerical computations of the viscosity solution, with a theoretical proof of convergence of the numerical result to the viscosity solution. In parallel, the viability theory (Aubin 1991) provided engineers with an equivalent approach to solve the same problems, leading to a new suite of numerical schemes (Saint-Pierre 1994) developed to solve differential game problems (Cardaliaguet et al. 1999). These numerical schemes have also been proven to converge to the viscosity solution of HJ PDE, providing similar guarantees as level set methods. These methods have now been extended to treat hybrid systems, which combine continuous state and discrete state dynamics (Tomlin et al. 2000, 2003; Bayen et al. 2002; Mitchell 2000).

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When actual implementations of these methods became operational in the late 1990s, the available computational power limited such computations to two-dimensional systems (Tomlin et al. 2000; Saint-Pierre 1994). Algorithmic improvements and the increase in computing power now enable calculations for systems with a continuous state dimension of up to four or more, depending on the mathematical characteristics of the considered dynamics. This is a major technological breakthrough that allows treatment of problems involving realistic models of physical systems. However, such computations are extremely expensive. Because of the high computational cost of solving reachability problems with converging methods (i.e., which attempt to compute the exact solution numerically), numerous approaches use approximate methods, such as ellipsoidal methods, which compute only approximations of the solution (under specific assumptions) at a lower cost.

This paper presents an application of an ellipsoidal reachability analysis to compute the bounds of response of an elastic single-degree-of-freedom (SDOF) structure to uncertain dynamic loads, including earthquakes, starting from uncertain initial conditions. Uncertainties are described by deterministic uncertainty ranges centered at the state of the system acquired by measurements, and simulate measurement noise. Ellipsoidal approximations of the reachable sets of a SDOF system under three canonical dynamic excitations, including pulse loading, are presented to demonstrate the applicability of this method. The principle of response superposition for the linear systems and the convolution method for linear system response computation are formulated as a concatenation of ellipsoidal reachable sets using their semigroup properties. Using such an extension, computation of the outside (worst-case) ellipsoidal approximation of reachable sets for a SDOF system under earthquake excitation is presented. Possible applications of this method to gauge the quality of numerical model calibration to experimental data and control error propagation in experimental methods, such as hybrid simulation, are discussed. A follow-up article presents an extension of the proposed method to multiple-degree-of-freedom systems.

Linear Time-Invariant Model of a Structural System and Reachability Theory

It is assumed that a structural system can be modeled as a linear elastic viscous-damped system of masses with N degrees of freedom. The equation of motion (dynamic equilibrium) governing the displacements $y(t) = [y_1(t), y_2(t), \ldots, y_N(t)]^T$ of such a structural system starting from an initial state $[y(t_0), \dot{y}(t_0)]^T$ subject to an external dynamic force $p(t) = [p_1(t), p_2(t), \ldots, p_N(t)]^T$ is

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = p(t) \tag{1}$$

where m, c, and k = mass matrix, damping coefficient matrix, and elastic stiffness constant matrix of the structure, respectively, and all are $N \times N$. Using a state-space description for dynamical systems, knowing that m is invertible and denoting the state vector $x(t) = [y(t), \dot{y}(t)]^T$, Eq. (1) can be restated in the state-space form for a linear time-invariant (LTI) dynamical system as

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \quad t \geq t_0 \\ x(t_0) &= x_0 \end{aligned} \tag{2}$$

with

$$A = \begin{pmatrix} 0 & I \\ -m^{-1}k & -m^{-1}c \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ m^{-1} \end{pmatrix}, \quad u(t) = p(t) \tag{3}$$

It is customary to designate $x(t) \in \mathbb{R}^h$ as the state of the system; $u(t) \in \mathbb{R}^q$ as the control input equal to the external dynamic excitation in this study; $A \in \mathbb{R}^{h\times h}$ as the dynamics matrix; and $B \in \mathbb{R}^{h\times q}$ as the input matrix. Dimensions of state-space formulation matrices are related to the number of degrees of freedom of the structural system as follows: $h = 2N$ and $q = N$. The state-matrix transition is defined by the following equations:

$$\begin{aligned} \frac{\partial}{\partial t}\Phi(t, t_0) &= A\Phi(t, t_0), \quad t \geq t_0 \\ \Phi(t_0, t_0) &= I \end{aligned} \tag{4}$$

The solution of the system of Eq. (2), the state trajectory of the system, is given by

$$x[t, t_0, x_0, u(\cdot)] = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau)Bu(\tau)d\tau, \quad t \geq t_0 \tag{5}$$

where $\Phi(t, t_0) = e^{A(t-t_0)}$.

Definition: Initial State and Inputs

The variable $\mathcal{X}_0$ is called the set of initial states and $\mathcal{U}$ is the set of control inputs. It is assumed that $\mathcal{X}_0 \subset \mathbb{R}^h$ and $\mathcal{U}_0 \subset \mathbb{R}^q$ are compact sets. Furthermore, it is assumed that the control input $u(t)$ and the initial condition $x(t_0)$ are restricted to the following sets: $u(t) \in \mathcal{U}$ and $x(t_0) \in \mathcal{X}_0$.

Definition: Input Functions

The space of control input functions $U(t)$ is given by $U(t) = \{\eta: [t_0, t] \to \mathbb{R}^q | \eta(\theta) \in \mathcal{U} \forall \theta \in [t_0, t]$ and η is measurable$\}$. The function $u(\cdot)$ is denoted a generic control input function and is assumed to be restricted to the functional space $u(\cdot) \in u(t)$.

Definition: Reachable Set

The reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, U(t)]$ at time $t > t_0$ from an initial set $\mathcal{X}_0$ is the set of all states $x(t)$ reachable at time t by the system [Eq. (2)] from at least one initial state $x_0 \in \mathcal{X}_0$ through at least one control input $u(\cdot) \in U(t)$. The reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, U(t)]$ can be expressed by $\mathcal{X}(t) = \mathcal{X}[t, t_0, \mathcal{X}_0, U(t)] = \{x[t, t_0, x_0, u(\cdot)] | x_0 \in \mathcal{X}_0$ and $u(\cdot) \in U(t)\}$.

Definition: Reachable Tube

The reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, U(t)]$ at time $t > t_0$ from an initial set $\mathcal{X}_0$ is the union of all reachable sets $\mathcal{X}[t, t_0, \mathcal{X}_0, U(t)]$ in the time interval $[t_0, t]$, as illustrated in Fig. 1. The reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, U(t)]$ can be expressed by

$$\mathcal{T}[t, t_0, \mathcal{X}_0, U(t)] = \bigcup_{\tau \in [t_0, t]} \{\tau\} \times \mathcal{X}[\tau, t_0, \mathcal{X}_0, U(\tau)]$$

Fig. 1 illustrates the preceding definitions.

Ellipsoidal Approximations of Reachable Sets

Using definitions from ellipsoidal calculus (Kurzhanski and Vályi 1997), an ellipsoidal reachability method is formulated for analysis of the response of structural systems to dynamic excitation using ellipsoidal approximations of reachable sets. Ellipsoidal techniques for the reachability analysis of LTI systems, introduced by Kurzhanski and Varaiya (2002), parametrize families of external

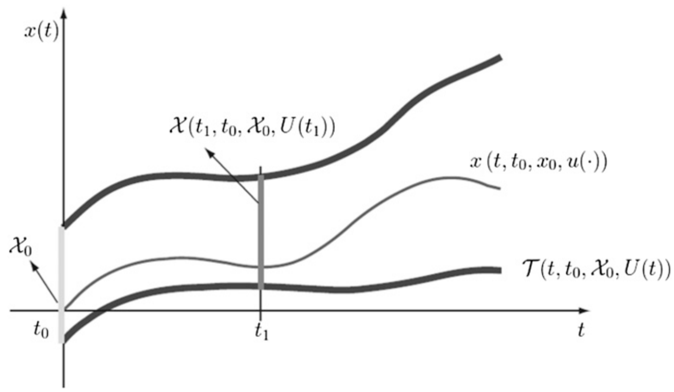

Fig. 1. Reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, U(t)]$ from the set $\mathcal{X}_0$ at time t_0; union of the sets $\mathcal{X}[t, t_0, \mathcal{X}_0, U(t)]$ for all times t is called the reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, U(t)]$

and internal ellipsoidal approximations of reachable sets by constructing them such that they are tangent to actual reachable sets at every point of their boundary at any instant in time. These approximations, described through ordinary differential equations, are implemented in *MATLAB 7.5.0.342 (R2007b)* using the ellipsoidal toolbox (ET) (Kurzhanskiy and Varaiya 2006).

Definition: Ellipsoid

A generic ellipsoid $\mathcal{E}(z, Z) \subset \mathbb{R}^h$ is defined as $\mathcal{E}(z, Z) = \{u: \langle (u - z), Z^{-1}(u - z)\rangle \le 1\}$, where $z \in \mathbb{R}^h$ is the center of the ellipsoid and $Z \in \mathbb{R}^{h\times h}$ is a positive definite matrix.

Definition: Affine Transformation of an Ellipsoid

An affine transformation $A\mathcal{E}(q, Q) + b$ of an ellipsoid $\mathcal{E}(q, Q)$ is an ellipsoid

$$A\mathcal{E}(q, Q) + b = \mathcal{E}\big(Aq + b, AQA^T\big), \quad A \in \mathbb{R}^{h\times h}, \quad b \in \mathbb{R}^h \tag{6}$$

where A = nonsingular matrix.

Definition: Geometric (Minkowski) Sum of k Ellipsoids

The geometric sum Ω of k ellipsoids is defined by

$$\Omega = \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) = \left[u: u = \sum_{i=1}^{k} e_i | e_i \in \mathcal{E}(q_i, Q_i)\right] \tag{7}$$

Definition: External Ellipsoidal Approximation of a Geometric Sum of k Ellipsoids

Given a vector $l \in \mathbb{R}^h$, an external ellipsoid approximation $\mathcal{E}(q, Q_l^+)$ of a geometric sum of k ellipsoids $\mathcal{E}(q_i, Q_i)$, where $i = 1, \ldots, k$, satisfies $\Omega \subseteq \mathcal{E}(q, Q_l^+)$, where the center is $q = q_1 + q_2 + \cdots + q_k$ and the shape matrix Q_l^+ is

$$Q_l^+ = \left(\langle l, Q_1 l\rangle^{1/2} + \cdots + \langle l, Q_k l\rangle^{1/2}\right) \times \left(\frac{1}{\langle l, Q_1 l\rangle^{1/2}} Q_1 + \cdots + \frac{1}{\langle l, Q_k l\rangle^{1/2}} Q_k\right)$$

Definition: Internal Ellipsoidal Approximation of a Geometric Sum of k Ellipsoids

Given a vector $l \in \mathbb{R}^h$, an internal ellipsoidal approximation $\mathcal{E}(q, Q_l^-)$ of a geometric sum of k ellipsoids $\mathcal{E}(q_i, Q_i)$, where $i = 1, \ldots, k$, satisfies $\mathcal{E}(q, Q_l^-) \subseteq \Omega$, where the center is $q = q_1 + q_2 + \cdots + q_k$ and the shape matrix Q_l^- is

$$Q_l^- = \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right)^T \times \left(Q_1^{1/2} + S_2 Q_2^{1/2} + \cdots + S_k Q_k^{1/2}\right)$$

with orthogonal matrices S_i, where $i = 2, \ldots, k$ ($S_i S_i^T = I$) and such that vectors $Q_1^{1/2} l, S_2 Q_2^{1/2} l, \ldots, S_k Q_k^{1/2} l$ are parallel.

The geometric sum of k ellipsoids $\mathcal{E}(q_i, Q_i)$, where $i = 1, \ldots, k$, is in general not an ellipsoid but can be approximated by families of external $\mathcal{E}(q, Q_l^+)$ and internal $\mathcal{E}(q, Q_l^-)$ ellipsoids parametrized by vector $l \in \mathbb{R}^h$. Varying vector l yields exact external and internal approximations

$$\bigcup_{\langle l,l\rangle = 1} \mathcal{E}\left(q, Q_l^-\right) = \mathcal{E}(q_1, Q_1) \oplus \cdots \oplus \mathcal{E}(q_k, Q_k) = \bigcap_{\langle l,l\rangle = 1} \mathcal{E}\left(q, Q_l^+\right)$$

In the present work, the external approximation $\mathcal{E}(q, Q_l^+)$ of a geometric sum of k ellipsoids $\mathcal{E}(q_i, Q_i)$, where $i = 1, \ldots, k$, is considered because this represents a conservative approximation of the bounds of system trajectories in its state space.

Definition: Support Function of a Set

Let K be a nonempty subset of a finite dimensional space $\mathbb{R}^h$. The support function ρ of the set K with $r \in \mathbb{R}^h$ is the function ρ_K: $\mathbb{R}^p \to \mathbb{R} \cup \{+\infty\}$ defined by

$$\rho_K(r) := \rho(K, r) := \sup_{x\in K} \langle r, x\rangle \in \mathbb{R} \cup \{+\infty\}$$

The support function ρ of the ellipsoid $\mathcal{E}(q, Q)$ with any vector $l \in \mathbb{R}^h$ is given by

$$\rho_{\mathcal{E}(q,Q)}(l) = \rho[\mathcal{E}(q, Q), l] = \langle l, q\rangle + \langle l, Ql\rangle^{1/2}$$

The system [Eq. (2)] in which the initial condition $x(t_0)$ and the control input $u(t)$ are restricted to the following sets, $x(t_0) \in \mathcal{X}_0$ and $u(t) \in \mathcal{P}(t)$, is considered, where $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)]$ are ellipsoidal sets. The following notions are defined.

Definition: Reachable Set by Ellipsoidal Technique

Let $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)] \subset \mathbb{R}^q$ be the ellipsoidal set of control inputs and $\Pi(t) = \{\xi: [t_0, t] \to \mathbb{R}^q | \xi(\theta) \in \mathcal{P}(\theta)\ \forall\, \theta \in [t_0, t]$ and ξ is measurable$\}$ be the space of control input functions. Given a set of initial positions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$, the ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)]$ at time $t \ge t_0$ from the set $\mathcal{X}_0$ is the set of all states $x[t, t_0, x_0, u(\cdot)]$ reachable at time t by the system [Eq. (2)] with $x(t_0) = x_0 \in \mathcal{X}_0$ through all possible controls $u(\cdot) \in \Pi(t)$. The ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)]$ can be expressed by $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)] = \{x[t, t_0, x_0, u(\cdot)] | x_0 \in \mathcal{X}_0$ and $u(\cdot) \in \Pi(t)\}$.

Definition: Reachable Tube by Ellipsoidal Technique

Let $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)] \subset \mathbb{R}^q$ be the ellipsoidal set of control inputs, and $\Pi(t) = \{\zeta: [t_0, t] \to \mathbb{R}^q | \zeta(\theta) \in \mathcal{P}(\theta)\ \forall\, \theta \in [t_0, t]$ and ζ is measurable$\}$

the space of control input functions. Given a set of initial positions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$, the ellipsoidal reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ at time $t > t_0$ from an initial set $\mathcal{X}_0$ is the union of all ellipsoidal reachable sets $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)]$ in the time interval $[t_0,\ t]$. The ellipsoidal reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ can be expressed by

$$\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)] = \bigcup_{\tau \in [t_0, t]} \{\tau\} \times \mathcal{X}[\tau, t_0, \mathcal{X}_0, \Pi(\tau)]$$

Although the initial set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and the control set $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)]$ are ellipsoidal sets, the ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)]$ will in general not be an ellipsoid. The ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)]$ may be approximated both externally and internally by ellipsoidal sets.

Definition: External Ellipsoidal Approximation of a Reachable Set

An external ellipsoidal approximation $\mathcal{E}^+$ of an ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)]$ satisfies $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)] \subseteq \mathcal{E}^+$. It is tight if a vector $l \in \mathbb{R}^h$ satisfying the adjoint equation $i = -A^T l, \forall l_0 \in \mathbb{R}^h$ exists, such that $\rho(\mathcal{E}^+,\ \pm l) = \rho\{\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)],\ \pm l\}$, where ρ is the support function defined earlier.

Remark

From the preceding definition, notation l means $l(t)$ with $l_0 = l(t_0)$. Note that $\forall l_0 \in \mathbb{R}^h$, $\pm l$ will be the directions in which the corresponding ellipsoidal approximation will be tight if the adjoint equation is satisfied.

Definition: External Ellipsoidal Approximation of a Reachable Tube

An external ellipsoidal approximation $\mathcal{T}^+$ of an ellipsoidal reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ satisfies $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)] \subseteq \mathcal{T}^+$. In this work, only the external ellipsoidal approximations $\mathcal{E}^+$ and $\mathcal{T}^+$ are considered because they represent conservative approximations of the exact reachable set and the exact reachable tube, respectively, by accounting for all possible worst-case perturbations in the allowed set of perturbations.

Semigroup Property of an Ellipsoidal Reachable Set

Given compact sets of control inputs $\mathcal{P}_1(t) = \mathcal{E}_1[p(t), p(t)] \subset \mathbb{R}^q$ and $\mathcal{P}_2(t) = \mathcal{E}_2[p(t), p(t)] \subset \mathbb{R}^q$, the spaces of control input functions are defined as $\Pi_1(t_0, \tau) = \{\xi_1 : [t_0, \tau] \to \mathbb{R}^q | \xi_1(\theta) \in \mathcal{P}_1(\theta)\ \forall\, \theta \in [t_0, \tau]$ and ξ_1 is measurable$\}$ and $\Pi_2(\tau, t) = \{\xi_2 \colon [\tau, t] \to \mathbb{R}^q | \xi_2(\theta) \in \mathcal{P}_2(\theta)\ \forall\, \theta \in [\tau, t]$ and ξ_2 is measurable$\}$. Let $\Pi(t)$ be the concatenation $[\Pi_1(\cdot,\cdot) \diamond_\tau \Pi_2(\cdot,\cdot)](t)$ of $\Pi_1(\cdot,\cdot)$ and $\Pi_2(\cdot,\cdot)$ at time τ given by

$$\Pi(t) = [\Pi_1(\cdot,\cdot) \diamond_\tau \Pi_2(\cdot,\cdot)](t) = \xi \colon [t_0, t] \to \mathbb{R}^q | \xi(\theta) \in \mathcal{P}_1(\theta) \quad \forall \theta \in [t_0, \tau]$$

$$\xi(\theta) \in \mathcal{P}_2(\theta) \quad \forall \theta \in [\tau, t] \text{ and } \xi \text{ is measurable}$$

Given a set of initial positions $\mathcal{X}_0$, the ellipsoidal reachable set $\mathcal{X}[\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)]$ at time $\tau \geq t_0$, from the set $\mathcal{X}_0$, is the ellipsoidal reachable set $\mathcal{X}[\tau, t_0, \mathcal{X}_0, \Pi(t_0, \tau)] = \{x[\tau, t_0, x_0, u(\cdot)] | x_0 \in \mathcal{X}_0$ and $u(\cdot) \in \Pi_1(t_0, \tau)\}$ The ellipsoidal reachable set $\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)]$ at time $t \geq \tau$, from the ellipsoidal reachable set $\mathcal{X}[\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)]$, satisfies the semigroup property (Fig. 2)

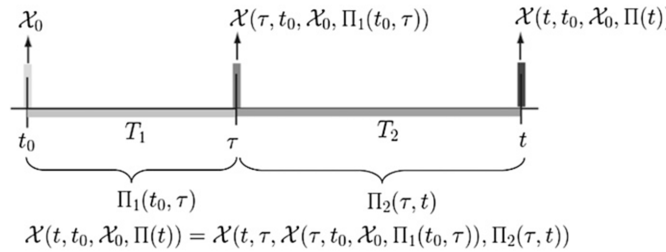

Fig. 2. Illustration of the semigroup property of an ellipsoidal reachable set

$$\mathcal{X}[t, t_0, \mathcal{X}_0, \Pi(t)] = \mathcal{X}\{t, \tau, \mathcal{X}[\tau, t_0, \mathcal{X}_0, \Pi_1(t_0, \tau)], \Pi_2(\tau, t)\},$$
$$t_0 \leq \tau \leq t$$

Model of Uncertainty

Uncertainty in the response of structural systems under dynamic excitation is assumed to originate with measurements of forces, accelerations, velocities, and displacements used to describe the excitation and states of a system. In this work, a deterministic model of uncertainty that simulates measurement noise is adopted. A range of possible values of system state variables and the excitation (control input) centered on the state or excitation value acquired by measurements is specified. Mechanical and geometric (stiffness, strength, damping, and mass) properties of the structural system are assumed to be known and certain.

The geometric characteristics of an ellipsoid (Boyd 2008) are used to formulate a model of uncertainty in the excitation (control inputs) and initial conditions for the LTI model of a structural system. The shape of a generic ellipsoid $\mathcal{E}(z, Z) = \{u \colon \langle (u - z), Z^{-1}(u - z) \rangle \leq 1\} \subset \mathbb{R}^h$ is characterized by the eigenvectors of its shape matrix Z. Eigenvectors define the principal directions of ellipsoid radii, and eigenvalues define the corresponding radii lengths.

Definition: Uncertainty of Initial State Set

Given an initial state set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$, the amount of uncertainty around the initial state x_0 is defined as $\mu_{x_0} = (\mu_{x_0 1}, \mu_{x_0 2}, \ldots, \mu_{x_0 h})$, where $\mu_{x_0 i}$ and $i = 1, \ldots, h$ are eigenvalues of the shape matrix X_0.

Definition: Uncertainty of Control Inputs Set

Given a control input (excitation) function set $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)]$, the amount of uncertainty around $p(t)$, the center of the ellipsoid $\mathcal{E}[p(t), p(t)]$, is defined as $\mu_{p(t)} = \{\max_t[\mu_{p(t)1}], \max_t[\mu_{p(t)2}], \ldots, \max_t[\mu_{p(t)q}]\}$, where $\max_t[\mu_{p(t)i}]$ and $i = 1, \ldots, q$ are eigenvalues of the shape matrix $p(t)$. The uncertainty of the initial state set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ of a structural system is modeled by selecting eigenvalues of its shape matrix X_0. Because the initial state of a structural system is described by the displacements (positions) and velocities of its masses, eigenvalues of the shape matrix X_0 represent deterministic bounds of displacement and velocity measurement errors at the time when system response begins. Because initial state uncertainty occurs at the start of the response, uncertainty in this initial state remains constant throughout the response time of the system (Fig. 3). Analogously, uncertainty of the control input (excitation) functions set $\mathcal{P}(t) = \mathcal{E}[p(t), p(t)]$ is modeled by choosing appropriate eigenvalues of its shape matrix $p(t)$. The control input for structural system models can be specified in terms of displacements, accelerations, or forces. The eigenvalues of the shape matrix $p(t)$

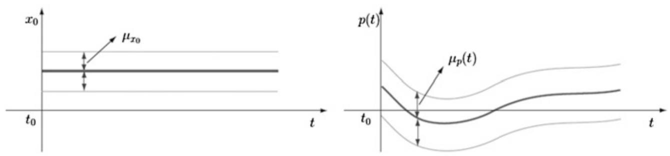

Fig. 3. Uncertainty models for structural system initial state and control input (excitation) for ellipsoidal reachability analysis

represent corresponding deterministic bounds of control input measurement errors. These bounds may vary with time during the system response, but in this work, the uncertainty in any control input is assumed to be constant (Fig. 3) and equal to the largest measurement error that occurred during dynamic excitation of the system.

Reachable Sets Computation for a SDOF System

In this section, ellipsoidal external approximations are used to compute conservative estimates of reachable tubes for trajectories of SDOF linear elastic structural systems excited by a finite force pulse. The state evolution equation for a SDOF system is

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{m} & -\dfrac{c}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{m} \end{bmatrix} [u] \tag{8}$$

where x_1 and x_2 = displacement and velocity of mass m. In these examples, $m = 1\,\mathrm{kg}$ and $k = 39.47\,\mathrm{N/m}$, giving the SDOF system a natural vibration frequency $\omega_n = \sqrt{k/m} = 6.28\,\mathrm{rad/s}$ and a natural period $T_n = 2\pi/\omega_n = 1\,\mathrm{s}$. The responses of two SDOF systems, an undamped system (damping ratio $\xi = 0$) and underdamped system ($\xi = 0.02$), were analyzed. The initial condition of the SDOF systems at $t = 0$ is zero displacement and zero velocity. The finite pulse excitation is a unit force pulse $P(t)$ with a duration or $t_p = 0.4\,\mathrm{s}$ specified as

$$P(t) = \begin{cases} P_1(t) = 0\,\mathrm{kN} & \text{for } t \in T_1 = [t_0, t_1] = [0, 3]\mathrm{s} \\ P_2(t) = 1\,\mathrm{kN} & \text{for } t \in T_2 = [t_1, t_2] = [3, 3.4]\mathrm{s} \\ P_3(t) = 0\,\mathrm{kN} & \text{for } t \in T_3 = [t_2, t_3] = [3.4, 10]\mathrm{s} \end{cases} \tag{9}$$

To compute ellipsoidal approximations of reachable sets, the semigroup property of reachable sets and concatenate responses of the SDOF system in three pulse subintervals is used. The caption of Fig. 4 describes the initial conditions and excitation used in the finite pulse reachability analysis. The amount of uncertainty is set by selecting eigenvectors and eigenvalues of the uncertainty ellipsoid shape matrices. The length of the initial state vector for a SDOF system is 2, making the size of the initial state shape matrix X_0 2×2. The initial state displacement uncertainty μ_{x_0} and velocity uncertainty $\mu_{\dot{x}_0}$ are set at $5 \cdot 10^{-4}$ for the undamped system and $5 \cdot 10^{-3}$ for the underdamped system, respectively, using appropriate units. The length of the control input vector for a SDOF system is 1. In this example, the uncertainty of the control input is assumed to be essentially zero; thus, the maximum $\mu_{p(t)}$ is set to $5 \cdot 10^{-16}$, which is machine zero in appropriate units.

The reachability analysis was conducted using *MATLAB* and ET (Kurzhanskiy and Varaiya 2006). Setting an input $u(\cdot) \in \mathcal{P}(t)$ in Eq. (8) and given the time interval $T = [t_0, t] = [0, 10]$ s, the set of initial conditions $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and the control input (excitation) $u(t) \in \mathcal{P}(t) = \mathcal{E}[p(t), p(t)]$, the external ellipsoidal reachable tube approximation $\mathcal{T} \subseteq \mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ of the reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ of the system [Eq. (8)] under finite pulse input was computed. Fig. 4 shows the computed responses of the undamped and underdamped SDOF systems. These graphs show the state-space trajectories of SDOF systems computed by numerically integrating their equations of motion [Eq. (8)] without any uncertainty and reachable tubes computed using ellipsoidal reachability analysis for the first 10 s (10 natural periods) of response time history. Also shown are reachable sets (cross sections of reachable tubes) at $t = 4$ s.

The SDOF system is at rest during the first 3 s of finite pulse excitation. However, because of the uncertainty in at-rest initial conditions, a reachable tube exists. The reachable tube contains the system trajectory computed without uncertainties showing that the ellipsoidal reachability analysis produces valid response bounds. The size of the reachable tube depends on the size of the uncertainty of the initial system state. In this example, the uncertainty is one order of magnitude smaller in the undamped case than in the underdamped case. In both the undamped and underdamped cases, the size of the reachable tube is roughly two orders of magnitude larger than the size of the magnitude of the initial state of uncertainty. The size of the reachable tube remains constant for the undamped SDOF system, whereas the size of the reachable tube decreases as the response of the underdamped SDOF system is damped out. The duration of the pulse is short enough (40% of the natural vibration period of SDOF systems) for the response to resemble that due to initial velocity obtained by momentum transfer, both with respect to the trajectories and sizes of the reachable tubes after the force pulse ($t > 3.4$ s). The cross sections of the reachable tubes at $t = 4$ s for the undamped and the underdamped SDOF systems show that the reachable sets enclose the exact solution. The sizes of the reachable set ellipsoid radii are typical, whereas the shape appears elongated because of the scale of the plot axes. Finally, this example demonstrates that the method of computing a reachable tube by concatenation using the semigroup property of reachable sets works.

SDOF System Response Reachability Analysis for Earthquake Excitation

This section applies the ellipsoidal reachability method for analysis of earthquake ground motion response of linear elastic SDOF systems. The earthquake ground motion acceleration record is represented as a sequence of finite pulses, and the reachable set semigroup property is applied to compute the reachable tube using concatenation. In this example, the 1940 El Centro earthquake ground motion acceleration record obtained from the Pacific Earthquake Engineering Research (PEER) strong motion database (PEER Center 2008) is used and treated as the control input $u(t)$ acting on the system [Eq. (8)]. The duration of the ground acceleration record $T = [t_0, t]$ is divided into n subintervals, where n is the number of samples of recorded earthquake excitation $u(t)$ such that

$$T = \bigcup_{i=1,\ldots,n} T_i, \quad T_i = [t_{i-1}, t_i], \quad i = 1, \ldots, n$$

where $t_n = t$. Then, in each subinterval T_i, earthquake excitation is represented as a finite-duration constant-intensity pulse $p_i(t) = -m \cdot a_{g,I}$, where the value of the ith pulse acceleration is the ith sample of the recorded ground motion acceleration data array and m is the mass of the SDOF system.

To introduce uncertainty into earthquake excitation as it is measured by accelerometers, each $p_i(t)$ is defined as $p_i(t) \in \mathcal{P}_i(t) = \mathcal{E}[p_i(t) P_i(t)]$. To introduce uncertainty into the state of the SDOF system as it is measured by displacement and velocity instruments, the initial state is defined at the beginning of each subinterval, and the

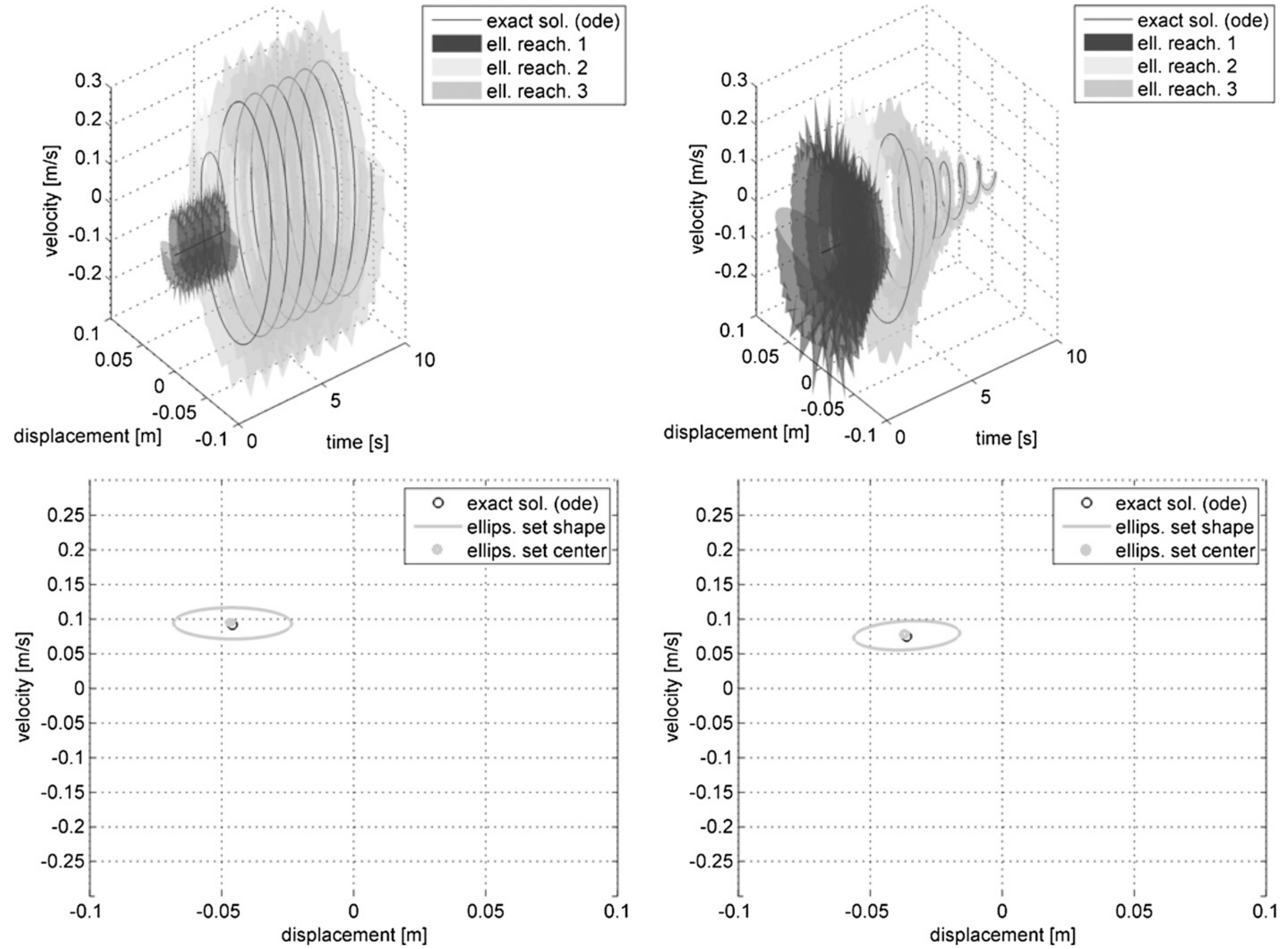

Fig. 4. (a) and (b) Reachable tubes and (c) and (d) reachable sets at $t = 4$ s for finite pulse response of an (a) and (c) undamped and (b) and (d) underdamped SDOF system computed using the initial state set $\mathcal{X}_0 = \mathcal{E}\left([0 \quad 0]^T, \begin{bmatrix} \mu_{x_0} & 0 \\ 0 & \mu_{\dot{x}_0} \end{bmatrix}\right)$ and the control input sets $\mathcal{P}_1(t) = \mathcal{E}\{[0], [\mu_{p(t)}]\}$ for $t \in T_1 = [t_0, t_1] = [0, 3]$ s, $\mathcal{P}_2(t) = \mathcal{E}\{[1], [\mu_{p(t)}]\}$ for $t \in T_2 = [t_1, t_2] = [3, 3.4]$ s, and $\mathcal{P}_3(t) = \mathcal{E}\{[0], [\mu_{p(t)}]\}$ for $t \in T_3 = [t_2, t_3] = [3.4, 10]$ s, with $\mu_{p(t)} = 5 \cdot 10^{-16}$, $\mu_{x_0} = \mu_{\dot{x}_0} = 5 \cdot 10^{-4}$ for undamped SDOF, and $\mu_{x_0} = \mu_{\dot{x}_0} = 5 \cdot 10^{-3}$ for underdamped SDOF

reachable state obtained at the end of the previous subinterval is defined with initial conditions at time t_0 specified as $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$. The magnitude of uncertainty is specified by prescribing the magnitude of the uncertainty ellipsoid shape matrix eigenvectors. Then, the external ellipsoidal reachable tube approximation $\mathcal{T}_i^+ \supseteq \mathcal{T}_i[t, t_0, \mathcal{X}_0, \Pi(t)]$, where $i = 1, \ldots, n$ for the SDOF system [Eq. (8)] for each of the n subintervals T_i is computed by computing reachable sets $\mathcal{X}_i(t)\ \forall t \in T_i$, where $i = 1, \ldots, n$ for each of the n subintervals. External ellipsoidal reachable tube approximation $\mathcal{T}^+$ of the reachable tube $\mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$ for the SDOF system under an earthquake excitation is then the union (in time) of the external ellipsoidal reachable tube approximations $\mathcal{T}_i^+ \supseteq \mathcal{T}_i(t)$, where $i = 1, \ldots, n$

$$\mathcal{T}^+ = \bigcup_{i=1,\ldots,n} \mathcal{T}_i^+ \supseteq \bigcup_{i=1,\ldots,n} \mathcal{T}_i(t) = \bigcup_{i=1,\ldots,n} \left[\bigcup_{t \in [t_{i-1}, t_i]} \{t\} \times \mathcal{X}_i(t) \right] = \mathcal{T}[t, t_0, \mathcal{X}_0, \Pi(t)]$$

where $\mathcal{X}_i(t) = \mathcal{X}[t, t_{i-1}, \mathcal{X}_{i-1}(t_{i-1}), \Pi_i(t_{i-1}, t_i)]$, where $i = 1, \ldots, n$ is the reachable set at time $t_i \geq t_{i-1}$ from the initial state $\mathcal{X}_{i-1}(t_{i-1})$ through any arbitrary control $p_i(\cdot) \in \Pi_i(t_{i-1}, t_i)$, where $\Pi(t_{i-1}, t_i) = \{\zeta_i: [t_{i-1}, t_i] \to \mathbb{R}^q | \zeta_i(\theta) \in \mathcal{P}_i(\theta) \forall \theta \in [t_{i-1}, t_i]$ and ζ_i is measurable$\}$, with $\mathcal{P}_i(t) = \mathcal{E}[q_i(t), Q_i(t)], t \in T_i$, and $i = 1, \ldots, n$. Subinterval external ellipsoidal reachable tube approximation is computed sequentially in chronological order of subintervals.

This algorithm is applied to investigate how uncertainty in a measured state and uncertainty in an excitation affect the earthquake response of a SDOF system. Fig. 5 shows the 1940 El Centro earthquake response trajectories of an undamped and underdamped (2% damping ratio) SDOF system computed by numerically solving state-space equations of motion [Eq. (8)] and using the time-history response analysis tool *Bispec* (Hachem 2010).

Uncertainty of an excitation is defined by setting the magnitude of uncertainty ellipsoid radii to $\mu_{p(t)} = 2.9 \cdot 10^{-3}\, g$. This magnitude was chosen assuming that the measurement error of a typical modern accelerometer is less than 0.1% of the maximum value of the measured acceleration (equal to $0.29g$ in this case). The same numerical value of uncertainty for the initial at-rest state of the system $\mu_{x_0} = \mu_{\dot{x}_0} = 2.9 \cdot 10^{-3} g$ is adopted. Figs. 6–8 depict the El Centro ground motion response reachable tubes and reachable sets for the

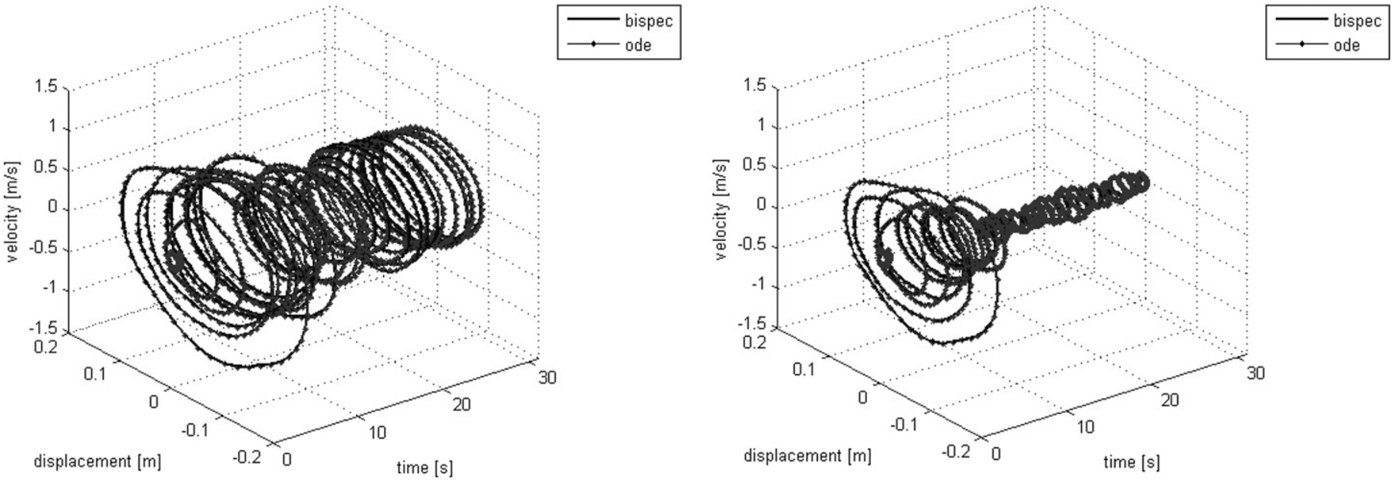

Fig. 5. Response of an (a) undamped and (b) underdamped SDOF system to 1940 El Centro earthquake ground motion record

undamped and underdamped SDOF systems computed using the initial state set

$$\mathcal{X}_0 = \mathcal{E}\left([0 \quad 0]^T, \begin{bmatrix} \mu_{x_0} & 0 \\ 0 & \mu_{\dot{x}_0} \end{bmatrix} \right)$$

with $\mu_{x_0} = \mu_{\dot{x}_0} = 2.9 \cdot 10^{-3} g$, and the control input set $\mathcal{P}(t) = \mathcal{E}\{[E(t)], [\mu_{p(t)}]\}$, with $\mu_{p(t)} = 2.9 \cdot 10^{-3} g$. Reachable sets (cross sections of the reachable tube) are shown at midpoints of the selected time intervals. The system trajectory remains within the computed reachable tube but not necessarily close to the origin of the reachable set ellipsoid. For example, at $t = 4.5$ s, the system state is relatively close to the reachable set ellipsoidal boundary. This shows that conservatism inherent to an external ellipsoidal reachable tube approximation is reasonable. Reachable tube size grows very rapidly during the initial 1-s-long time interval, even though the excitation itself, and therefore the response of the system, is quite small. At $t = 0.5$ s, eigenvalues of reachable set ellipsoids are two orders of magnitude larger than magnitudes of eigenvalues of the initial uncertainty state. The size of the reachable tube for an undamped SDOF system continues to grow during response analysis; conversely, the size of the reachable tube for an underdamped SDOF system remains similar to that attained at the time when maximum displacement response was reached during the [4, 5] s time interval. This observation indicates that the accumulation of uncertainty during the duration of an entire earthquake ground motion excitation is likely to lead to extremely conservative reachable tube estimates. Fig. 9 shows that state trajectories for both undamped and underdamped SDOF systems are contained within the reachable set computed at the instant when the two SDOF systems attain maximum displacement during their response to the 1940 El Centro ground motion record. This points to a promising way to compute reasonably conservative deterministic state trajectory bounds for linear elastic SDOF systems under earthquake excitation.

Applications of Ellipsoidal Reachability Analysis

Theorem: External Ellipsoidal Approximation of a Reachable Set for an LTI System

Given an uncertain initial state set $\mathcal{X}_0 = \mathcal{E}(x_0, X_0)$ and an uncertain control input (excitation) function set $\mathcal{P}(t) = \mathcal{E}[p(t), P(t)]$, the ellipsoidal reachable set for a LTI system [Eq. (2)] at time $t \geq t_0$ is given by

$$\mathcal{X}[t, t_0, \mathcal{E}(x_0, X_0), \Pi(t)] = \Phi(t, t_0)x_0 + \Phi(t, t_0)\mathcal{E}(0, X_0) + \int_{t_0}^{t} \Phi(t, s)B(s)p(s)ds + \int_{t_0}^{t} \Phi(t, s)\mathcal{E}\left[0, B(s)P(s)B^T(s)\right]ds \quad (10)$$

where $\Phi(t, t_0) = e^{A(t-t_0)}$.

Proof

The proof follows trivially from the LTI system state trajectory [Eq. (5)] and affine and semigroup properties of reachable sets. Eq. (10) shows that the ellipsoidal reachable set of an LTI system is an ellipsoid that encloses the state of the system at time t computed using Eq. (5) with axes defined by a transformation of the initial state and control input uncertainty ellipsoids through the dynamics of the LTI system.

The following examples compute how reachable tubes and reachable sets grow for an undamped SDOF system with a natural vibration period, $T_n = 1$ s, responding in free vibration, given different values of uncertainty in the initial state and in the control input, as reported in the caption of Fig. 10. The reachable sets are computed at $t = 2$ s. The reachable sets in Figs. 10(a and b) show that the uncertainty in the initial state has a transient and relatively small effect on their size. Conversely, continuously present uncertainty in control input has a significant effect on the size of the reachable sets, shown in Figs. 10(c and d), with their size being roughly proportional to the magnitude of control input uncertainty. To investigate the effect of the SDOF system damping ratio on the size of the reachable tube, the reachable set is computed at $t = 2$ s for two underdamped $T_n = 1$ s SDOF systems, with damping ratios equal to 0.02 and 0.05 in free vibration. The initial conditions and uncertainties used in this analysis are reported in the caption of Fig. 10, whereas the reachable sets shown in Figs. 10(e and f) indicate that damping reduces the size of a reachable set of ellipsoids, diminishing the effect of both transient and continuously present uncertainties.

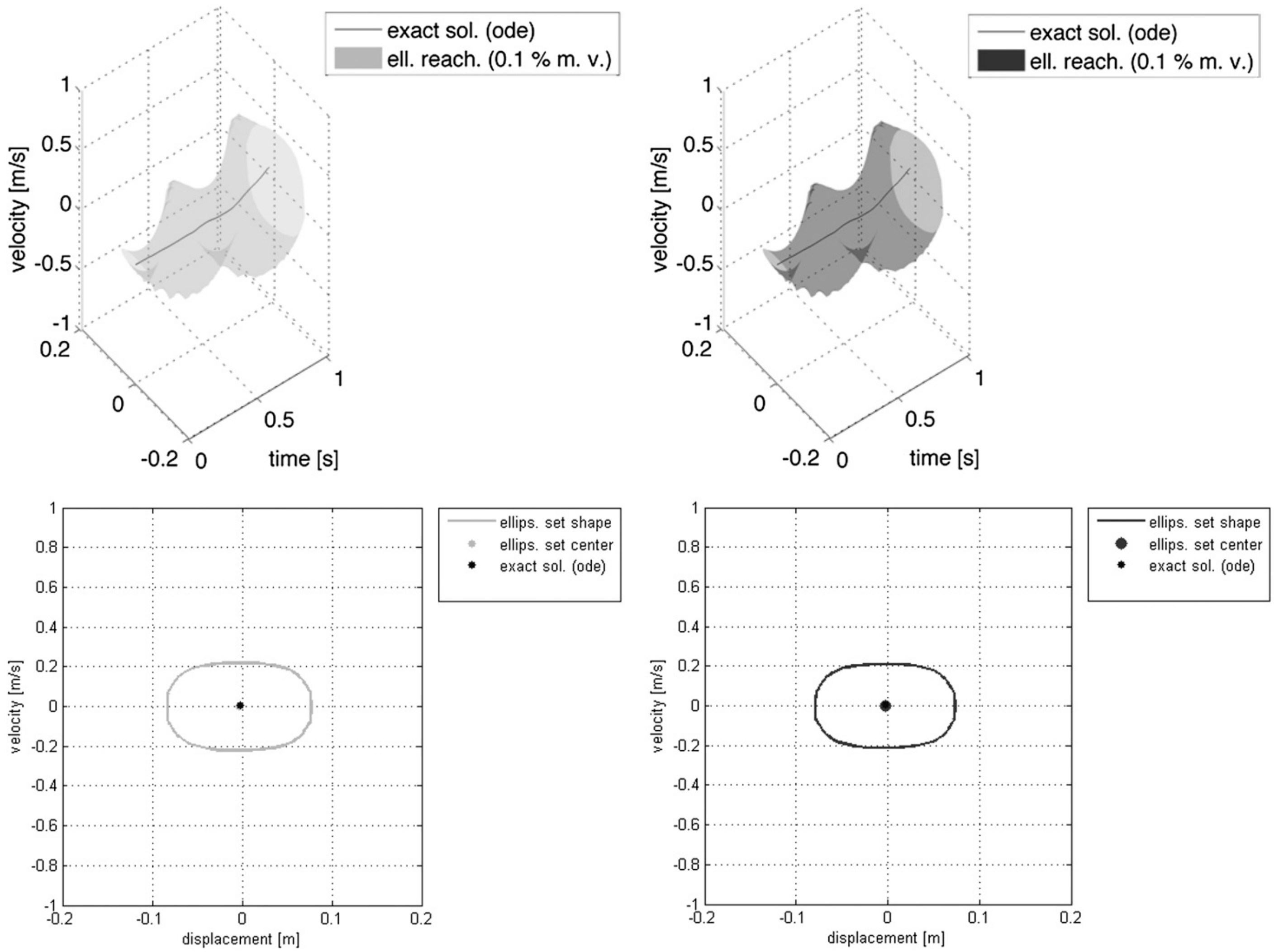

Fig. 6. 1940 El Centro ground motion response reachable tubes in the time interval (a) and (b) [0 1] s and reachable sets at (c) and (d) $t = 0.5$ s for the (a) and (c) undamped and (b) and (d) underdamped SDOF systems

Reachable tube size information can be used in the process of validation of computer simulation tools, such as FEM software for modeling the dynamic response of structures against experimental data. A common validation methodology comprises a dynamic response experiment on a prototypical structure followed by development of a numerical model and an analysis of model response to the same excitation. A comparison of the experimentally observed and simulated system state trajectories is used to evaluate the quality of the numerical simulation. Common evaluation criteria are based on measures of instantaneous, averaged or cumulative, and absolute or relative prototype versus model signal mismatch errors in the time and frequency domains. However, the amount of experimental measurement error is rarely explicitly taken into account. Deterministic reachable tube bounds computed using the ellipsoidal reachability analysis method presented here provide a way to explicitly include information about the accuracy of the instruments used in the experiments, thus giving a realistic measure of acceptable error levels for numerical simulation tools.

Ellipsoidal reachability analysis can also be used to improve the reliability of hybrid simulation (Stojadinović et al. 2006). Hybrid simulation is an experiment-based method for investigating the dynamic response of structures to time-varying excitation using a hybrid model. A hybrid model is an assemblage of one or more numerical and one or more experimental consistently scaled substructures. The response of a hybrid model to a time-varying excitation is obtained by solving its equations of motion using a time-stepping integration procedure that dynamically incorporates measured and computed data. This integration procedure is conducted in the presence of disturbances (Shing and Mahin 1990), such as model abstractions and approximations, random measurement, systematic experiments, actuation servocontrol, numerical integration algorithm errors, and time delay. A time-stepping integration procedure developed for hybrid simulation (Stojadinović et al. 2006) is comprised of two phases: (1) a predictive phase, in which the target state at the end of a time step is determined based on the current state and the past trajectory of the hybrid model by extrapolation; and (2) a corrective phase, in which the end of the time step target state computed (with some delay) for the entire hybrid model using a time-stepping integration algorithm becomes known, and the system state trajectory is corrected to arrive at the desired target state. Ellipsoidal reachability analysis can be used to examine state trajectories computed in predictive and corrective phases of a time step. A typical duration of a time step in seismic response hybrid simulations is 0.01 or 0.02 s, governed by the time step when the earthquake

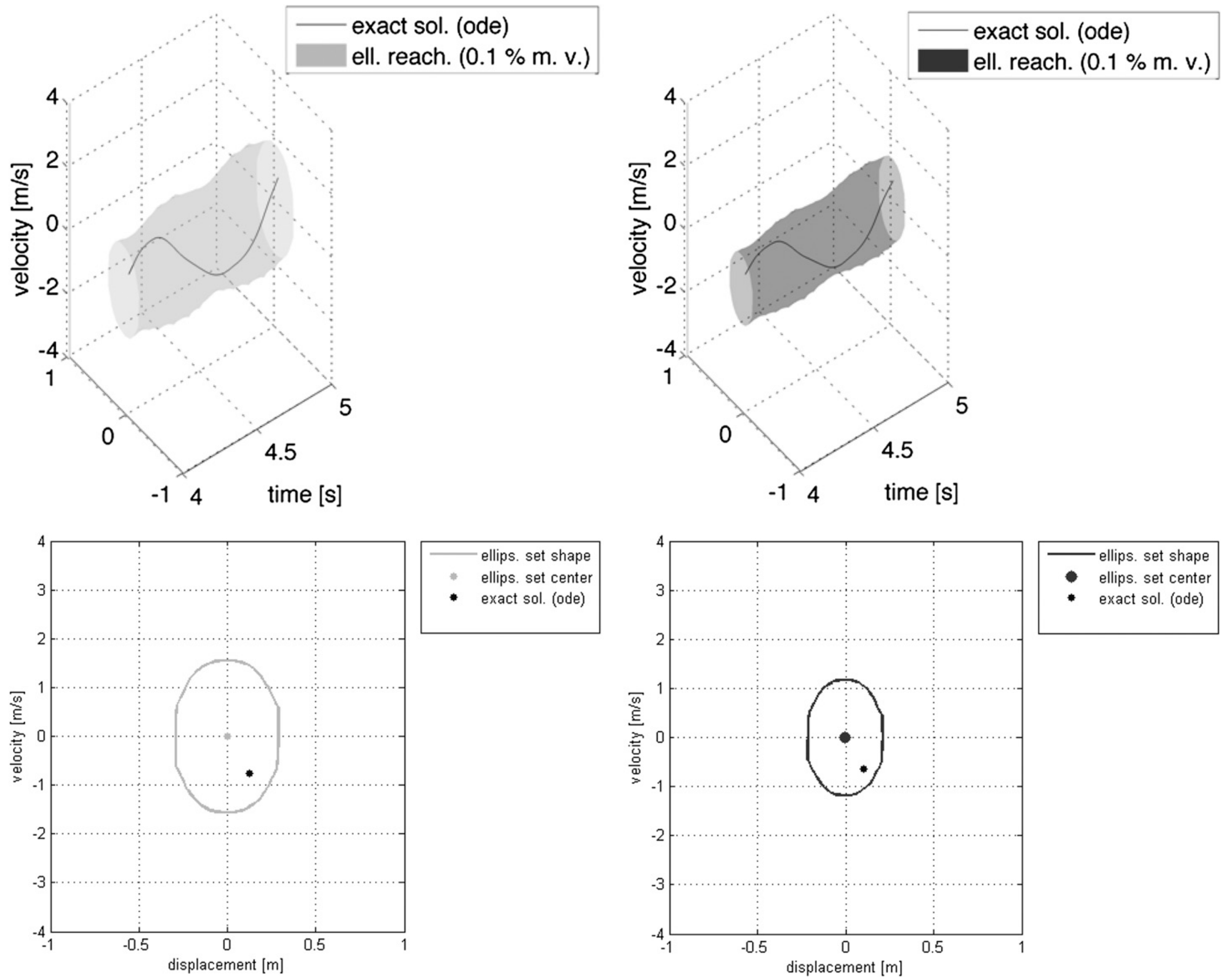

Fig. 7. 1940 El Centro ground motion response reachable tubes in the time interval (a) and (b) [4 5] s and reachable sets at (c) and (d) $t = 4.5$ s for the (a) and (c) undamped and (b) and (d) underdamped SDOF systems

ground motion was acquired. Assuming a linearized model based on the secant or tangent characteristics of the hybrid model, a reachable tube for the time step can be computed using uncertain excitation during the time step and starting from an uncertain current state of the hybrid model. The hybrid model state trajectory computed during the predictive phase should be inside this reachable tube. Similarly, a reachable tube computed for the corrective phase of the time step should also contain the hybrid simulation corrective phase state trajectory. Trajectory tests formulated in this way can be used to avoid errors in the predictive and corrective phases of a hybrid simulation time step and thus increase the reliability of this simulation method.

Conclusion

In this article, a reachability analysis is introduced and an application of an ellipsoidal reachability analysis is presented to compute bounds of response of a linear elastic SDOF system under uncertain dynamic excitation starting from an uncertain initial state. A deterministic description of the initial state and excitation uncertainty is used to represent state and excitation measurement errors. In an example, an external ellipsoidal reachable set approximation is to obtain a conservative, worst-case estimate of the sets of states a linear SDOF system can reach under the prescribed uncertainties in response to finite pulse loading. A method to concatenate ellipsoidal reachable sets for a sequence of impulse loads to compute the ellipsoidal reachable set approximation for SDOF system response to earthquake excitation was also formulated. Finally, possible applications of reachability analysis were discussed to gauge the quality of numerical model calibration to experimental data and control error propagation in experimental methods such as hybrid simulation.

A follow-up article presents an extension of the ellipsoidal reachability analysis to linear multiple-degree-of-freedom structural dynamic systems. Additional research is needed to understand (1) the effects of the magnitude of the initial state and excitation uncertainty, magnitude of response, and response characteristics of the system on the rate of growth and final size of reachable tubes; and (2) options for using reachability analysis to bound response state trajectories of nonlinear structural systems under dynamic excitation.

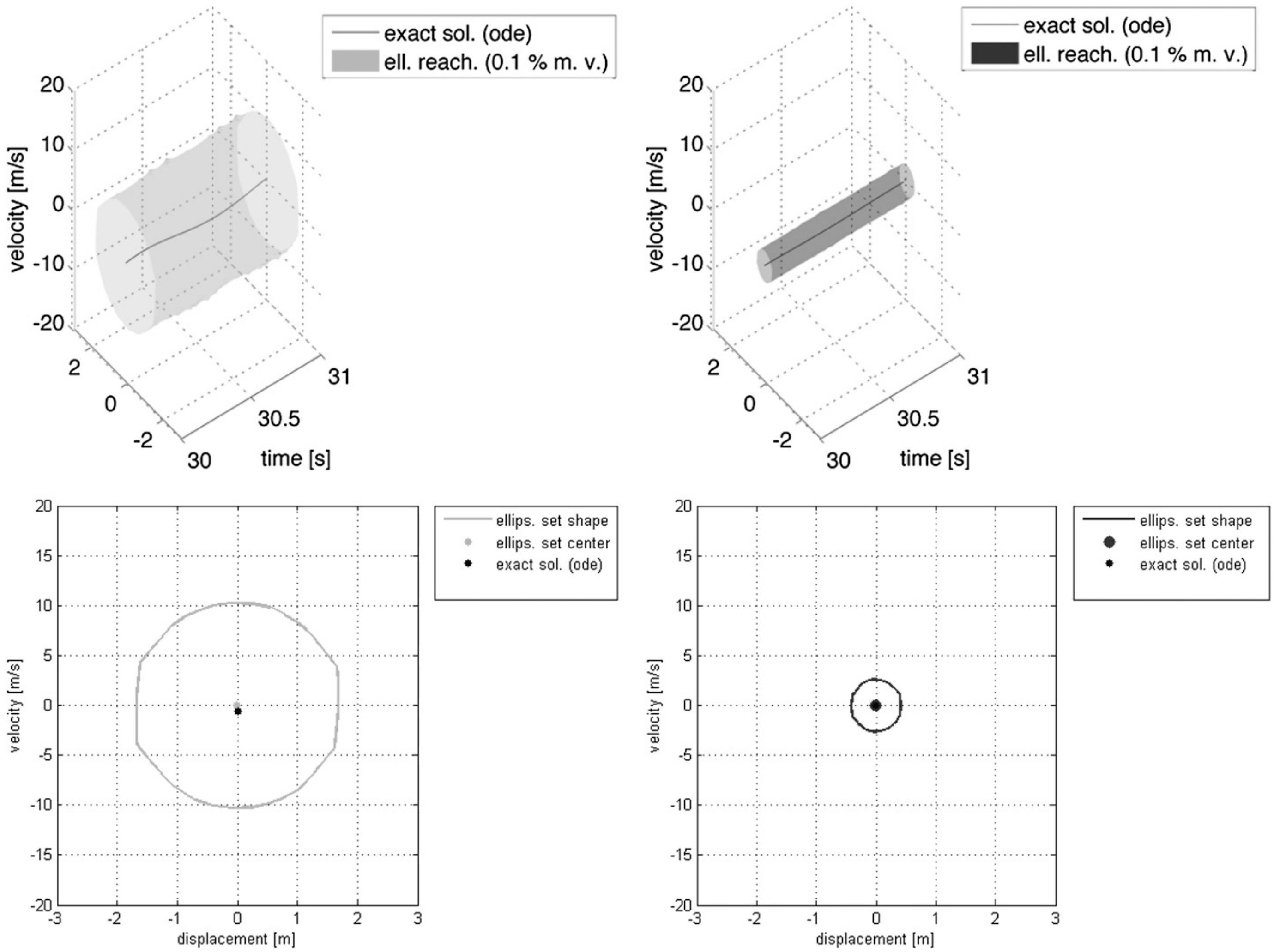

Fig. 8. 1940 El Centro ground motion response reachable tubes in the time interval (a) and (b) [30 31] s and reachable sets at (c) and (d) $t = 30.5$ s for the (a) and (c) undamped and (b) and (d) underdamped SDOF systems

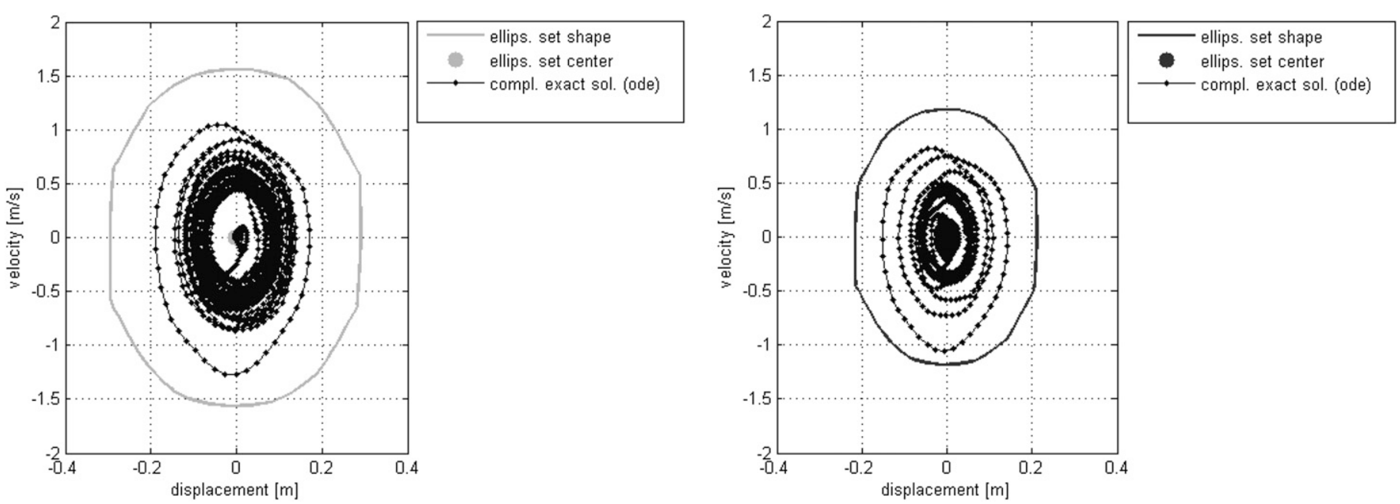

Fig. 9. Reachable sets at the instant of maximum displacement enclosing the state trajectories for the (a) undamped and (b) underdamped SDOF systems for the 1940 El Centro earthquake ground motion record

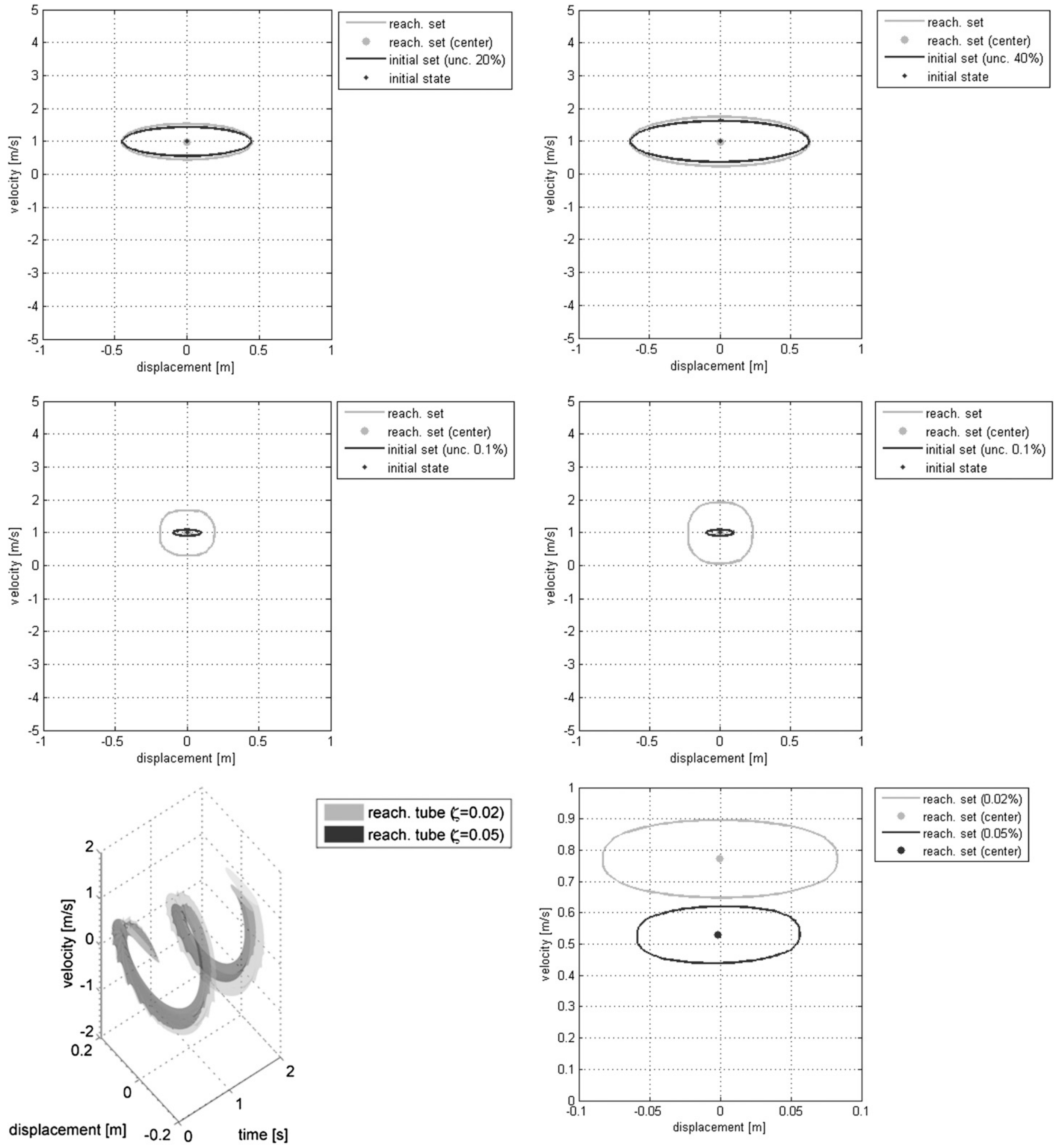

Fig. 10. Reachable sets for an undamped SDOF in free vibration at $t = 2$ s calculated using the initial state set $\mathcal{X}_0 = \mathcal{E}\left([0 \quad 1]^T, \begin{bmatrix} \mu_{x_0} & 0 \\ 0 & \mu_{\dot{x}_0} \end{bmatrix}\right)$, with $\mu_{x_0} = \mu_{\dot{x}_0}$ and the control input set $\mathcal{P} = \mathcal{E}\{[0], [\mu_{p(t)}]\}$, with control input uncertainty $\mu_{p(t)} = 5 \cdot 10^{-16}$ and initial state uncertainty (a) $\mu_{x0} = 0.2$ and (b) $\mu_{x0} = 0.4$, and with initial state uncertainty $\mu_{x_0} = 5 \cdot 10^{-3}$ and control input uncertainty (c) $\mu_{p(t)} = 0.2$ and (d) $\mu_{p(t)} = 0.4$; (e) reachable tubes and (f) sets for underdamped SDOF systems in free vibration with state uncertainty $\mu_{x_0} = 5 \cdot 10^{-3}$ and control input uncertainty $\mu_{p(t)} = 5 \cdot 10^{-16}$ and with different damping ratios $\zeta = 0.02$ and $\zeta = 0.05$

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