# Analytic and Numerical Solutions to the Kinematic Wave equation

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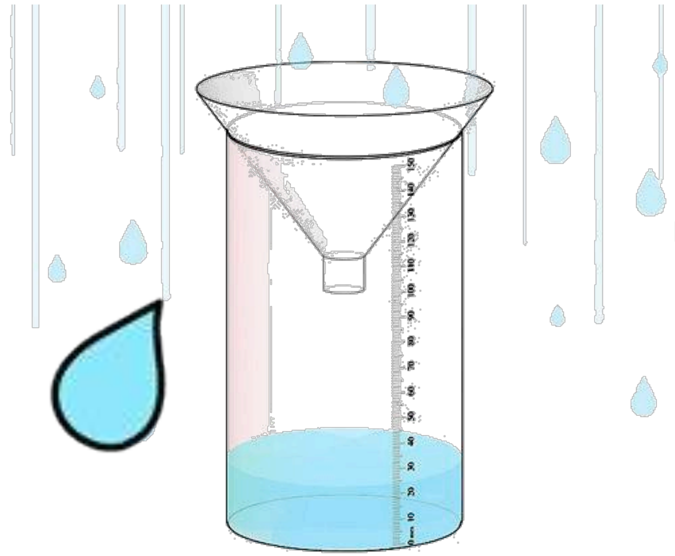
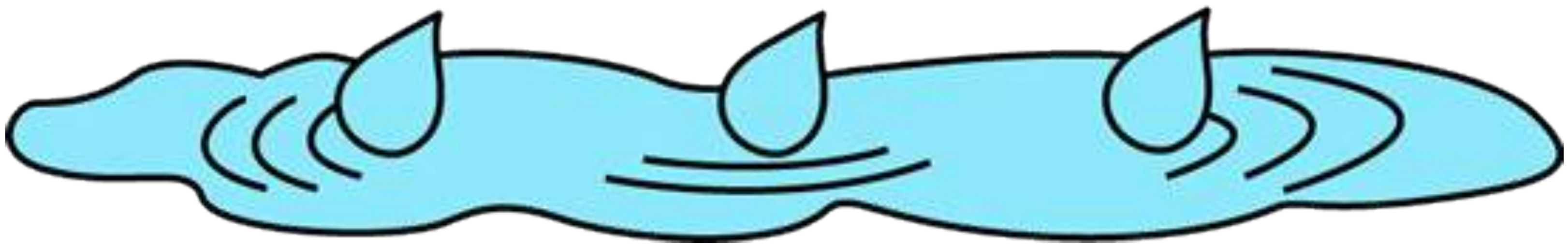
$$\rho \frac{D\overrightarrow{U}}{Dt} = -\nabla p + \mu \nabla (\nabla \overrightarrow{U}) + \rho g \delta_{i,3}$$

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r - f$$

# INTRODUCTION:

## PROBLEM MOTIVATION

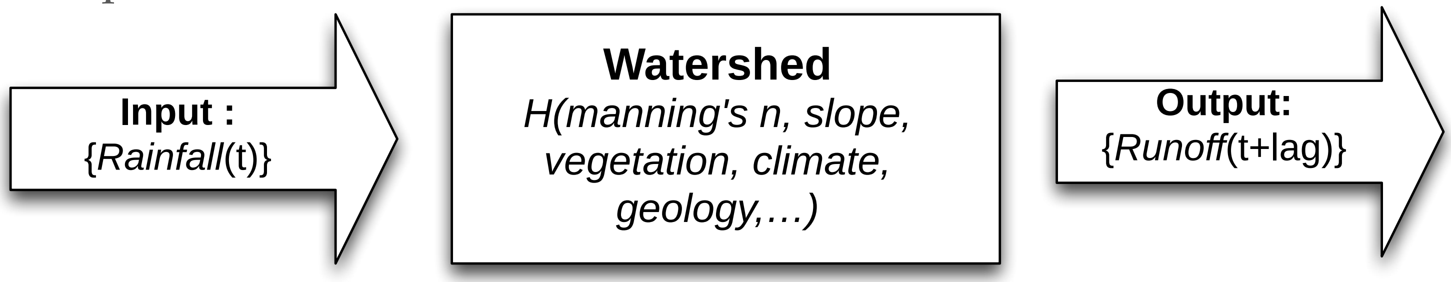





# INTRODUCTION:
## HYDROGRAPH GENERATION

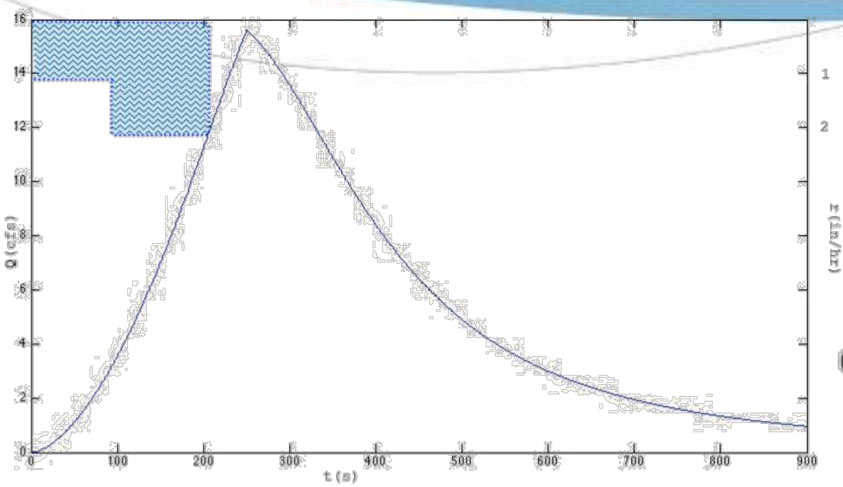
- Usual assumption: Watershed is an LTI System
  - Input: rainfall (hyetograph)
    - Measured by rainfall gauge.
  - Output: runoff (hydrograph)
    - Measured by stream gauge at watershed outlet.
  - Watershed defines a system transfer function
    - Finding transfer function allows prediction of flood events!

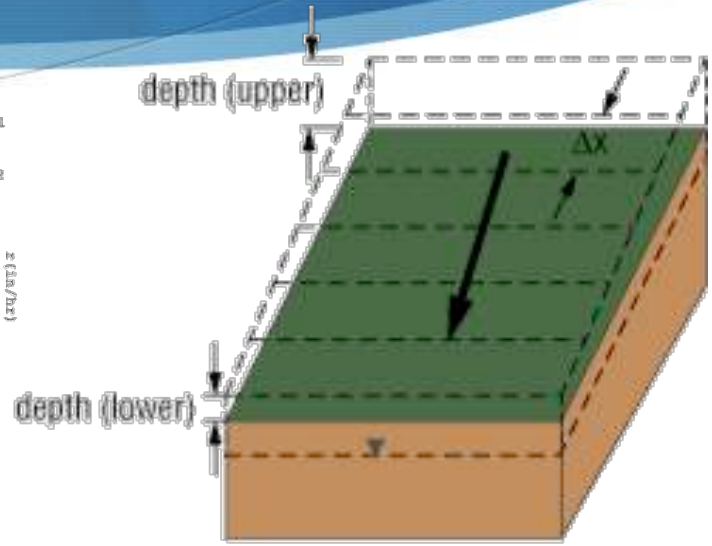
- Many methods to find:
  - Distributed parameter physical system; let's model it!
  - Objective is to use minimal number of gauges to estimate watershed response.

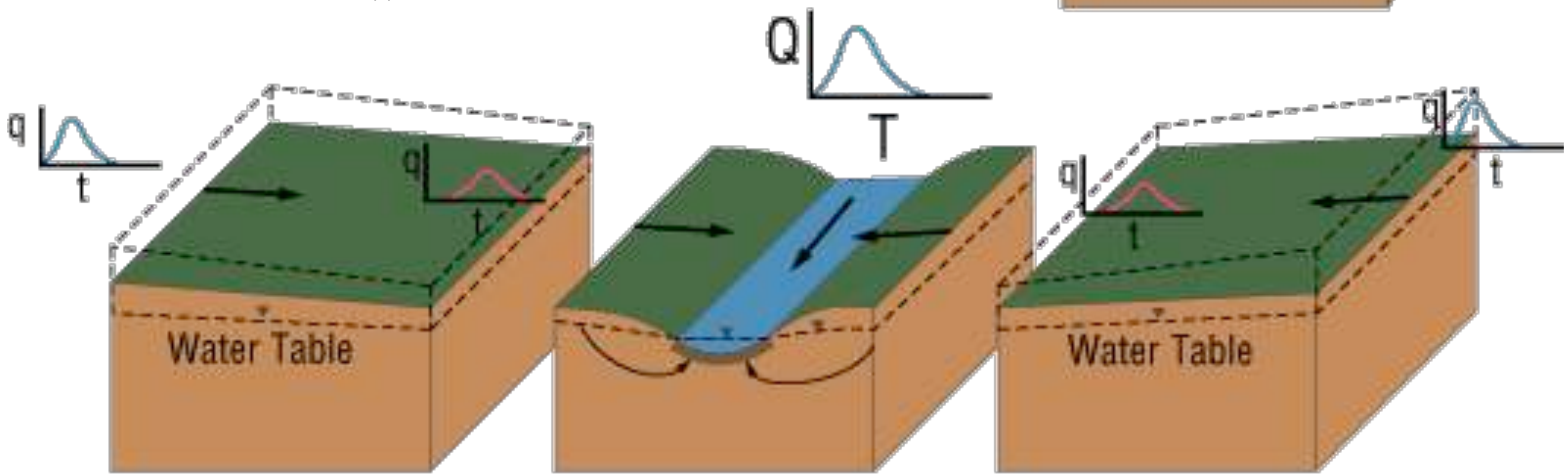


# INTRODUCTION

## PROBLEM SETUP



# INTRODUCTION:
## PRIMARY PROJECT OBJECTIVES

- **Analytical** derivation:
  - Overland Flow
  - Method of Characteristics.
- **Numerical** simulations of solution:
  - Steady rainfall and runoff rate
  - Temporally- and spatially- varying rainfall and runoff rate

# ANALYTIC SOLUTION
## FROM NAVIER-STOKES TO SAINT-VENANT IN 1 MINUTE

Conservation of momentum in Navier-Stokes:

$$\frac{\partial \bar{u}}{\partial t} + u\left(\frac{\partial \bar{u}}{\partial x} + \phi_1\right) + \bar{w}\frac{\partial \bar{u}}{\partial z} = -g\sin\theta - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\nabla^2\bar{u} - \overline{(\nabla \cdot v')u'} = 0$$

Use Leibniz's rule, the free-surface condition and $\int_0^h \overline{u^2}\, dz = V^2 h$

$$\int_0^h \frac{\partial\left(\bar{u}^2\right)}{\partial x} dz = \frac{\partial}{\partial x} V^2 h - \bar{w}\bar{u}\,|_{z=h} + \bar{u}_{z=h}\frac{\partial h}{\partial t}$$

For laminar and transitional flows, use KE correction coefficient:

$$\beta_c = \int_0^h (\bar{u}/V)^2\, dz / h$$

# ANALYTIC SOLUTION

## From Navier-Stokes to Saint-Venant in 1 minute

Some additional assumptions yield:

$$\frac{\partial h}{\partial x}\left(V^2 h\right)+\frac{\partial}{\partial t}(Vh)+hg\left(\frac{\partial h}{\partial x}+S_f-S_o\right)=0$$

Shear stress; bed slope

More commonly written as:

$$V\frac{\partial h}{\partial x}+\frac{\partial V}{\partial t}+g\left(\frac{\partial h}{\partial x}+S_f-S_o\right)+\frac{iV}{h}=0$$

Small departure from uniform steady flow by substitution of $V = V_0 + V_p$ and $h = h_0 + h_p$

$$V_0\frac{\partial h_p}{\partial x}+h_0\frac{\partial V_p}{\partial x}+\frac{\partial h_p}{\partial t}-i=0$$

Simplified form of continuity equation is:

$$\frac{\partial h_p}{\partial t}+\frac{\partial q_p}{\partial x}=i$$

# ANALYTIC SOLUTION

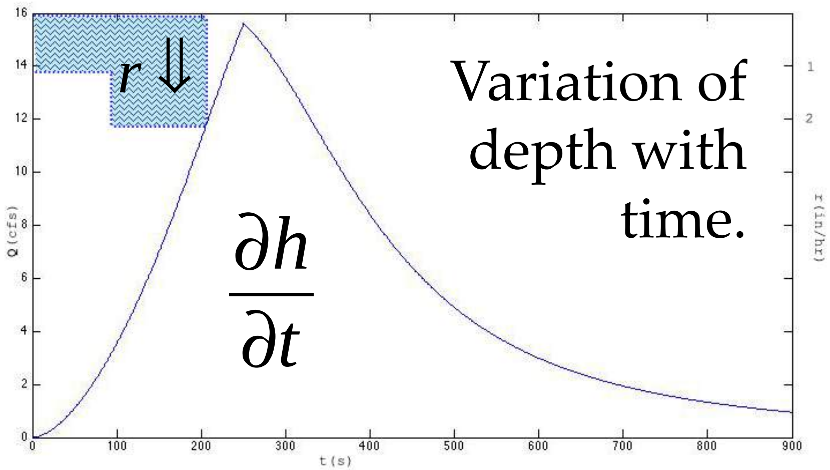
## FROM NAVIER-STOKES TO SAINT-VENANT IN 1 MINUTE

Take

inflow = **rainfall** – **infiltration**

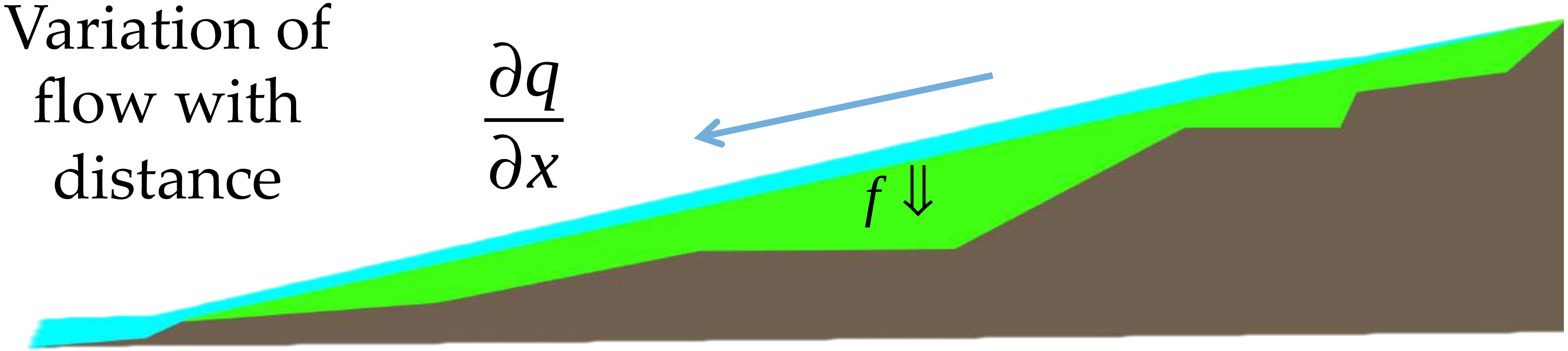
to get:

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = r - f$$




Variation of flow with distance

$$\frac{\partial q}{\partial x}$$



# ANALYTIC SOLUTION
## ASSUMPTIONS, BC/IC, AND CHARACTERISTICS

**Explicit form:**

$$q = \alpha h^m \qquad \alpha = \frac{1}{n}\sqrt{S_0}$$

Flow vs. Height    Manning's Law

**Initial BC/IC**

$$h(0,t) = 0, \quad h(x,0) = 0$$

Zero flow at initial time and distance

**Characteristic Equations**

$$dh/dt = r - f$$

$$dx/dt = \alpha m h^{m-1}$$

**Characteristic Domains**

Domain 1: $t = t_p, \quad 0 < x < L.$

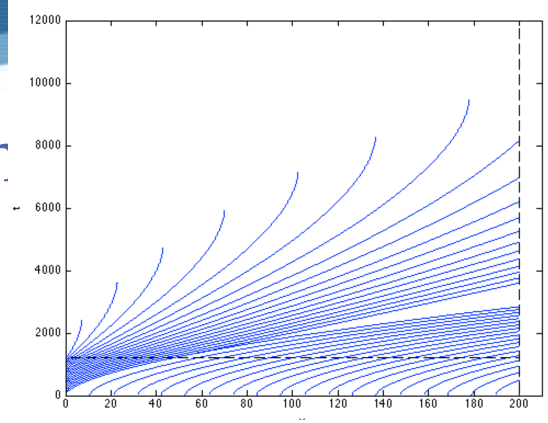
Domain 2: $x = 0, \quad 0 < t_0 < t_r.$

Domain 3: $t_r = t.$

# ANALYTIC SOLUTION

## CHARACTERISTIC DOMAIN I

$$t = t_p, \quad 0 < x < L$$




**Ponding Time**

$$t_p = \frac{S^2}{2rK_s} \ln\left(\frac{r}{r-K_s}\right)$$

**Smith-Parlange Infiltration**

$$t - t_p = \frac{S^2/2}{K_r^2} \ln\left[\left(\frac{r-K_s}{r}\frac{f}{f-K_s}\right) - \frac{K_s}{f} + \frac{K_s}{r}\right]$$

$$\int_0^h dh = \int_{t_p}^t (r-f)dt = (r-f)\left(t-t_p\right) - \int_f^r (t-t_p)df$$

$$\int_0^x dx = m\alpha \int_{t_p}^t h^{m-1}dt = m\alpha \int_0^h \frac{h^{m-1}}{r-f} dh(t-t_p) = \; -m\alpha \frac{S^2}{2} \int_f^r \frac{h^{m-1}}{f^2(f-K_s)} df$$

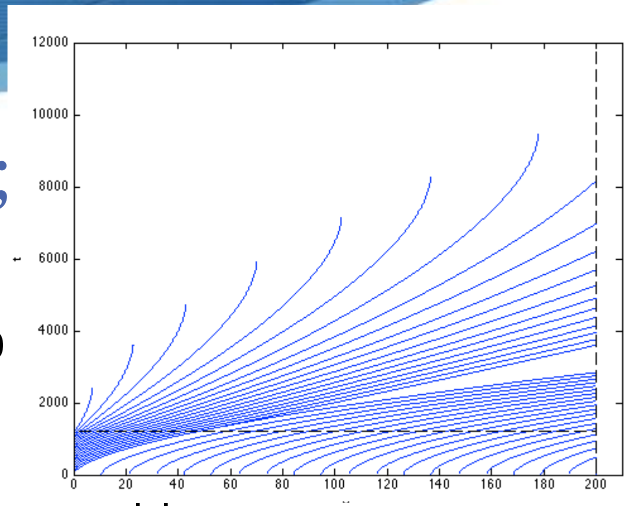
$$m = 2$$

$$x - x_0 = 2\alpha \left(\frac{S^2/2}{K_s}\right)^2 \left\{\frac{r^2-f^2}{2rf^2} + \frac{r-K_s}{K_s} \cdot \left[-\frac{1}{f} + \frac{1}{2K_s}\ln\left(\frac{r-K_s}{r}\frac{f}{f-K_s}\right)\right] \cdot \ln\left(\frac{r-K_s}{r}\frac{f}{f-K_s}\right)\right\}$$

# ANALYTIC SOLUTION

## CHARACTERISTIC DOMAIN II

$$x = 0, \quad t_p < t < t_r$$




Characteristics emanate from time axis;

$$h = \int_{t_p}^{t} (r - f)dt = (r - f)\left(t - t_0\right) - \int_{f}^{t_0} (t - t_0$$

$$h = \frac{S^2/2}{K_r^2}\left[\left(r - k\right)\ln\left(\frac{f_0 - K_s}{f_0}\frac{f}{f - K_s}\right) - rK_s\left(\frac{1}{f} - \frac{}{f_0}\right)\right]$$

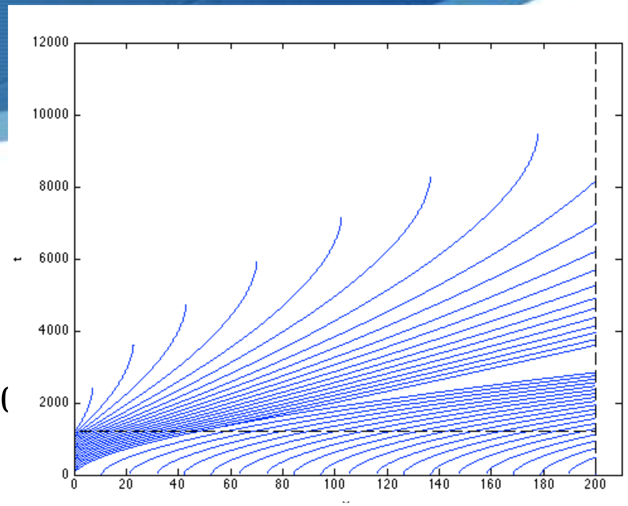
$$\int_0^x dx = m\alpha \int_{t_p}^{t} h^{m-1} dt \qquad m = 2$$

$$x = -2\alpha\left(\frac{S^2/2}{K_s}\right)^2 \left\{\frac{r - f_0}{f_0}\left(\frac{1}{f} - \frac{1}{f_0}\right) - \frac{r}{2}\left(\frac{1}{f^2} - \frac{1}{f_0^2}\right) + \left[\left(\frac{r - K_s}{fK_s} - \frac{r - f_0}{f_0 K_s}\right) - \frac{r - K_s}{2K_s^2}\cdot\ln\left(\frac{f_0 - K_s}{f_0}\frac{f}{f - K_s}\right)\right]\cdot\ln\left(\frac{f_0 - K}{f_0}\frac{f}{f - K}\right)\right\}$$

# ANALYTIC SOLUTION

## CHARACTERISTIC DOMAIN III

$$t_r = t_f$$




Now no rain, so: $dh/dt = -f$

$$h - h_* = \int_{t_r}^{t} -f dt = -f\left(t - t_r\right) - \int_{f}^{f_*}(t - t_($$

$$h - h_* = -\frac{S^2/2}{K_r^2}\ln\left(\frac{f_r - K_s}{f_r}\frac{f}{f - K_s}\right)$$

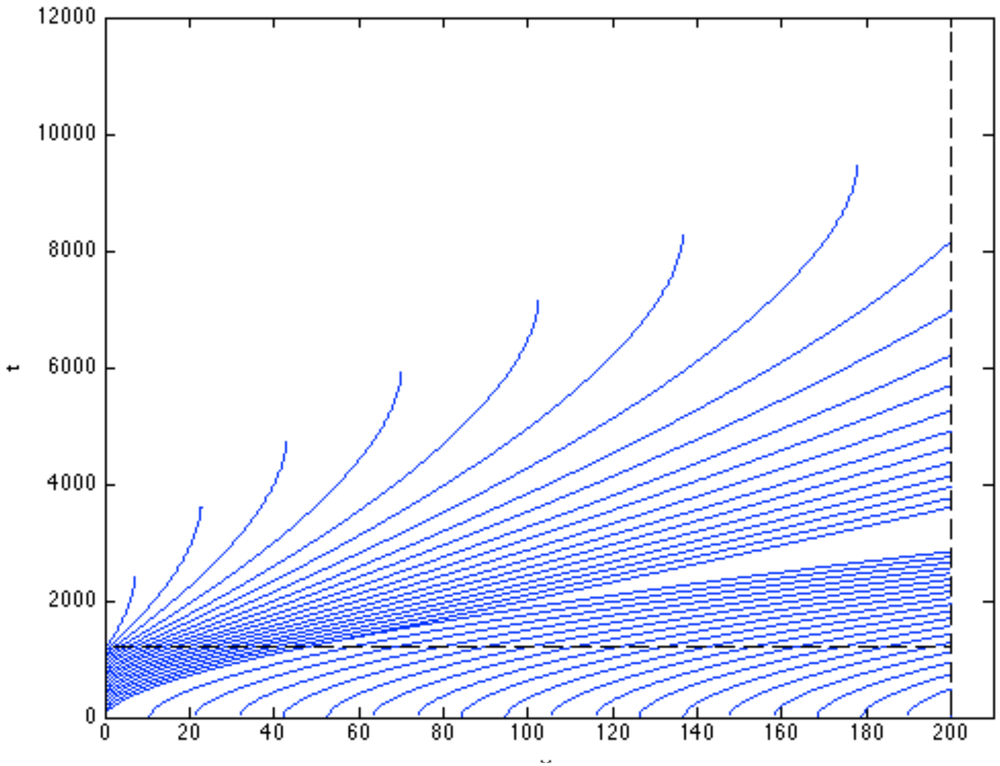
m = 2

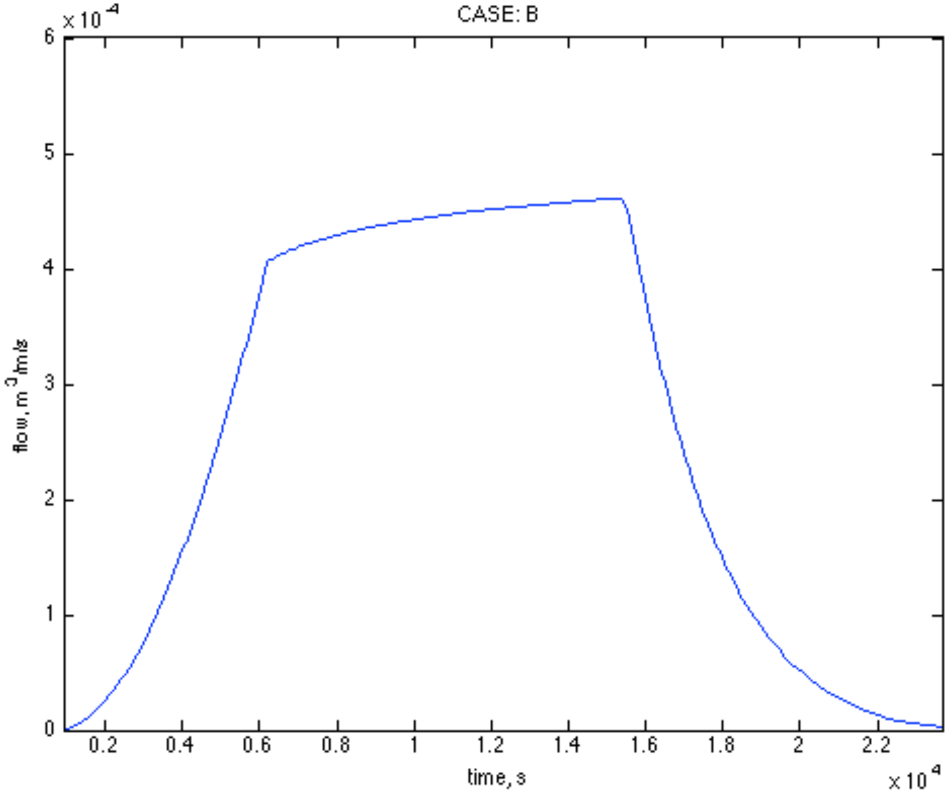
$$f_{zw} = K_s\left[1 - \frac{f_r - K_s}{f_r}\exp\left(\frac{h_* K_s}{S^2/2}\right)\right]^{-1}$$

$$x - x_* = 2\alpha\left(\frac{S^2/2}{K_s}\right)\left\{\left(h_* - \frac{S^2/2}{K_s}\right)\left(\frac{K_s}{f} - \frac{K_s}{f_r}\right) + \left[\frac{S^2/2}{K_s}\left(\frac{f - K_s}{f} + \frac{1}{2}\ln\left[\frac{f_r - K_s}{f_s}\frac{f}{f - K_s}\right]\right) - h_*\right]\cdot\ln\left(\frac{f_r - K_s}{f_r}\frac{f}{f - K_s}\right)\right\}$$

# ANALYTIC SOLUTION

## RESULTS

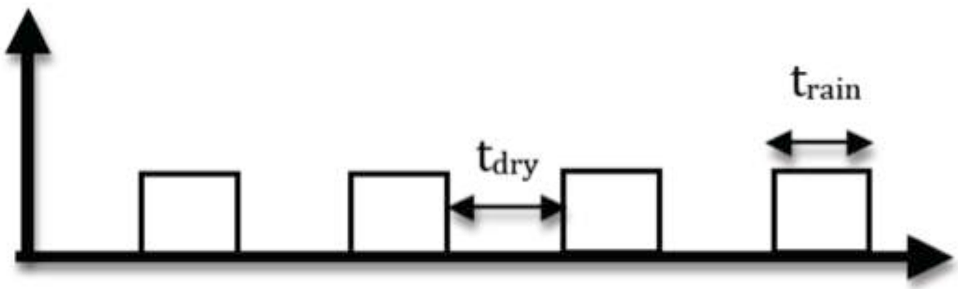


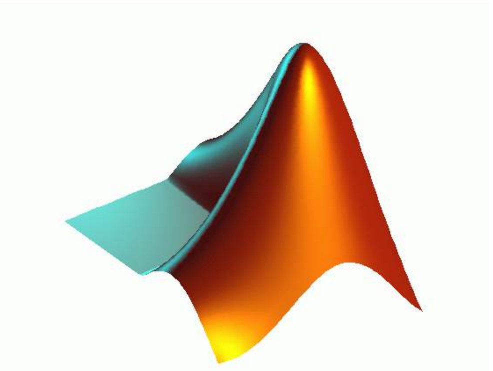



# ANALYTIC SOLUTION

## DISCUSSION

- Can only work for very specific cases of physical settings
- Try to solve analytically with changes will become a mental gymnastic exercise. (we tried to solve it with two storms)
- Rescue ? Go Numerical – finite difference schemes



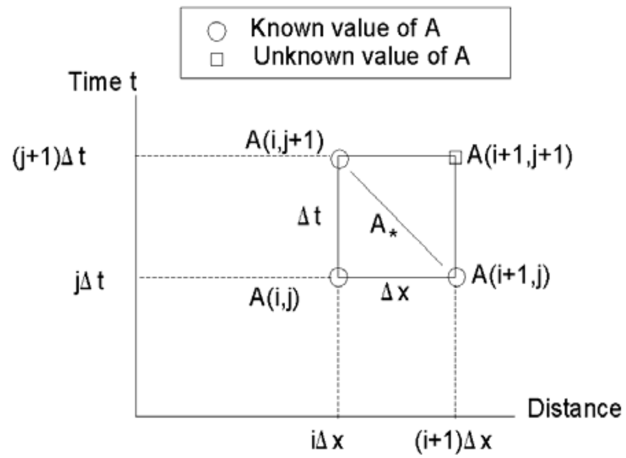



# NUMERIC SOLUTION

## Explicit Finite-Differences Solution

Start with discrete uniform rainfall and infiltration:

$$q_{i+1}^{j+1} = \alpha (y_{i+1}^{j+1})^m$$




$$\frac{q_{i+1}^{j+1} - q_{i+1}^{j+1}}{\Delta x} + \frac{y_{i+1}^{j+1} - y_{i+1}^{j}}{\Delta t} = (i - f)_{i+1}^{j},$$

Write as an update function

$$f(y_{i+1}^{j+1}) = \frac{\Delta t}{\Delta x}\alpha\left(y_{i+1}^{j+1}\right)^m + y_{i+1}^{j+1} - \left[\frac{\Delta t}{\Delta x}\alpha\left(y_i^{j+1}\right)^m + y_{i+1}^{j} + \Delta t\left(i - f\right)_{i+1}^{j+1}\right]$$

# NUMERIC SOLUTION

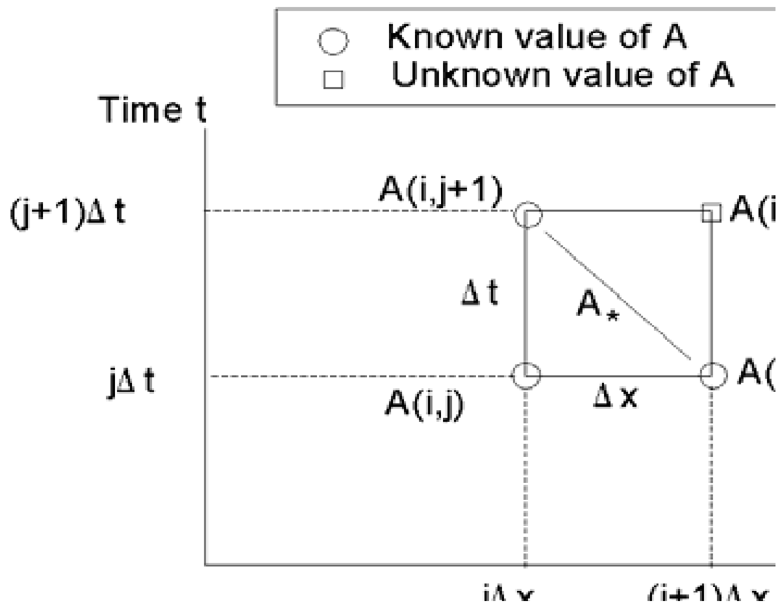
## Explicit Finite-Differences Solution

$$\frac{\partial A}{\partial t} + \alpha \frac{\partial A^m}{\partial x} = \bar{q}$$

$$Q = \alpha \; A^m$$

$$\frac{\partial A}{\partial x} \approx \frac{A_{i+1}^{j} - A_{i}^{j}}{\Delta x} \qquad A_* \simeq \frac{A_{i+1}^{j} + A_{i}^{j+1}}{2}$$

$$\frac{\partial A}{\partial t} \simeq \frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} \qquad \bar{q} \simeq \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2}$$




$$\frac{A_{i+1}^{j+1} - A_{i+1}^{j}}{\Delta t} + \alpha\left[\frac{\left(A_{i+1}^{j}\right)^m - \left(A_{i}^{j}\right)^m}{\Delta x}\right] = \frac{q_{i+1}^{j+1} + q_{i+1}^{j}}{2} = \bar{q}$$

Write as an update function $A_{i+1}^{j+1} = \bar{q}\,\Delta t + A_{i+1}^{j}\left[1 - \frac{\alpha\,\Delta t}{\Delta x}\left(A_{i+1}^{j}\right)^{m-1}\right] + \frac{\alpha\,\Delta t}{\Delta x}\left(A_{i}^{j}\right)^{m}$

# NUMERIC SOLUTION

## 4-POINT IMPLICIT FINITE DIFFERENCE

Use weighting parameter for spatial derivative:

$$\frac{\partial y}{\partial x} = \theta \frac{y_{i+1}^{j+1} - y_i^{j+1}}{\Delta x} + (1-\theta)\frac{y_{i+1}^{j} - y_i^{j}}{\Delta x}, \qquad \frac{\partial y}{\partial t} = \frac{1}{2}\left[\frac{y_i^{j+1} - y_i^{j}}{\Delta t} + \frac{y_{i+1}^{j+1} - y_{i+1}^{j}}{\Delta t}\right]$$

Plug into previous update function:

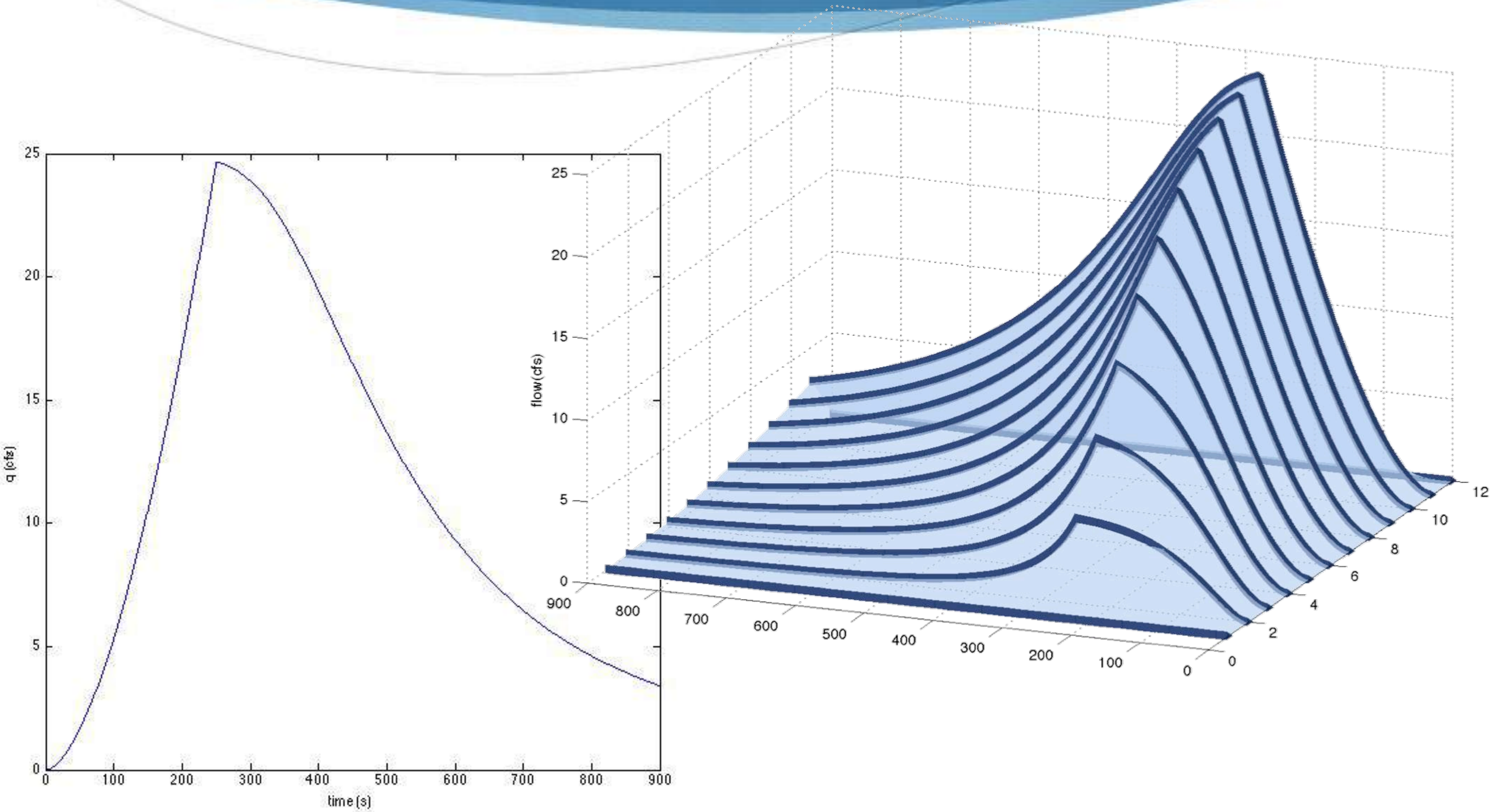
$$y_{i+1}^{j+1} - y_{i+1}^{j} + y_i^{j+1} - y_i^{j} + \ldots$$

$$\ldots + \frac{2\Delta t}{\Delta x}\left\{\theta\alpha\left[\left(y_{i+1}^{j+1}\right)^m - \left(y_i^{j+1}\right)^m\right] - (1-\theta)\alpha\left[\left(y_{i+1}^{j}\right)^m - \left(y_i^{j}\right)^m\right]\right\} - \ldots$$

$$\ldots - \Delta t\left[(i-f)_{i+1} + (i-f)_i\right] = 0$$

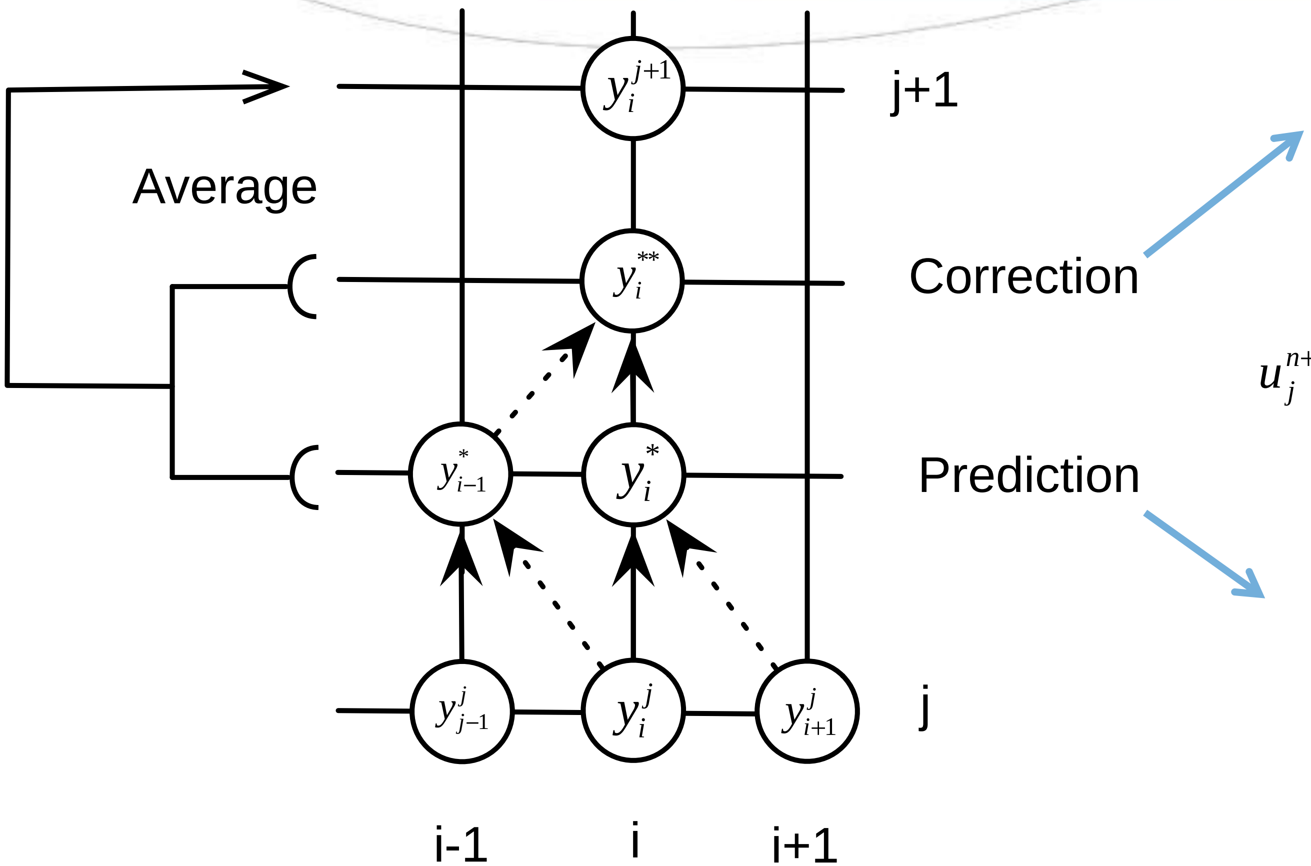
# NUMERIC SOLUTION

## EXPLICIT FINITE-DIFFERENCES SOLUTION



# NUMERIC SOLUTION

## McCormack Splitting Method Concept




$$\frac{u_j^{n+1}-u_j^{**}}{\left(\Delta t/2\right)} = -a\left(\frac{u_j^{*}-u_{j-1}^{*}}{\Delta x}\right)$$

where $u_j^{**} = \frac{1}{2}\left(u_j^{n}-u_j^{*}\right)$

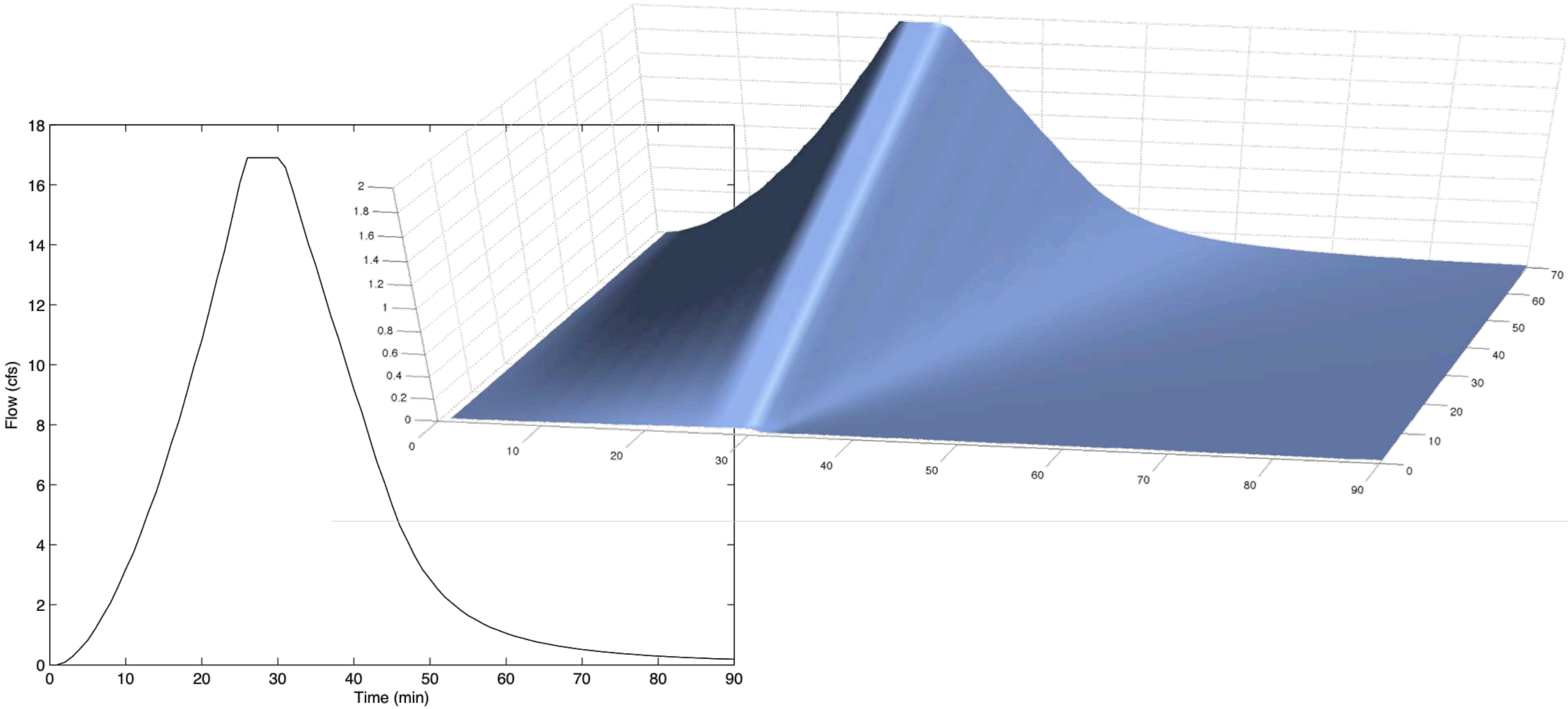
$$u_j^{n+1} = \frac{1}{2}\left[\left(u_j^{n}+u_j^{*}\right)-\frac{a\Delta t}{\Delta x}\left(u_j^{*}+u_{j-1}^{*}\right)\right]$$

$$\frac{u_j^{*}-u_j^{n}}{\Delta t} = -a\left(\frac{u_{j+1}^{n}-u_j^{n}}{\Delta x}\right)$$

**Courant Condition:** $\frac{a\Delta t}{\Delta x} \leq 1$

# NUMERIC SOLUTION

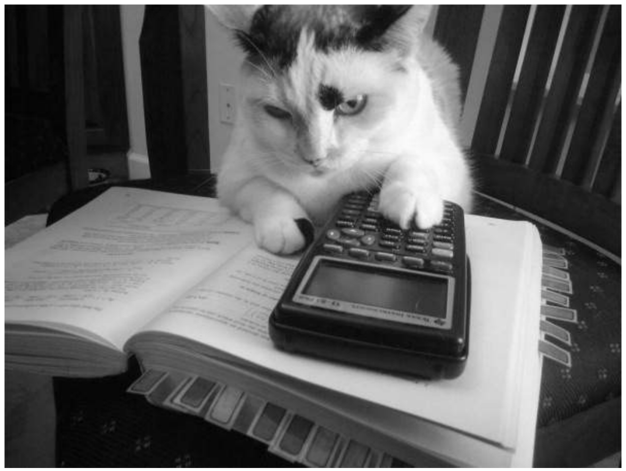
## McCormack Splitting Method Results



# DISCUSSION

## NUMERIC VS. ANALYTIC

- Work in Progress
- Numerical errors expected
- Initial condition and the forcing turn
- Unsteadiness



ANALYTICAT

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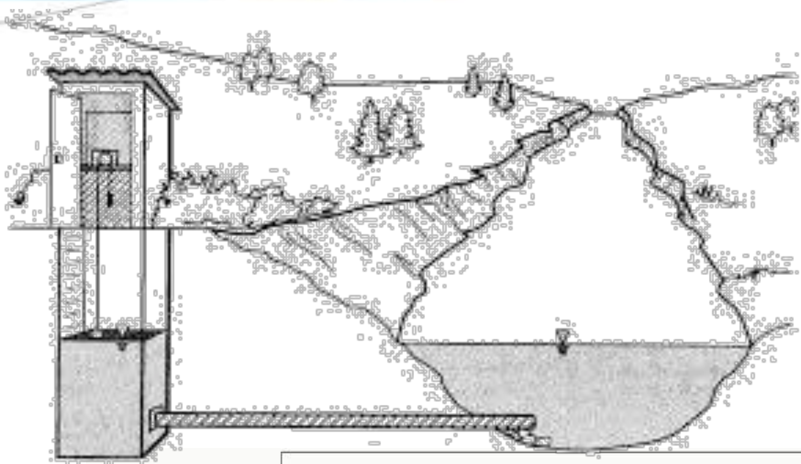
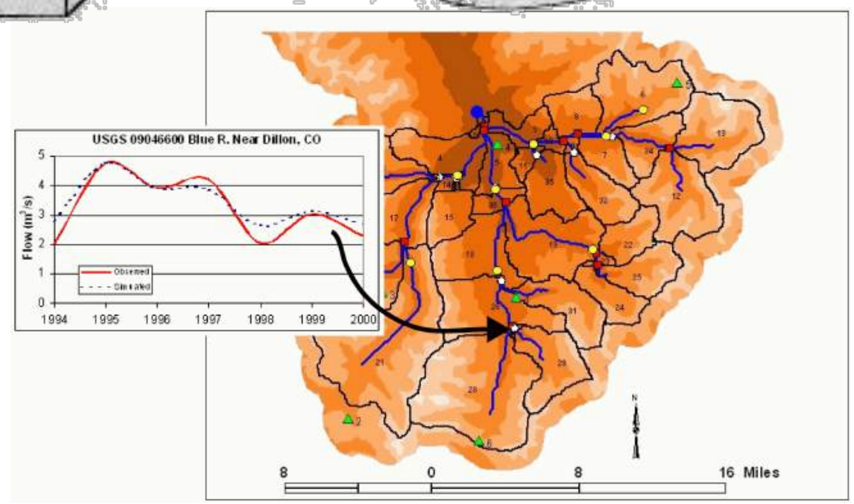
# CONCLUSIONS
## &
## FUTURE DIRECTIONS

- Analytic solution is elegant, but limited
- Numerical schemes are powerful but subject to approximation errors
- Validate using empirical unit hydrograph approach
- Use a spatial grid of interpolated, real infiltration data and rainfall





# THANK YOU!

## QUESTIONS?

*e.g., Who Are Those People?*

# Selected References:


[1] W. Brutsaert. Water on the Land Surface: Fluid Mechanics of Free Surface Flow. In *Hydraulic Theory*, pages 161–194. 1995.

[2] J V Girjldez and D A Woolhiser. Analytical integration of the kinematic equation for runoff on a plane under constant rainfall rate and Smith and Parlange infiltration. *Water Resources Research*, 32(11):3385–3389, 1996.

[3] Vijay P Singh and David A Woolhiser. Mathematical Modeling of Watershed Hydrology. *Journal of Hydrologic Engineering*, 7:270–292, 2002.

[4] R E Smith and J-Y Parlange. A Parameter-Efficient Hydrologic Infiltration Model. *Water Resources Research*, 14(3):533–538, 1978.