THE TRANSPORT EQUATION:
ANALYTICAL SOLUTION AND APPLICATION FOR A CONTROL PROBLEM

CE291 Term Project

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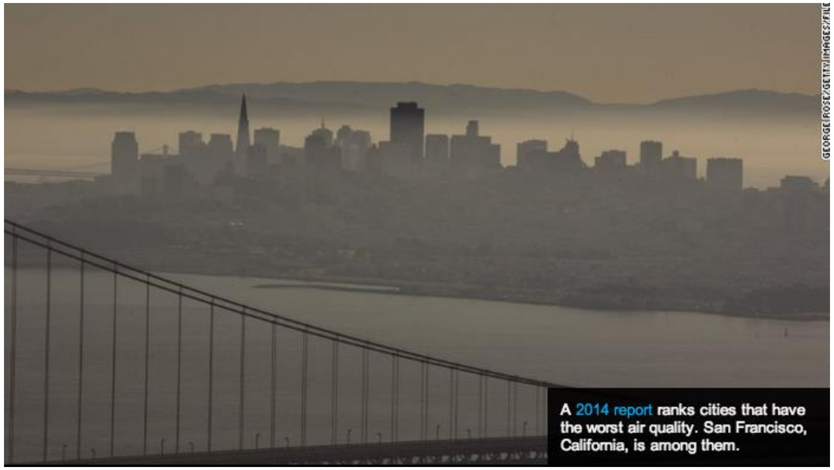
Motivation and Strategy

- Predict air pollutant concentration by solving 2D transport equation
- Apply the solution to a control problem



Advection-Dispersion Equation

$$\frac{\partial c}{dt} = D_L \frac{\partial^2 c}{\partial x^2} + D_T \frac{\partial^2 c}{\partial y^2} - \frac{u \partial c}{\partial x} - \frac{v \partial c}{\partial y} + q(x, y, t)$$

Initial condition: **c(x**,**y**,0)=**f**(**x**,**y**).

C: solute concentration.

D_L, D_T: the longitudinal and transverse dispersion coefficients

u, **v:** wind speed in the x and y direction

q: a source/sink term

Overview of Solving Strategy

- Non-dimensionalization
- Transformation

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial Y^2} + Q(X, Y, T)$$

- Analogy of Green's Function
- Fourier Transform of Green's Function
- Solution of Green's Function

- *Ref: Aral, Mustafa M., and Boshu Liao. "Analytical solutions for two-dimensional transport equation with time-dependent dispersion coefficients." Journal of Hydrologic Engineering 1.1 (1996): 20-32.*

Non-Dimensionalization

$$x = \frac{\boldsymbol{x}}{L}, \qquad y = \frac{\boldsymbol{y}}{L}, \qquad D_L = \frac{\boldsymbol{D_L}}{D_r}, \qquad D_T = \frac{\boldsymbol{D_T}}{D_r},$$

$$t = \frac{t\boldsymbol{D_T}}{D_r}, \qquad u = \frac{\boldsymbol{u}L}{D_r}, \qquad v = \frac{\boldsymbol{v}L}{D_r}, \qquad c = \frac{\boldsymbol{c}}{c_r}, \qquad q = \frac{\boldsymbol{q}L^2}{c_r D_r}, \qquad f(x,y) = \frac{\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y})}{c_r}$$

$$\frac{\partial c}{\partial t} = \frac{D_L \partial^2 c}{\partial x^2} + \frac{D_T \partial^2 c}{\partial y^2} - \frac{u \partial c}{\partial x} - \frac{v \partial c}{\partial y} + q(x,y,t)$$

Assumptions and Transformation

Assume $D_L = a^2 D_T = D$, and D scales with L^2/T, we have:

$$\frac{L_x^2}{T} = \frac{a^2 L_y^2}{T} \quad and \quad L_x = aL_y$$

$$x' = x, \quad y' = ay, \quad v' = av$$

Let X = x'-ut, Y = y'-v't

$$\frac{\partial c(x', y', t)}{\partial t} = \frac{\partial c(X, Y, t)}{\partial t} + (-u)\frac{\partial c}{\partial X} + (-v')\frac{\partial c}{\partial Y}$$

Introduce T:

$$T = \alpha(t) = \int_0^t D dt' = D * t$$

$$\frac{\partial c}{\partial t} = \frac{\partial c}{\partial T}\frac{\partial T}{\partial t} = D\frac{\partial c}{\partial T}$$

Assumption and Transformation

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial Y^2} + Q(X, Y, T)$$

Where:

$$Q(X,Y,T) = \frac{q\{X + U\alpha^{-1}(T), \frac{[Y + V\alpha^{-1}(T)]}{a}, \alpha^{-1}(T)\}}{D[\alpha^{-1}(T)]}$$

Initial condition:

$$c(X,Y,0) = f\left(X, \frac{Y}{a}\right), -\infty < X, Y < \infty$$

Green's Function

- Our Equation

$$\frac{\partial c}{\partial T} = \frac{\partial^2 c}{\partial X^2} + \frac{\partial^2 c}{\partial Y^2} + Q(X, Y, T)$$

- Heat equation with sources on an infinite domain

$$\frac{\partial u}{\partial t} = k\nabla^2 u + Q(\boldsymbol{x}, t)$$

- Green's Function for the heat equation without boundaries satisfies

$$\frac{\partial G}{\partial t} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x_0})\delta(t - t_0)$$

subject to:

$$G(\boldsymbol{x}, t; \boldsymbol{x_0}, t_0) = 0 \qquad for\ t < t_0$$

- To solve this, apply Fourier Transform. The reason is it will transform the PDE into an ODE, which is much easier to solve.

Fourier Transform

- Introduce Fourier Transform of the Green's Function

$$\bar{G}(\boldsymbol{\omega}, t; \mathbf{x_0}, t_0) = \frac{1}{(2\pi)^n} \int_{-\infty}^{\infty} G(\boldsymbol{x}, t; \mathbf{x_0}, t_0) e^{i\boldsymbol{\omega}\cdot\boldsymbol{x}}\, d^n x$$

$$G(\boldsymbol{x}, t; \mathbf{x_0}, t_0) = \int_{-\infty}^{\infty} \bar{G}(\boldsymbol{\omega}, t; \mathbf{x_0}, t_0) e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\, d^n \boldsymbol{\omega}$$

- Transforming $\frac{\partial G}{\partial t} = k\nabla^2 G + \delta(\boldsymbol{x} - \boldsymbol{x_0})\delta(t - t_0)$ gives

$$\frac{\partial \bar{G}}{\partial t} = -k\omega^2 \bar{G} + \frac{e^{i\boldsymbol{\omega}\cdot\mathbf{x_0}}}{(2\pi)^n} \delta(t - t_0)$$

- For $t > t_0$

$$\frac{\partial \bar{G}}{\partial t} = -k\omega^2 \bar{G}$$

- Solving this we will have:

$$\bar{G} = C(\boldsymbol{\omega}) e^{-k\omega^2(t - t_0)}$$

Fourier Transform

- From $G(\mathbf{x}, t; \mathbf{x_0}, t_0) = 0,\ for\ t < t_0$

$$\bar{G}(\boldsymbol{\omega}, t; \mathbf{x_0}, t_0) = 0 \quad for\ t < t_0$$

- Integrate $\frac{\partial \bar{G}}{\partial t} = -k\omega^2 \bar{G} + \frac{e^{i\boldsymbol{\omega}\cdot\mathbf{x_0}}}{(2\pi)^n}\delta(t - t_0)$ from $t = t_{0-}$ to $t = t_{0+}$

$$\bar{G}(t_{0+}) = \frac{e^{i\boldsymbol{\omega}\cdot\mathbf{x_0}}}{(2\pi)^n}$$

$$\bar{G}(\boldsymbol{\omega}, t; \mathbf{x_0}, t_0) = \frac{e^{i\boldsymbol{\omega}\cdot\mathbf{x_0}}}{(2\pi)^n} e^{-k\omega^2(t-t_0)}$$

$$G(\mathbf{x}, t; \mathbf{x_0}, t_0) = \frac{1}{(2\pi)^n}\left[\frac{\pi}{k(t - t_0)}\right]^{\frac{n}{2}} e^{\frac{(-\mathbf{x}-\mathbf{x_0})^2}{4k(t-t_0)}}$$

- From conversion between u and G:

$$u(\boldsymbol{x}, t) = \int_0^t \int_{-\infty}^{\infty} \left[\frac{1}{4\pi k(t - t_0)}\right]^{\frac{n}{2}} e^{\frac{-(\boldsymbol{x}-\boldsymbol{x_0})^2}{4k(t-t_0)}} Q(\boldsymbol{x}_0, t_0) d^n x_0 dt_0 + \int_{-\infty}^{\infty} \left(\frac{1}{4\pi kt}\right)^{\frac{n}{2}} e^{\frac{-(\boldsymbol{x}-\boldsymbol{x_0})^2}{4kt}} f(\boldsymbol{x}_0) d^n \boldsymbol{x_0}$$

where n is the number of dimensions

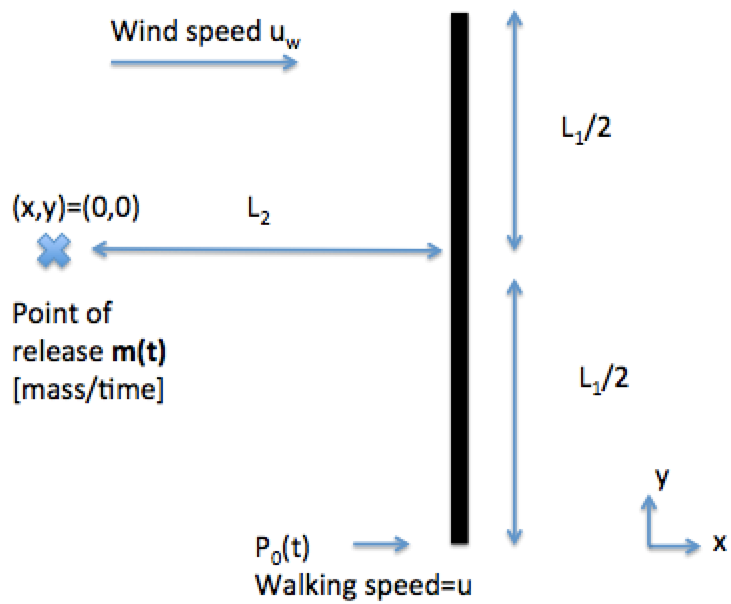
Solution

- Point Source Transport:

$$C(x,y,t) = \frac{M}{4\pi Dt} \exp\left[-\frac{(x-Ut)^2 + (y-Vt)^2}{4Dt}\right]$$

where M is the mass of the release.

Control Problem-Definition



- Factory releasing pollutants at m(t)
- Number of people entering the road during the day $P_0(t)$
- Health impact
  - cost function

$$\int_{t_0}^{t_f} \int_{-L_1/2}^{L_1/2} c(y,t) * p(y,t)\, dy\, dt$$

- Want: m(t) that minimizes the cost function
  - Constraints:

$$\int_{t_0}^{t_f} m\, dt = M, \;\; m \le m_{max}$$

Analytical Approach

1) Superposition to model continuous source

$$c(x,y,t)_{instant\_point} = \frac{M}{4\pi Dt} \exp\left(-\frac{(x-u_x t)^2}{4Dt}\right) \exp\left(-\frac{(y-u_y t)^2}{4Dt}\right)$$

$$c(x,y,t)_{continous} = \int_{t_0}^{t} \frac{m(t')}{4\pi Dt} \exp\left(-\frac{(x-u_x t')^2}{4Dt'}\right) \exp\left(-\frac{(y-u_y t')^2}{4Dt'}\right) dt'$$

Substituting x = L_2, $u_x = u_w$, $u_y = 0$:

$$c(y,t)_{continous} = \int_{t_0}^{t} \frac{m(t')}{4\pi Dt} \exp\left(-\frac{(L_2-u_w t')^2}{4Dt'}\right) \exp\left(-\frac{y^2}{4Dt}{}'\right) dt'$$

Analytical Approach

2) Rewrite the cost function (product of number of people and concentration, integrated over time and space):

$$\int_{t_0}^{t_f} \int_{-L_1/2}^{L_1/2} c(y,t) * p(y,t)\, dy\, dt$$

Applying method of characteristics, we obtain:

$$p(y,t) = p(t - \frac{(y + \frac{L_1}{2})}{u})$$

We can rewrite the cost function as:

$$\int_{t_0}^{t_f} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} (\int_{t_0}^{t} \frac{m(t')}{4\pi D t'} \exp\left(-\frac{(L_2 - u_w t')^2}{4Dt'}\right) \exp\left(-\frac{y^2}{4Dt'}\right) dt') * p\left(t - \frac{(y + \frac{L_1}{2})}{u}\right) dy\, dt'$$

Numerical Approach- Discretization

Superposition principle: treat continuous release as **i** individual releases of $\mathbf{m_i}$ at infinitesimal time intervals **δt**, where **δt=24hr/i**

$$total\ exposure = \sum_{i=1}^{i=k} m_i \delta t * k_i$$

$$where\ k_i = \sum_{ti=i\delta t+\delta t}^{t_i=tf} \sum_{y_j=-L_1/2}^{y_j=L_1/2} \frac{1}{4\pi D(t_i - i\delta t)} \exp\left(-\frac{(L_2 - u_w(t_i - i\delta t))^2}{4D(t_i - i\delta t)}\right) \exp\left(-\frac{y_j{}^2}{4D(t_i - i\delta t)}\right) p\left(t_i - \frac{(y_j + \frac{L_1}{2})}{u}\right) \delta t \delta y$$

(Recall the cost function:)

$$\int_{t_0}^{t_f} \int_{-\frac{L_1}{2}}^{\frac{L_1}{2}} \left(\int_{t_0}^{t} \frac{m(t')}{4\pi D t'} \exp\left(-\frac{(L_2 - u_w t')^2}{4Dt'}\right) \exp\left(-\frac{y^2}{4Dt'}\right) dt'\right) * p\left(t - \frac{(y + \frac{L_1}{2})}{u}\right) dy\, dt'$$

Numerical Approach- Discretization

constant

$$total\ exposure = \sum_{i=1}^{i=k} m_i(\delta t * k_i)$$

Recall the constraints: $\int_{t_0}^{t_f} m\, dt = M,\ \ m \leq m_{max}$

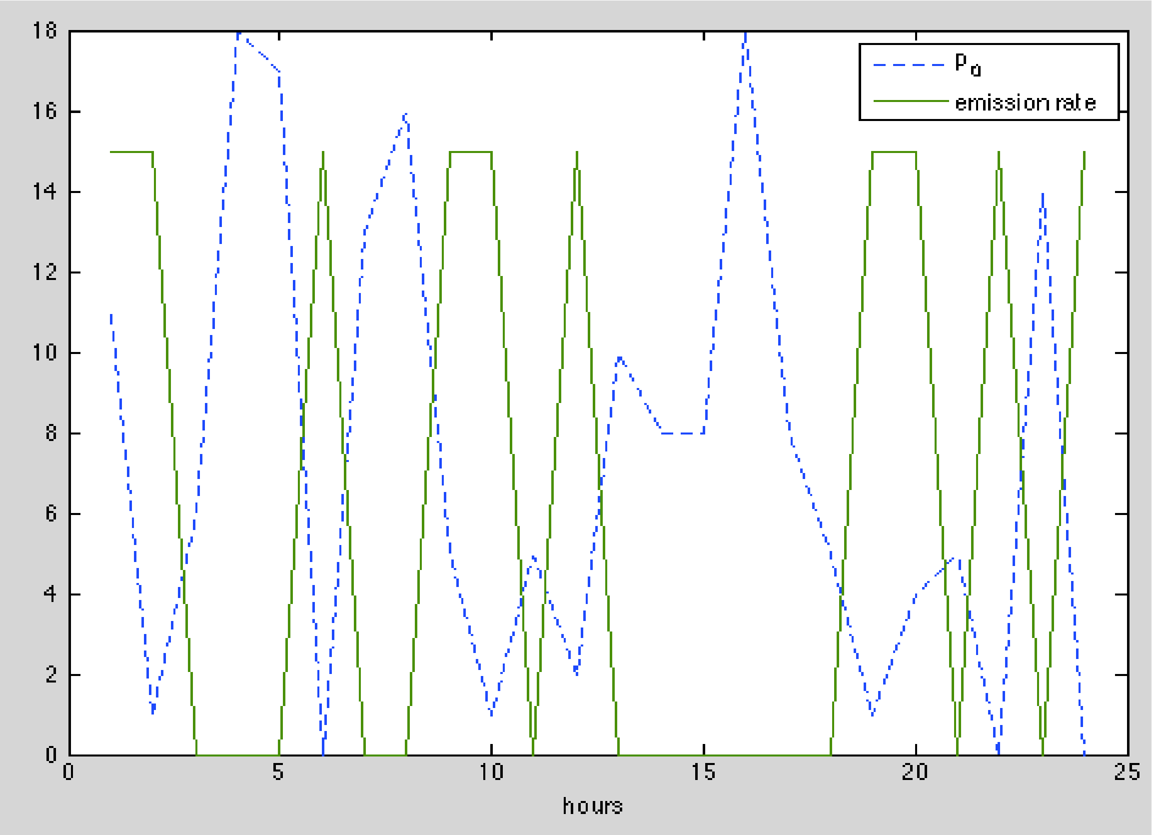
Solution: rank k_i from smallest to largest, then assign m_{max} to m_i in this order until $\sum_{i=1}^{i=k} m_i \delta t * k_i$=M

Numerical Solution

Given:

u=100; u_w=10; D=100; L_1=1000; L_2=200; m_{max}=15; M=150; δt=1; $\delta y = u*\delta t$=100;

P_0 generated by random function



References

- Aral, M. M.,& Liao, B.(1996). Analytical Solutions for Two-Dimensional Transport Equation with Time-Dependent Dispersion Coefficient. *Journal of Hydrologic Engineering,1:20-32.*

- Prentice-Hall, Inc. (1987). *Elementary Applied Partial Differential Equations (2nd ed.)*. New Jersey: Richard Haberman