A Study on Options Pricing

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What are financial options?

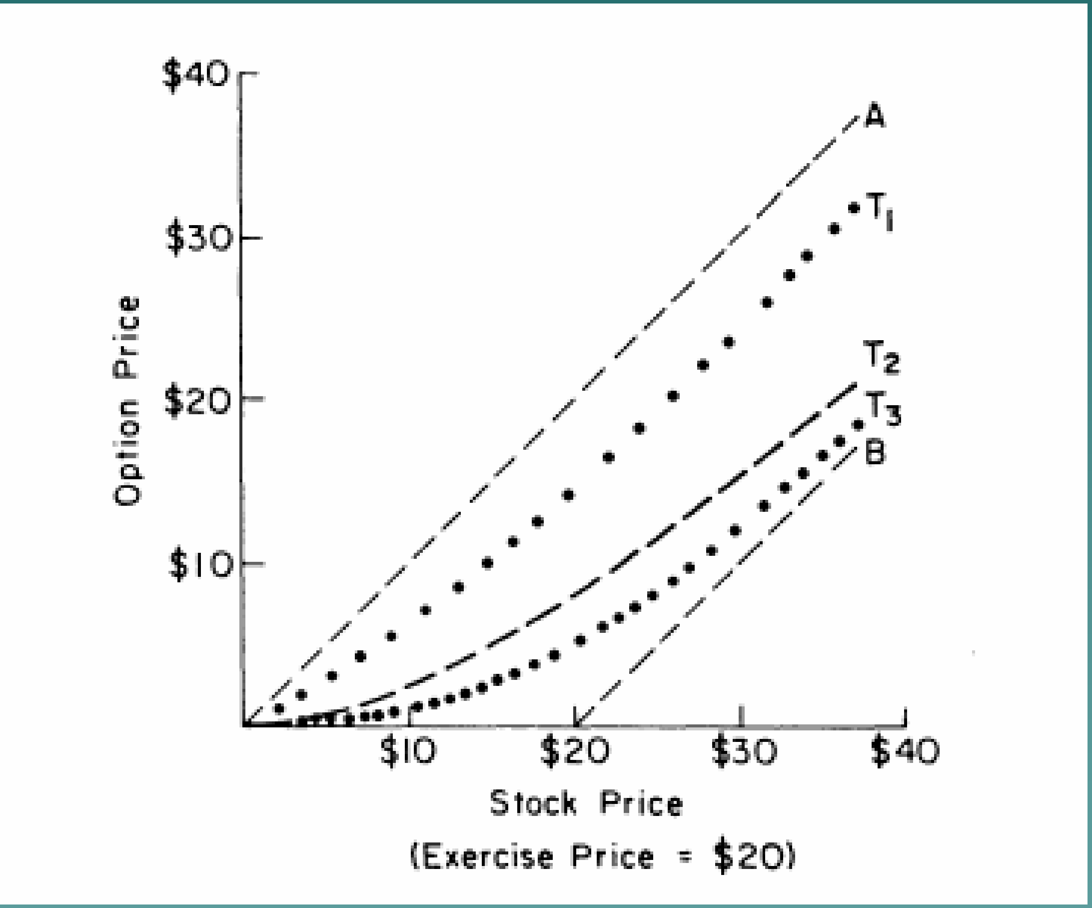
- Financial instruments that give their owner the right to sell/purchase an underlying asset at a price specified in advance
- Call Option: Right to purchase
- Put Option: Right to sell

Assumptions of the B&S PDE

1. Constant interest rate
2. Stock Price follows random walk
3. Stock pays no dividends
4. European stock option
5. No transaction costs
6. Possible to hold fraction of security

Fischer Black & Myron Scholes [1973]

The relation between option price and stock price



Black-Scholes PDE

$$V_t + \frac{\sigma^2 S^2}{2} V_{ss} + rSV_s - rV = 0$$

$$V = value\ of\ option$$

$$\sigma = volatility\ in\ underlying\ asset$$

$$S = price\ of\ underlying\ asset$$

$$r = risk - free\ rate\ of\ return$$

Solution of B&S PDE

$$V_t = -\frac{\sigma^2 S^2}{2} V_{ss} - rSV_s + rV$$

Terminal Condition

$$V\left(S,T^{\times}\right) = S - K \quad if\ S \geq K$$

$$V\left(S,T^{\times}\right) = 0 \quad if\ S < K$$

Variable Change

$$x = \ln\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(T - t)$$

$$\tau = T - t$$

$$u = Ve^{r}(T - t)$$

$$u_{\tau} = \frac{\sigma^2}{2}(u_{xx})$$

Solution of B&S PDE

$$u(x,\tau) = \frac{1}{\sigma\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0(y)\, e^{-\frac{(x-y)^2}{2\sigma^2\tau}}\, dy$$

$$u(x,\tau) = Ke^{x+\sigma^2\frac{\tau}{2}}\,\Phi\left(d_1\right) - K\Phi\left(d_2\right)$$

$$where\ d_1 = \frac{x+\sigma^2\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = \frac{x}{\sigma\sqrt{\tau}}$$

Φ *is normal CDF*

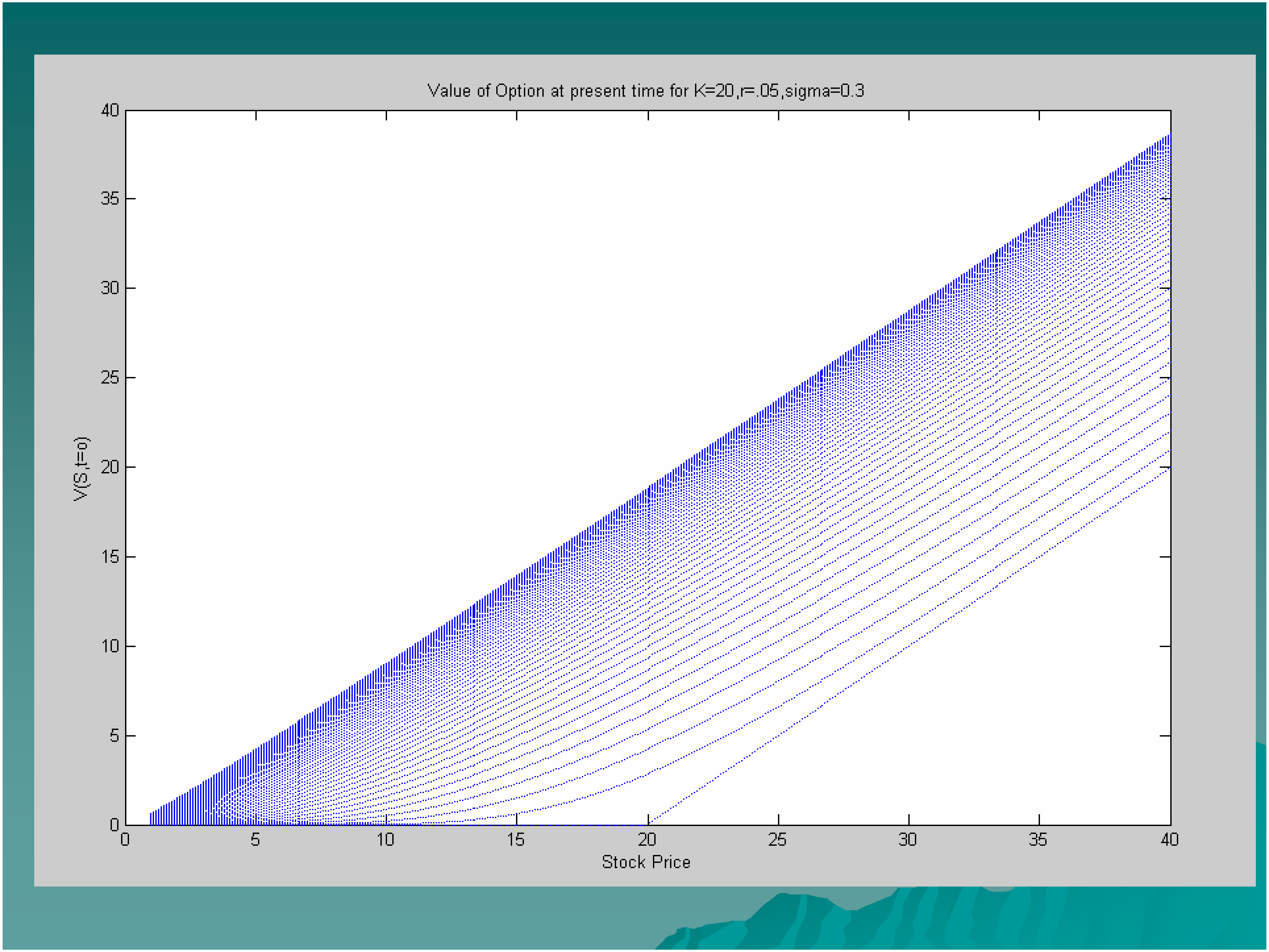
Solution of B&S PDE

$$V\left(S,t\right)=S\Phi\left(d_1\right)-Ke^{-r(T-t)}\Phi\left(d_2\right)$$

$$where\ d_1=\frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

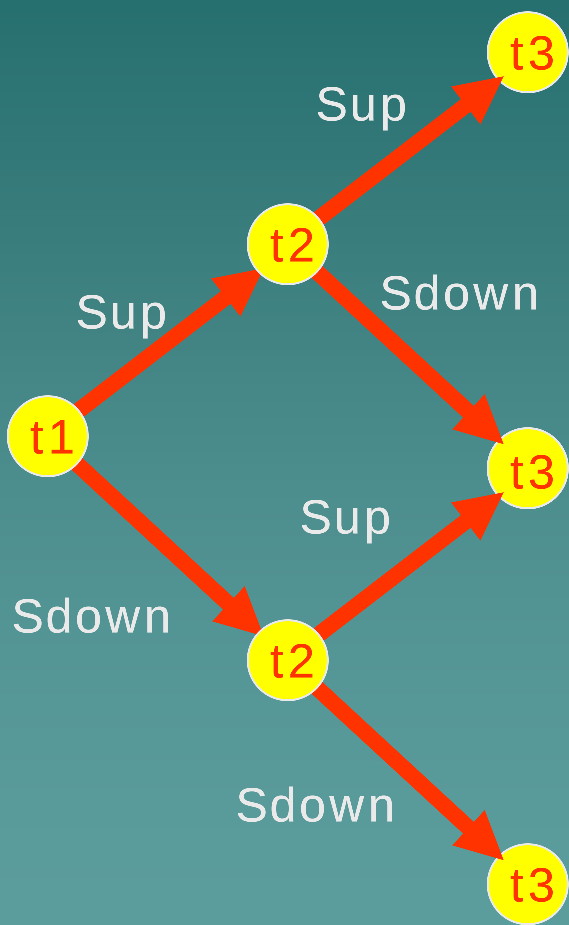
$$d_2=\frac{\ln\left(\frac{S}{K}\right)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

Φ *is normal CDF*



Value of Option at present time for K=20,r=.05,sigma=0.3
V(S,t=o)
Stock Price
0
5
10
15
20
25
30
35
40

Binomial Options Pricing Model



$$S_{t_j} = S_{t_{j-1}} e^{\sigma\sqrt{t}}$$

$$S_{t_j} = S_{t_{j-1}} e^{-\sigma\sqrt{t}}$$

$$\sigma = volatility$$

Cox, Ross & Rubenstein [1979]

Viability Approach

Viability Condition

$$\forall n \le N, \quad W^{n} \ge \boldsymbol{b}\left(T - t^{n}, S^{n}\right) \Leftrightarrow \left(T - t^{n}, S^{n}, W^{n}\right) \in Epigraph\left(\boldsymbol{b}\right)$$

$$where\, \boldsymbol{b} = 0$$

Capturability Condition

$$\left(T - t^{n^{*}}, S^{n^{\times}}, W^{n^{\times}}\right) \in Epigraph\left(\boldsymbol{c}\right)$$

$$where\, \boldsymbol{c} = \begin{Bmatrix} \boldsymbol{u}\left(s\right) & if & t = 0 \\ +\infty & if & t \ne 0 \end{Bmatrix}$$

$$where\, \boldsymbol{u} = \max\left(S - K, 0\right)$$

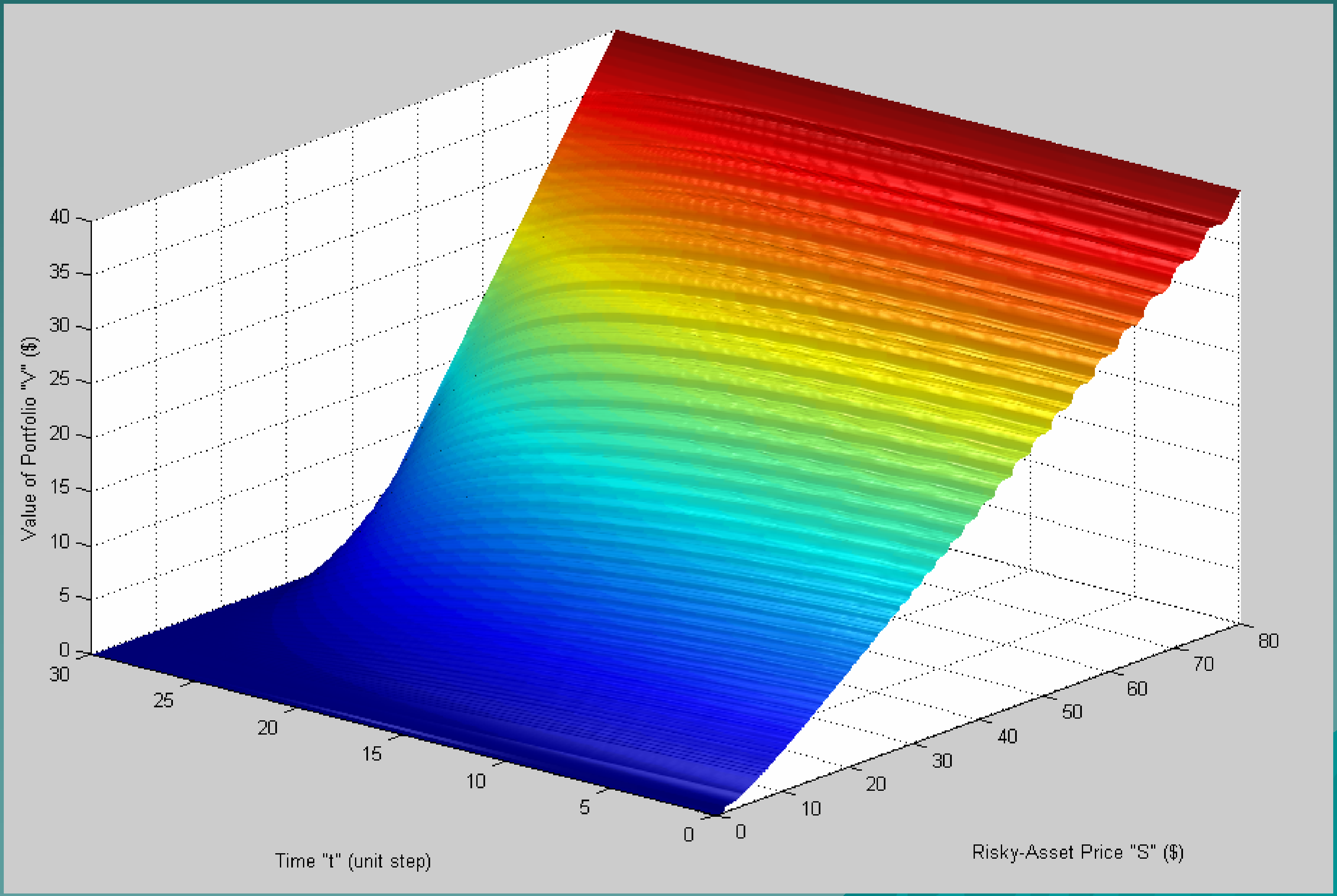
Guaranteed Capture Basin Algorithm

$$V_{\rho}^{n+1}\left(t,S_{1}\right)=\max\left(V_{\rho}\left(t,S_{1}\right),\ \inf_{p_{1}\subseteq\left[0,1\right]}\right.$$

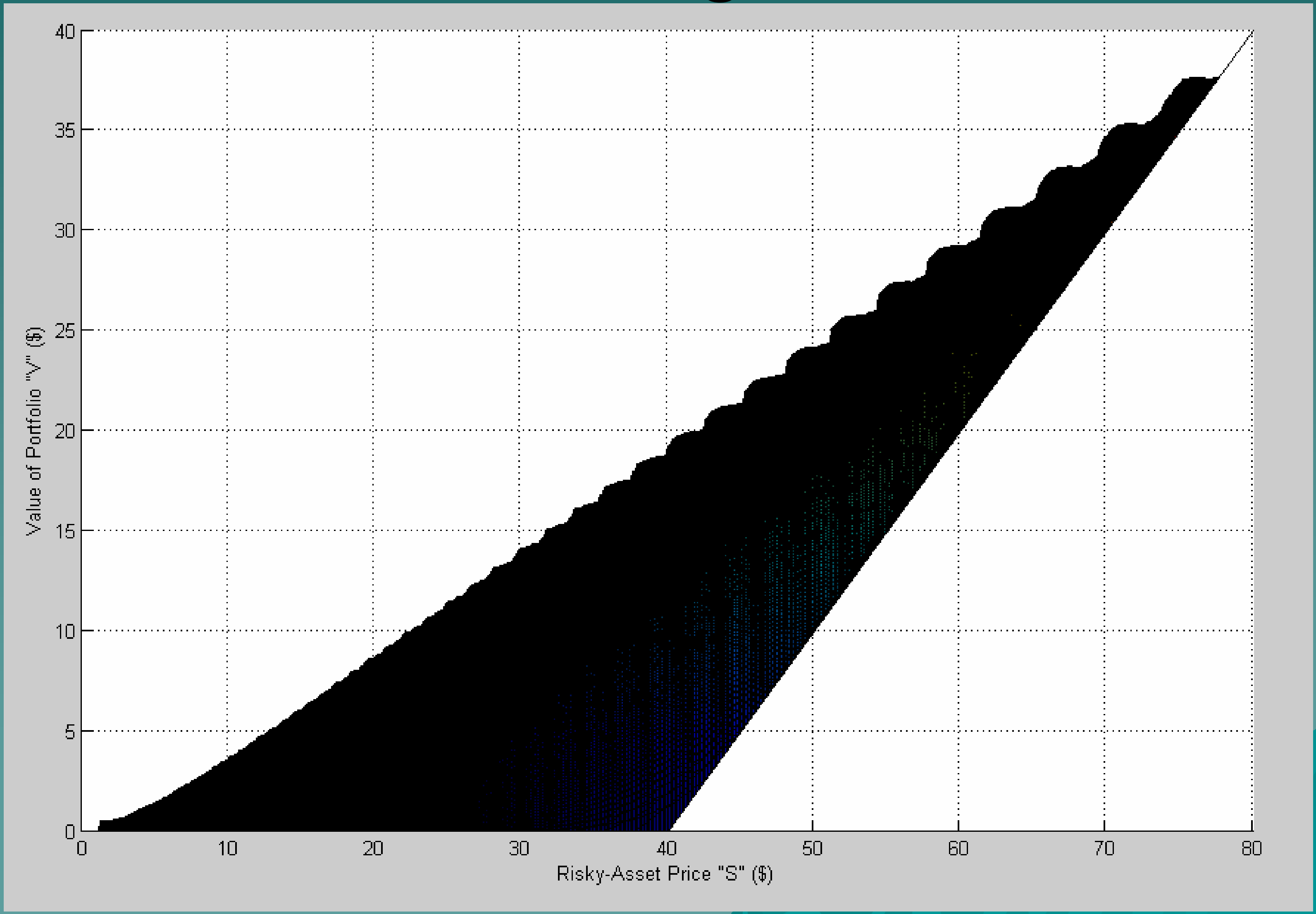
$$\sup_{v\subseteq\left[v_{m},v_{M}\right]}\frac{V_{\rho}^{n}\left(t-\rho,\,S_{1}\left(1+\gamma_{\rho_{1}}\left(S_{1},v\right)\right)\right)-\ \rho_{1}S_{1}\left(\gamma_{\rho_{1}}\left(S_{1},v\right)-\gamma_{\rho}\left(S_{1},v\right)-\gamma_{\rho_{0}}\right)}{1+\gamma_{\rho_{o}}}$$

$$\left[-\sigma\sqrt{\rho},\sigma\sqrt{\rho}\right]$$

Preliminary Results of the Capture Basin Algorithm



Preliminary Results of the Capture Basin Algorithm



On the horizon…

- Fix coding for capture basin algorithm
- Obtain more numerical examples
- Fix coding for Cox, Ross, & Rubenstein Binomial Tree

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